TROPICALIZING THE MODULI SPACE OF SPIN CURVES

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Abstract. We study the tropicalization of the moduli space of algebraic spin curves, $\overline{\mathcal{S}}_{g,n}$. We exhibit its combinatorial stratification and prove that the strata are irreducible. We construct the moduli space of tropical spin curves $\overline{S}_{g,n}^{\text{trop}}$, prove that is naturally isomorphic to the skeleton of the analytification, $\overline{S}_{g,n}^{\text{an}}$, of $\overline{\mathcal{S}}_{g,n}$, and give a geometric interpretation of the retraction of $\overline{S}_{g,n}^{\text{an}}$ onto its skeleton in terms of a tropicalization map $\overline{S}_{g,n}^{\text{an}} \to \overline{S}_{g,n}^{\text{trop}}$.

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Contents

1. Introduction and preliminaries

1.1. **Introduction.** In recent years the geometry of toroidal compactifications of moduli spaces has been treated more and more from the point of view of tropical geometry. If $\mathcal{U} \hookrightarrow \mathcal{Y}$ is a toroidal embedding of Deligne-Mumford stacks, the boundary, $\mathcal{Y} \smallsetminus \mathcal{U}$, is endowed with a stratification whose structure is encoded in the skeleton, $\Sigma(\mathcal{Y})$, of the Berkovich analytification of $\mathcal{Y}$. This skeleton is a topological retraction of the analytification and has the structure of a generalized cone complex. On the other hand, if $\mathcal{Y}$ is a moduli space, $\Sigma(\mathcal{Y})$ has been described, in some remarkable cases, as the moduli space of objects which can be viewed as the tropical counterpart of the objects parametrized by $\mathcal{Y}$. See [ACP15], [Uli15] [CMR16], [AP18], for example.

This paper studies the moduli space of theta characteristics on smooth curves of genus g, written $\mathcal{S}_g$, and its compactification via stable spin curves. The moduli space of stable spin curves, $\overline{\mathcal{S}}_g$, was constructed by Cornalba in [Cor89]. It is endowed with a finite (ramified) map of degree 2^{2g}, written $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$, onto the moduli space of stable curves, $\overline{\mathcal{M}}_g$. The fiber of π over a curve X parametrizes theta characteristics on partial normalizations of X. The space $\overline{\mathcal{S}}_g$ has two connected components, $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, parametrizing theta characteristics whose space of global sections has even or odd dimension, respectively. There has been in the years a constant interest in the geometry of $\overline{\mathcal{S}}_g$ and its applications; see [J98], [J00], [JKV01], [BF06], [F10], [FV14], for example. Here we pursue the following route:

(1) We lay the combinatorial groundwork by defining spin graphs and (extended) spin tropical curves. We construct the moduli space of tropical spin curves, $\overline{S}_g^{\text{trop}}$, and the natural map $\pi^{\text{trop}} \colon \overline{S}_g^{\text{trop}} \to \overline{M}_g^{\text{trop}}$ to the moduli space of tropical curves.
(2) We describe the stratification of the toroidal embedding $\mathcal{S}_g \hookrightarrow \overline{\mathcal{S}}_g$ using spin graphs and exhibit a recursive presentation of the stata. We prove that the strata are irreducible.
(3) We give a modular interpretation of the skeleton of the analytification of $\overline{\mathcal{S}}_g$, by showing that it is isomorphic to $\overline{S}_g^{\text{trop}}$ as an extended generalized cone complex.

Theta characteristics on a tropical curve, Γ, of genus g have been studied in [Zha10], in case Γ is "pure" (i.e. such that $g = b_1(\Gamma)$, see 2.4). They are defined as divisors D such that $2D$ is equivalent to the canonical divisor of Γ, and are in bijection with the elements of $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$, so there are 2^g of them, of which exactly one is non-effective.

Theta characteristics on smooth algebraic curves tropicalize to theta characteristics on the skeleton (a tropical curve) and the tropicalization map has an interesting behaviour. Indeed, each effective theta characteristic in the skeleton is the image of 2^g theta characteristics, exactly half of which are odd; the unique non-effective theta characteristic is the image of 2^g even theta characteristics; this has been proved in [JL18] generalizing results of [P16] in the case of hyperelliptic curves.

We briefly mention that, in the case of plane quartic curves, an interesting relation has been discovered between classical and tropical bitangents in [BLMPR16], [LM17], and [CJ17].

On a different vein, the behaviour of the tropicalization map indicates that a moduli space parametrizing theta characteristics on tropical curves defined as above could not be the skeleton of the moduli space of stable spin curves, as odd and even theta characteristics, which live in different connected components in the algebraic world, may tropicalize to the same object. For this reason, our notion of a spin structure on a tropical curve is a bit different in that it consists of a theta characteristic enriched by a parity function.

Much less is known about higher spin curves, or arbitrary roots of divisors, on the tropical side. On the other hand the algebro-geometric counterpart has been the object of much interest, and the corresponding moduli spaces have been widely studied together with their compactifications; see for example [J00], [AJ03], [CCC07], [C08]. The combinatorial structure of these compactifications is quite intricate and we believe it would be of great interest to study it from the tropical, and non-Archimedean, point of view.

1.2. **Outline of the paper.** In Section 2 we introduce the notion of spin graph, tailored to represent the combinatorial data associated to an algebraic spin curve. A spin graph is a triple (G, P, s) where G is a stable graph and (P, s) a spin structure, consisting of a cyclic subgraph P of G, and a parity function defined on the vertices of the contracted graph G/P. We work with graphs with legs; although the legs are irrelevant to the spin structure, their presence is essential as it enables us to use recursive arguments.

Then we define a spin tropical curve as a tropical curve whose underlying graph is endowed with a spin structure. For every tropical curve Γ, we denote by S_Γ^{trop} the set of spin structures of Γ.

There is a natural graded poset structure on the set of spin graphs of fixed genus, given by the contractions of spin graphs, that is, contractions of graphs which are compatible with the spin structures. As is well known, contractions of graphs correspond to specializations of tropical curves, and in fact we use this poset structure to construct a moduli space of spin tropical curves, $\overline{S}_{g,n}^{\mathrm{trop}}$, as an extended generalized cone complex. The cells of this space parametrize extended spin tropical curves with a fixed underlying spin graph. The following statement summarizes the main results of Section 2.

Theorem A. *The extended generalized cone complex $\overline{S}_{g,n}^{\mathrm{trop}}$ has pure dimension $3g-3+n$ and has two connected components corresponding to the parity of the spin structure. There is a natural map $\pi^{\mathrm{trop}} \colon \overline{S}_{g,n}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$ which is a morphism of extended generalized cone complexes, and for every extended tropical curve Γ we have $(\pi^{\mathrm{trop}})^{-1}(\Gamma) = S_\Gamma^{\mathrm{trop}}/\mathrm{Aut}(\Gamma)$.*

In Section 3 we consider the moduli space of stable spin curves. As mentioned earlier, we actually consider more generally the moduli space of n-pointed stable spin curves $\overline{\mathcal{S}}_{g,n}$: for $n = 0$ this is the space constructed by Cornalba in [Cor89], while for $n > 0$ this is a particular case of the construction of Jarvis in [J00]. A stable spin curve over a stable curve X is given by a theta characteristic on a partial normalization of X. So, a stable spin curve has a combinatorial type which is given by the dual graph, G, of X, the set, P, of nodes of the partial normalization, and a parity function, s, corresponding to the parity of the space of sections on each connected component of the partial normalization. These data are encoded in the dual spin graph of the stable spin curve. We set $\mathcal{S}_{(G,P,s)}$ to be the locus in $\overline{\mathcal{S}}_{g,n}$ parametrizing stable spin curves with combinatorial type (G, P, s). The following statement summarizes informally some results of Section 3.

Theorem B. *We have a toroidal open embedding $\mathcal{S}_{g,n} \hookrightarrow \overline{\mathcal{S}}_{g,n}$ which admits a stratification by loci of the form $\mathcal{S}_{(G,P,s)}$, and such that the poset of the closures of the strata is governed by the graded poset of spin graphs.*

We devote Section 4 to prove that the strata $\mathcal{S}_{(G,P,s)}$ are irreducible. For our further applications we actually need a stronger statement, Theorem 4.2.3, from which the irreducibility of $\mathcal{S}_{(G,P,s)}$ follows easily. The proof is by induction, based on a direct proof in some special cases.

In Section 5 we describe the skeleton of the Berkovich analityfication $\overline{S}^{\mathrm{an}}_{g,n}$ of $\overline{\mathcal{S}}_{g,n}$ and show that there is an isomorphism, $\overline{\Sigma}(\overline{\mathcal{S}}_{g,n}) \cong \overline{S}^{\mathrm{trop}}_{g,n}$, between this skeleton and our moduli space of spin tropical curves. Our construction is consistent with the tropicalization map for curves $\mathrm{Trop}_{\overline{\mathcal{M}}_{g,n}} : \overline{M}^{\mathrm{an}}_{g,n} \to \overline{M}^{\mathrm{trop}}_{g,n}$ and the results of the Section 5 imply the following

Theorem C. *There is a tropicalization map,* $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}} : \overline{S}^{\mathrm{an}}_{g,n} \to \overline{S}^{\mathrm{trop}}_{g,n}$, *fitting in the following commutative diagram*

$$\begin{array}{ccc}
\overline{S}^{\mathrm{an}}_{g,n} & \xrightarrow{\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}} & \overline{S}^{\mathrm{trop}}_{g,n} \\
{\scriptstyle \pi^{\mathrm{an}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^{\mathrm{trop}}} \\
\overline{M}^{\mathrm{an}}_{g,n} & \xrightarrow{\mathrm{Trop}_{\overline{\mathcal{M}}_{g,n}}} & \overline{M}^{\mathrm{trop}}_{g,n}
\end{array}$$

1.3. **Graphs.** Throughout the paper, $G = (V, E, L, w)$ is a *weighted graph with legs*, or simply a *graph*, defined by the following set of data:

(1) A non empty finite set, $V = V(G)$, of *vertices*, endowed with a *weight* function $w \colon V \to \mathbb{N}$.
(2) A finite set $H = H(G)$ of *half-edges* endowed with an involution $\iota : H \to H$, and an endpoint map $\epsilon : H \to V$.
(3) A set $E = E(G)$ of *edges* of G, defined as the set of pairs $e = \{h, h'\}$ of (distinct) elements of H interchanged by ι.
(4) A set $L = L(G)$ of *legs* of G defined as the set of fixed points of ι.

If the weight function w is identically zero we say G is *weightless*.

We set $n = |L(G)|$; if $n = 0$ we say G is *leg-free*. To a graph G we associate the leg-free graph $[G] := G - L$, obtained from G by removing the legs.

Given $v \in V$, we denote by $\ell(v) = |\epsilon^{-1}(v) \cap L|$ the number of legs ending in v, and by $\deg(v) = |\epsilon^{-1}(v) \smallsetminus L|$ the number of half-edges other than legs ending in v.

The *genus* of G is $g(G) := \sum_{v \in V} w(v) + b_1(G)$, where $b_1(G) = |E| - |V| + c(G)$, and $c(G)$ is the number of connected components of G. Obviously, $g(G) = g([G])$.

If $F \subset E$, we write $\langle F \rangle$ for the leg-free subgraph of G spanned by F. We will abuse notation and write $b_1(F) = b_1(\langle F \rangle)$. Next, $G - F$ denotes the graph obtained by removing F, so that G and $G - F$ have the same vertices and the same legs. We denote by $G - F^o$ the graph obtained by replacing every edge $e = \{h, h'\}$ in F by a pair of legs having the same endpoint as h and h'. Hence $[G - F^o] = [G - F]$ and

$$|L(G - F^o)| = |L(G)| + 2|F|.$$

The group of divisors on G, written $\mathrm{Div}(G)$, is the free abelian group generated by V. We write $\underline{d} = (\underline{d}_v,\ v \in V)$ for a divisor of G. The *degree* of a divisor $\underline{d}$ is the integer $|\underline{d}| := \sum_{v\in V} \underline{d}_v$. The *canonical divisor* of G, written $\underline{k}_G$, is defined, for all $v \in V$, as follows

$$(\underline{k}_G)_v = 2w(v) - 2 + \deg(v).$$

Notice that $\underline{k}_G$ has degree $2g(G) - 2c(G)$.

We say that G is *stable* (respectively, *semistable*) if it is connected and if for every vertex $v \in V(G)$ we have

$$2w(v) - 2 + \deg(v) + \ell(v) > 0 \quad \text{(respectively,} \geq 0).$$

The isomorphism classes of stable graphs of genus g with n legs form a finite set denoted by $\mathcal{G}_{g,n}$. We will use the following two well known facts:

(1) $\mathcal{G}_{g,n}$ is not empty if and only if $2g - 2 + n > 0$.
(2) If G is stable and $F \subset E(G)$, then every connected component of $G - F^o$ is stable.

If G is weightless and $\deg(v) + \ell(v) = 3$ for every vertex $v \in V(G)$, we say that G is 3-*regular*. It is easy to check that a 3-regular graph of genus g with n legs has $3g - 3 + n$ edges.

1.4. **Cycles and Contractions.** Let G be a graph. Following [D97], we denote by $\mathcal{E}_G$ (respectively, $\mathcal{V}_G$) the vector space over $\mathbb{F}_2$ spanned by E (respectively, by V). Every element of $\mathcal{E}_G$ can be identified with a subset, F, of E, and we shall abuse notation by using the same symbol, F, for the vector and the subgraph, $\langle F\rangle$. If G has legs, then the subgraph spanned by E is $[G]$, but we shall abuse notation again and denote by $G \in \mathcal{E}_G$ (instead of $[G]$) the element corresponding to E.

We have a linear map $\partial\colon \mathcal{E}_G \to \mathcal{V}_G$ such that $\partial(e) = u + v$ where u and v are the ends of e. The kernel of ∂ is called the *cycle space* of G, and denoted $\mathcal{C}_G := \mathrm{Ker}\partial$.

It is well known that $|\mathcal{C}_G| = 2^{b_1(G)}$, and that $F \in \mathcal{E}_G$ lies in $\mathcal{C}_G$ if and only if the graph $\langle F\rangle$ has no vertex of odd degree (such graphs are called *Eulerian* if they are connected). We refer to elements of $\mathcal{C}_G$, and to the subgraphs of G they span, as *cyclic*. If $F \in \mathcal{C}_G$ and $b_1(F) = 1$ we say that F is a *cycle*.

A *morphism* $\eta\colon G \to G'$ between two graphs is given by a map $\eta\colon V(G) \cup H(G) \to V(G') \cup H(G')$, compatible with the graph structure. That is, writing ϵ' and ι' for the endpoint map and the involution of G', for every $x \in V(G) \cup H(G)$ we have

$$\eta \circ (id_{V(G)} \cup \epsilon)(x) = (id_{V(G')} \cup \epsilon') \circ \eta(x)$$

$$\eta \circ (id_{V(G)} \cup \iota)(x) = (id_{V(G')} \cup \iota') \circ \eta(x).$$

Hence η induces maps $\eta^V\colon V(G) \to V(G')$, $\eta^E\colon E(G) \to E(G') \cup V(G')$ and $\eta^L\colon L(G) \to L(G') \cup V(G')$. An *isomorphism* is a morphism η such that η^V is a weight-preserving bijection, η^E is a bijection between $E(G)$ and $E(G')$, and η^L an order-preserving bijection between $L(G)$ and $L(G')$.

We write $\mathrm{Aut}(G)$ for the group of automorphisms of G (i.e., isomorphisms of G with itself). Notice that an automorphism of G is the identity on L.

Given a subset $F \subset E$, the *contraction* of F is the graph G/F together with the morphism

$$\gamma\colon G \longrightarrow G/F$$

given by contracting all edges $e \in F$. The legs of G/F are identified with the legs of G, and the edges of G/F are identified with $E(G) \smallsetminus F$. The weight function $V(G/F) \to \mathbb{N}$ maps $v \in V(G/F)$ to $g(\gamma^{-1}(v))$.

Notice that G is connected if and only if so is G/F. We have

$$b_1(G/F) = b_1(G) - b_1(F),$$

and the genus of G equals the genus of G/F.

A contraction $\gamma\colon G \to H$ induces a homomorphism $\gamma_*\colon \mathrm{Div}(G) \to \mathrm{Div}(H)$ such that for $\underline{d} \in \mathrm{Div}(G)$ and $v \in V(H)$ we have $(\gamma_*\underline{d})_v = \sum_{u\in\gamma^{-1}(v)} \underline{d}_u$.

1.5. **Algebraic curves.** In this paper, k denotes an algebraically closed field of characteristic not equal to 2, and g, n non-negative integers.

A *curve* is a reduced, projective variety of dimension one, not necessarily connected, over k. We assume that our curves have at most nodes as singularities. The genus of a curve is the arithmetic genus. We write ω_X for the dualizing sheaf of a curve X.

A *pointed curve* is a pair (X, σ), where X is a curve and σ is an ordered finite set of smooth and distinct points of X.

As usual, the *dual graph* $G = (V, E, L, w)$ of an n-pointed curve (X, σ) is defined so that V is the set of irreducible components of X, the weight of a vertex is the genus of the normalization of the corresponding component, E is the set of nodes of X, and to every point in σ there corresponds a leg ending at the vertex corresponding to the component to which the point belongs. For $v \in V$ we write $C_v \subset X$ for the corresponding component, while we use the same notation for edges (resp. legs) of G and nodes (resp. points) of X. The genus of G is equal to the genus of X.

We define (X, σ) to be *stable*, or *semistable*, if so is its dual graph.

Let X be a nodal curve and $E \subset X$ a smooth rational component; we say E is *exceptional* if $|E \cap \overline{X \smallsetminus E}| = 2$ and $|\sigma \cap E| = 0$. A node of X is *exceptional* if it lies on an exceptional component of X. A stable curve has no exceptional component.

A pointed curve is *quasistable* if it is semistable and if two exceptional components of X do not intersect.

Let $(\widehat{X}, \sigma)$ be a quasistable curve. There is a stable curve, written $st(\widehat{X}, \sigma)$, obtained by contracting all the exceptional components of $\widehat{X}$; we have a unique contraction morphism $\widehat{X} \to st(\widehat{X})$. Conversely, given a stable curve (X, σ) and a set of nodes R of X, we construct the quasistable curve $(\widehat{X}_R, \sigma)$, called the *blow up* of X at R, such that $X = st(\widehat{X}_R)$, where the set of exceptional components of $\widehat{X}_R$ is contracted to R. We write X^ν_R for the normalization of X at R, so that

$$\widehat{X}_R = X_R^\nu \cup (\cup_{r\in R} E_r) \tag{1}$$

with E_r an exceptional component. We shall view X_R^ν as a pointed curve, (X_R^ν, σ_R), as follows. Let $\nu_R\colon X_R^\nu \to X$ be the normalization map, then $\sigma_R = \sigma \cup \nu_R^{-1}(R)$. So, the dual graph of (X_R^ν, σ_R) is $G - R^o$. Notice that every connected component of (X_R^ν, σ_R) is stable, as a pointed curve.

The dual graph of $\widehat{X}_R$ will be denoted by $\widehat{G}_R$. Clearly $\widehat{G}_R$ is obtained from G by inserting a vertex, $\widehat{v}_r$, in the interior of every edge in R. We have a natural inclusion $V(G) \subset V(\widehat{G}_R)$ and we refer to the vertices of type $\widehat{v}_r$ as the *exceptional vertices* of $\widehat{G}_R$. Summarizing, we have

$$V(\widehat{G}_R) = V(G) \cup \{\widehat{v}_r,\ \forall r \in R\}.$$

Let $\mathrm{Pic}(X)$ be the Picard scheme of a curve X, parametrizing line bundles on X up to isomorphism. There is a homomorphism, $\underline{\deg}\colon \mathrm{Pic}(X) \to \mathrm{Div}(G)$ sending a line bundle to its multidegree. We have $\underline{\deg}\,\omega_X = \underline{k}_G$.

A *theta characteristic* is a line bundle L on X such that $L^2 \cong \omega_X$. We set

$$\mathcal{S}_X^0 := \{L \in \mathrm{Pic}(X) :\ L^2 \cong \omega_X\}. \tag{2}$$

We have

$$\deg_{C_v} \omega_X = 2w(v) - 2 + \deg(v) \tag{3}$$

for every vertex v of G. Hence a theta characteristic on X exists if and only if $\deg(v)$ is even for every v, if and only if G is cyclic.

Recall that the parity of a theta characteristic, L, is the parity of $h^0(X, L)$, and it is deformation invariant. We write $\mathcal{S}_X^{0,+}$ for the even theta characteristics and $\mathcal{S}_X^{0,-}$ for the odd ones, so that $\mathcal{S}_X^0 = \mathcal{S}_X^{0,+} \sqcup \mathcal{S}_X^{0,-}$.

The moduli stack of stable n-pointed curves of genus g is denoted by $\overline{\mathcal{M}}_{g,n}$. We let $\mathcal{M}_{g,n}$ be the substack parametrizing smooth curves. The map $\overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}$, sending a stable pointed curve to its dual graph, is a surjection to which we will return later.

From now on, $X = (X, \sigma)$ will be a stable curve of genus g with n marked points, and G its dual graph. We shall always assume that $2g - 2 + n > 0$.

2. Spin graphs and spin tropical curves

2.1. **Spin graphs.** We want to define spin structures on a graph similarly to what is done for curves. As we shall see, a spin structure on a curve X is essentially a theta characteristic on a partial normalization of X. Since partial normalizations of X correspond to spanning subgraphs of G, we will define a spin structure on a graph to be a theta characteristic on a spanning subgraph, plus some extra structure.

Let G be a graph and $\underline{k}_G$ its canonical divisor. If every vertex of G has even degree, with (3) in mind we consider the divisor

$$(\underline{k}_G/2)_v = w(v) - 1 + \deg(v)/2, \quad \forall v \in V.$$

This divisor makes sense if and only if G is cyclic, and it plays the role of a theta characteristic on G. Using the notation introduced in Subsection 1.4, we introduce the following definition.

Definition 2.1.1. A *spin graph* is a triple (G, P, s) such that G is a graph, $P \in \mathcal{C}_G$ and $s\colon V(G/P) \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is a *sign* (or *parity*) function such that $s(v) = 0$ for every vertex v of G/P of weight 0. We call (P, s) a *spin structure* on G.

The *parity* of a spin graph (G, P, s) is the parity of $\sum_{v \in V(G/P)} s(v)$.

Recall that G is assumed to have n legs. Yet, a spin structure on G is the same as a spin structure on the leg-free graph $[G]$.

We denote by SP_G the set of spin structures on G, and we write

$$SP_G = SP_G^+ \sqcup SP_G^-$$

where SP_G^+ (resp. SP_G^-) denotes the set of even (resp. odd) spin structures.

If the sign function is identically zero, we denote it by s_0. The trivial spin structure, $(0, s_0)$, is of course even, hence $SP_G^+ \neq \emptyset$.

Observe that $g(G) = 0$ if and only if $SP_G = SP_G^+$, if and only if $|SP_G| = 1$.

If G has one vertex and no edges, a spin graph on it is of type $(G, 0, s)$; in this case we denote such a spin graph also by (G, G, s).

We have a forgetful map $\phi\colon SP_G \longrightarrow \mathcal{C}_G$ mapping (P, s) to P. Set

$$SP_{(G,P)} = \phi^{-1}(P), \quad SP^+_{(G,P)} = SP_{(G,P)} \cap SP_G^+, \quad SP^-_{(G,P)} = SP_{(G,P)} \cap SP_G^-.$$

Remark 2.1.2. Let G be a graph and G_0 the underlying weightless graph. Then $\mathcal{C}_G = \mathcal{C}_{G_0}$ and $SP_{G_0} \subset SP_G$, preserving the parity.

For $P \in \mathcal{C}_G$ we set $R := E \smallsetminus P$. With the notation in 1.3, define

$$\overline{P} := G - R^o$$

so that $\overline{P}$ has the same vertices as G and is endowed with $(n + 2|R|)$ legs.

Notice that $\overline{P}$ is connected if and only if P is a spanning subgraph of G, if and only if $|V(G/P)| = 1$. More precisely, the connected components of $\overline{P}$ correspond to the vertices of G/P, so that we write $\{\overline{P}_v,\ v \in V(G/P)\}$ for the set of connected components of $\overline{P}$.

Definition 2.1.3. Let (G, P, s) be a spin graph. For any $v \in V(G/P)$ we define the spin structure $(\overline{P}_v, s_v)$ on $\overline{P}_v$ such that $s_v(v) := s(v)$ for $v \in V(G/P)$ (we identify the vertex of $\overline{P}_v/\overline{P}_v$ with $v \in V(G/P)$).

We call $(\overline{P}_v, s_v)$, for $v \in V(G/P)$, the *connected components* of (G, P, s).

Connecting with the initial observation, for any spin structure (P, s) on G, we define the divisor $\underline{d}^P \in \operatorname{Div}(\overline{P})$ as follows

$$\underline{d}^P_v = w(v) - 1 + \deg_{\overline{P}}(v)/2, \tag{4}$$

where $\deg_{\overline{P}}(v)$ is the degree of v as a vertex of $\overline{P}$. Hence $2\underline{d}^P = \underline{k}_{\overline{P}}$ so that $\underline{d}^P$ will be viewed as the theta characteristic of $\overline{P}$.

We denote by $c^+(\overline{P})$ the number of connected components of $\overline{P}$ of positive genus, i.e. the number of vertices of G/P of positive weight.

Lemma 2.1.4. *Let $P \in \mathcal{C}_G$. Then $|SP_{(G,P)}| = 2^{c^+(\overline{P})}$. Moreover,*

$$|SP^+_{(G,P)}| = |SP^-_{(G,P)}| = 2^{c^+(\overline{P})-1},$$

unless $P = 0$ and G is weightless, in which case $|S^+_{(G,0)}| = 1$ and $S^-_{(G,0)} = \emptyset$. In particular

$$|SP_G| = \sum_{P\in\mathcal{C}_G} 2^{c^+(\overline{P})} \geq 2^{b_1(G)+1} - 1,$$

with equality if and only if G is weightless and $c^+(\overline{P}) = 1$ for every $P \neq 0$.

Proof. A vertex of G/P has weight nonzero if and only if it is obtained contracting a connected component of P of positive genus. The statement follows trivially from this. ♣

2.2. **Contractions of spin graphs.** Let $\gamma\colon G \to G'$ be a morphism of graphs. The maps γ^V and γ^E (see Subsection 1.4) induce two linear maps

$$\gamma^V_*\colon \mathcal{V}_G \longrightarrow \mathcal{V}_{G'} \qquad \text{and} \qquad \gamma^E_*\colon \mathcal{E}_G \longrightarrow \mathcal{E}_{G'},$$

where $\gamma^E_*(e) = e$ if $\gamma^E(e) \in E(G')$ and $\gamma^E_*(e) = 0$ otherwise.

The following easy lemmas give some useful functorial properties of a contraction $\gamma\colon G \to G'$ with respect to the maps γ^V_* and γ^E_*. The same statements are easily seen to hold also when γ is an isomorphism.

Lemma 2.2.1. *Let $\gamma\colon G \to G' = G/F$ be a contraction.*

(1) The following diagram of linear maps is commutative

$$\begin{array}{ccc} \mathcal{E}_G & \xrightarrow{\gamma^E_*} & \mathcal{E}_{G'} \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathcal{V}_G & \xrightarrow{\gamma^V_*} & \mathcal{V}_{G'} \end{array}$$

(2) The restriction of γ^E_* to $\mathcal{C}_G$ induces a surjection, denoted as follows

$$\gamma_*\colon \mathcal{C}_G \longrightarrow \mathcal{C}_{G'}.$$

(3) If $\widetilde{\gamma}\colon G' \to G''$ is a contraction, then $(\widetilde{\gamma}\gamma)_* = \widetilde{\gamma}_*\gamma_*$.

(4) Let $P \in \mathcal{C}_G$ and set $P' := \gamma_* P$. The following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & G' \\ \downarrow & & \downarrow \\ G/P & \xrightarrow{\overline{\gamma}} & G'/P' \end{array}$$

where $\overline{\gamma}$ is a contraction and $G'/P' = G/(F \cup P)$.

Proof. The first three parts are clear. By definition, $P' = P \smallsetminus F$. Hence $G'/P' = (G/F)/P' = (G/F)/(P \smallsetminus F) = G/(F \cup P)$ ♣

Lemma 2.2.2. *Let $\gamma\colon G \to G'$ be a contraction.*

(1) There is a parity preserving map (abusing notation)

$$\gamma_*\colon SP_G \to SP_{G'}; \qquad (P, s) \mapsto (P', s')$$

where $P' = \gamma_* P$ and s' maps $v' \in V(G'/P')$ to $\sum_{v \in \overline{\gamma}^{-1}(v')} s(v)$.

(2) If $\widetilde{\gamma}\colon G' \to G''$ is a contraction, then $(\widetilde{\gamma}\gamma)_* = \widetilde{\gamma}_* \gamma_*$.

Proof. $\gamma_* P$ is cyclic by Lemma 2.2.1 . Hence to show that the image of γ_* is in $SP_{G'}$ it suffices to check that $s'(v') = 0$ for every vertex v' in G'/P' of weight zero. Since $\overline{\gamma}\colon G/P \to G'/P'$ is a contraction, if v' has weight zero then every vertex v of G/P mapping to v' has weight zero, hence $s(v) = 0$ and hence $s'(v') = 0$.

The fact that γ_* preserves the parity, and part (2), are clear. ♣

Definition 2.2.3. A *contraction* (resp. an *isomorphism*) from a spin graph (G, P, s) to a spin graph (G', P', s') is a contraction (resp. an isomorphism) of graphs $\gamma\colon G \to G'$ such that $\gamma_*(P, s) = (P', s')$.

We write $\gamma\colon (G, P, s) \to (G', P', s')$.

Recall that $\mathcal{G}_{g,n}$ is the set of isomorphism classes of stable graphs of genus g with n legs. The contraction of a graph in $\mathcal{G}_{g,n}$ is again in $\mathcal{G}_{g,n}$, and $\mathcal{G}_{g,n}$ is a poset with respect to the following partial order

$$G \geq G', \text{ if } G' = G/F \text{ for some } F \subset E(G)$$

(i.e. if there is a contraction $G \to G'$). It is well known that $\mathcal{G}_{g,n}$ is graded, and a rank function on it is $G \mapsto |E(G)|$. Moreover, $\mathcal{G}_{g,n}$ is connected as a poset (i.e. the associated graph is connected).

Remark 2.2.4. A way to see that $\mathcal{G}_{g,n}$ is connected, useful for later, is to consider the graph $G_{g,n}$ made of a single vertex of weight g with n legs. Then for every $G \in \mathcal{G}_{g,n}$ we have $G \geq G_{g,n}$, and hence $\mathcal{G}_{g,n}$ is connected.

2.3. **Posets of spin graphs of genus g.** The poset of cyclic graphs of genus g with n legs is the set

$$\mathcal{C}_{g,n} := \bigsqcup_{G \in \mathcal{G}_{g,n}} \mathcal{C}_G$$

partially ordered as follows: $(G, P) \geq (G', P')$ if there exists a contraction $\gamma\colon G \to G'$ such that $\gamma_* P = P'$.

Next, the poset of stable spin graphs of genus g is the set

$$\mathcal{SP}_{g,n} := \bigsqcup_{G \in \mathcal{G}_{g,n}} SP_G,$$

endowed with the following partial order: $(G, P, s) \geq (G', P', s')$ if there exists a contraction $\gamma\colon G \to G'$ such that $\gamma_*(P, s) = (P', s')$.

It is easy to see that the above are partial orders.

Proposition 2.3.1. *(1) The poset $\mathcal{C}_{g,n}$ is connected and the forgetful map $\mathcal{C}_{g,n} \to \mathcal{G}_{g,n}$ is a quotient of posets.*

(2) *If $g > 0$, the poset $\mathcal{SP}_{g,n}$ has two connected components, $\mathcal{SP}^+_{g,n}$ and $\mathcal{SP}^-_{g,n}$ corresponding to, respectively, even and odd spin graphs.*
If $g = 0$, then $\mathcal{SP}_{g,n}$ is connected and $\mathcal{SP}_{g,n} = \mathcal{SP}^+_{g,n}$.

(3) *The forgetful maps below are quotients of posets, the first and the second one for every g, while the third one for $g > 0$,*

$$\mathcal{SP}^+_{g,n} \longrightarrow \mathcal{C}_{g,n}, \quad \mathcal{SP}^+_{g,n} \longrightarrow \mathcal{G}_{g,n}, \quad \mathcal{SP}^-_{g,n} \longrightarrow \mathcal{G}_{g,n}.$$

(4) *The map sending (G, P) to $|E|$ (respectively, (G, P, s) to $|E|$) is a rank on $\mathcal{C}_{g,n}$ (respectively, on $\mathcal{SP}_{g,n}$).*

Proof. Let $G_{g,n} \in \mathcal{G}_{g,n}$ be the graph introduced in Remark 2.2.4. We have $\mathcal{C}_{G_{g,n}} = \{0\}$; moreover, if $g > 0$, we have $SP_{G_{g,n}} = \{(0, s_0), (0, s_1)\}$ (so that $(0, s_0)$ is even and $(0, s_1)$ is odd), and if $g = 0$, we have $SP_{G_{g,n}} = \{(0, s_0)\}$.

Let $G \in \mathcal{G}_{g,n}$ and let $\gamma\colon G \to G_{g,n} = G/G$. For every $P \in \mathcal{C}_G$ we have $\gamma_* P = 0$, hence for every $(G, P) \in \mathcal{C}_{g,n}$ we have $(G, P) \geq (G_{g,n}, 0)$. This implies that $\mathcal{C}_{g,n}$ is connected. To prove that the forgetful map is a quotient it suffices to observe that if $G \geq G'$ then $(G, 0) \geq (G', 0)$.

Let now $(G, P, s) \in \mathcal{SP}_{g,n}$. By definition

$$\gamma_*(P, s) = \begin{cases} (0, s_0), & \text{if } (P, s) \text{ is even;} \\ (0, s_1), & \text{if } (P, s) \text{ is odd.} \end{cases}$$

Therefore every even spin graph (G, P, s) satisfies $(G, P, s) \geq (G_{g,n}, 0, s_0)$, hence $\mathcal{SP}^+_{g,n}$ is connected; similarly for $\mathcal{SP}^-_{g,n}$, when non-empty.

Let us show show that if $g > 0$, then SP^-_G is not empty for every $G \in \mathcal{G}_{g,n}$. This is clear if G is not weightless. If G is weightless then $b_1(G) = g > 0$, hence there exists $P \in \mathcal{C}_G$ such that $b_1(P) > 0$, hence G/P is not weightless. Therefore $SP^-_{(G,P)}$ is not empty.

It is clear that the forgetful maps in (3) are quotients.

We know that the map $G \mapsto |E|$ is a rank on $\mathcal{G}_{g,n}$. The partial order on the fibers of $\mathcal{C}_{g,n} \to \mathcal{G}_{g,n}$ is trivial, hence the rank on $\mathcal{G}_{g,n}$ lifts to a rank on $\mathcal{C}_{g,n}$, proving (4) for $\mathcal{C}_{g,n}$. The proof for $\mathcal{SP}_{g,n}$ is the same. ♣

Consider now $\mathrm{Aut}(G)$, the automorphism group of G. By Definition 2.2.3, if $\alpha \in \mathrm{Aut}(G)$ and $(P, s) \in SP_G$ then α maps P to $\alpha_* P$ and (P, s) to $\alpha_*(P, s)$, which is also an element of SP_G. We set

$$\mathrm{Aut}(G, P, s) := \{\alpha \in \mathrm{Aut}(G) : \ \alpha_*(P, s) = (P, s)\}.$$

We need to take into account the action of $\mathrm{Aut}(G)$ on $\mathcal{C}_G$ and on SP_G. Set

$$[\mathcal{C}_{g,n}] := \bigsqcup_{G \in \mathcal{G}_{g,n}} \mathcal{C}_G/\mathrm{Aut}(G), \qquad [\mathcal{SP}_{g,n}] := \bigsqcup_{G \in \mathcal{G}_{g,n}} SP_G/\mathrm{Aut}(G). \tag{5}$$

Recall that an automorphism of G preserves the weights. Hence $\mathrm{Aut}(G)$ leaves the sets SP^+_G and SP^-_G invariant and we write $[\mathcal{SP}^+_{g,n}]$ and $[\mathcal{SP}^-_{g,n}]$ so that $[\mathcal{SP}_{g,n}] = [\mathcal{SP}^+_{g,n}] \sqcup [\mathcal{SP}^-_{g,n}]$.

In what follows the isomorphism class of an object will be denoted using square brackets. For example we write $[P, s] \in SP_G/\mathrm{Aut}(G)$ for the class of (P, s) and $[G, P, s] \in [\mathcal{SP}_{g,n}]$ for the class of (G, P, s).

It is clear that the partial orders defined on $\mathcal{C}_{g,n}$ and on $\mathcal{SP}_{g,n}$ descend on $[\mathcal{C}_{g,n}]$ and $[\mathcal{SP}_{g,n}]$ (since two spin graphs in the same class are not comparable). So we shall use the notation

$$[G, P, s] \geq [G', P', s']$$

to mean that there exist $(G, P, s) \in [G, P, s]$ and $(G, P', s') \in [G, P', s']$ such that $(G, P, s) \geq (G', P', s')$. Similarly for $[\mathcal{C}_{g,n}]$. The forgetful maps below are, again, quotients of posets, the first three for every g, the last for $g > 0$

$$[\mathcal{C}_{g,n}] \to \mathcal{G}_{g,n}, \quad [\mathcal{SP}^+_{g,n}] \to [\mathcal{C}_{g,n}], \quad [\mathcal{SP}^+_{g,n}] \to \mathcal{G}_{g,n}, \quad [\mathcal{SP}^-_{g,n}] \to \mathcal{G}_{g,n}.$$

Remark 2.3.2. Proposition 2.3.1 holds if we replace $\mathcal{C}_{g,n}$ and $\mathcal{SP}_{g,n}$ by $[\mathcal{C}_{g,n}]$ and $[\mathcal{SP}_{g,n}]$ (mutatis mutandis).

2.4. **Tropical curves and their spin structures.** We recall the definition of a tropical curve, and some properties of the associated moduli spaces; see [BMV11] and [ACP15] for details and references.

A *(n-pointed) tropical curve* is a pair $\Gamma = (G, \ell)$ where G is a graph (with n legs) and $\ell\colon E \to \mathbb{R}_{\geq 0}$ is a length function. We need to consider the more general case of an *extended tropical curve*, which is defined as above with the difference that $\ell\colon E \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. To simplify the notation we set $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, endowed with the compact one-point topology. We say that Γ is *stable* if so is G and that Γ is *pure* if G is weightless. The *genus* of Γ is the genus of G.

The moduli space of stable n-pointed tropical curves of genus g is denoted by $M^{\mathrm{trop}}_{g,n}$, and the moduli space of extended ones by $\overline{M}^{\mathrm{trop}}_{g,n}$. The first is a generalized cone complex, the second an extended generalized cone complex. They have both dimension $3g-3+n$ and $M^{\mathrm{trop}}_{g,n}$ is open and dense in $\overline{M}^{\mathrm{trop}}_{g,n}$. For any $G \in \mathcal{G}_{g,n}$ we write

$$M^{\mathrm{trop}}_G := \{[\Gamma] \in M^{\mathrm{trop}}_{g,n} : \Gamma = (G, \ell)\}, \quad \overline{M}^{\mathrm{trop}}_G := \{[\Gamma] \in \overline{M}^{\mathrm{trop}}_{g,n} : \Gamma = (G, \ell)\}.$$

The spaces $M^{\mathrm{trop}}_{g,n}$ and $\overline{M}^{\mathrm{trop}}_{g,n}$ are constructed as colimits of suitable diagrams of cone complexes built from the poset $\mathcal{G}_{g,n}$. If for every graph G we denote by $\sigma_G = \mathbb{R}^E_{\geq 0}$, $\overline{\sigma}_G = \overline{\mathbb{R}}_{\geq 0}$, and by σ^o_G, $\overline{\sigma}^o_G$ their interiors, then we have *graded stratifications*

$$M^{\mathrm{trop}}_{g,n} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M^{\mathrm{trop}}_G \cong \bigsqcup_{G \in \mathcal{G}_{g,n}} \sigma^o_G/\mathrm{Aut}(G),$$

$$\overline{M}^{\mathrm{trop}}_{g,n} = \bigsqcup_{G \in \mathcal{G}_{g,n}} \overline{M}^{\mathrm{trop}}_G \cong \bigsqcup_{G \in \mathcal{G}_{g,n}} \overline{\sigma}^o_G/\mathrm{Aut}(G).$$

The terminology "graded stratification" means the folowing. The closure of a stratum is a union of strata, and the stratum corresponding to G contains the stratum corresponing to G' if and only if $G \geq G'$. Moreover, the map associating to each stratum its dimension is a rank on the poset of strata, i.e. on $\mathcal{G}_{g,n}$; see [CC18, Def. 1.3.2].

Before defining spin tropical curves, let us take a brief detour. A divisor on a tropical curve Γ is a formal sum $D = \sum_{p\in\Gamma} D(p)p$, where $D(p) \in \mathbb{Z}$ is non-zero only for a finite number of points $p \in \Gamma$. A rational function on Γ is a continuous piecewise-linear function on Γ with integer slopes; a principal divisor on Γ is the divisor associated to a principal function. Two divisors are equivalent if their difference is a principal divisor. The canonical divisor, K_Γ, of Γ is the canonical divisor of the underlying graph G, seen as a divisor on Γ. A *theta characteristic* on Γ is a divisor D such that $2D - K_\Gamma$ is equivalent to a principal divisor on Γ. Theta characteristics on pure tropical curves have been studied by Zharkov, and they correspond to elements of the cycle space $\mathcal{C}_G$ of G (see [Zha10, Thm. 7]), so a pure tropical curve of genus g admits exactly $2^{b_1(G)} = 2^g$ theta characteristics.

Let us define the theta characteristics on any tropical curve of genus g. Given an edge e of G, we denote by $p_e \in \Gamma$ the mid-point of e. For $P \in \mathcal{C}_G$, let D^P be the following divisor

$$D^P(x) = \begin{cases} \underline{d}^P_x, & \text{if } x \in V; \\ 1, & \text{if } x = p_e \text{ for } e \in E \smallsetminus P; \\ 0, & \text{otherwise} \end{cases}$$

where $\underline{d}^P$ is the divisor defined in (4). It is easy to check that D^P is a theta characteristic on Γ, and that there are $2^{b_1(G)}$ of them. Moreover, if $w = 0$, one can show that D^P is equivalent to the divisor $\mathcal{K}_P$ described in [Zha10]. In particular, we have

Remark 2.4.1. The divisors D^P associated to the elements P in $\mathcal{C}_G$ are (non-equivalent) representatives for the $2^{b_1(G)}$ theta characteristics of Γ.

We will not include the proof as it is not necessary here. Our definition of spin tropical curves can be seen as an enriched version of tropical theta characteristics.

Definition 2.4.2. A *(extended) spin tropical curve* is a triple $\Psi = (\Gamma, P, s)$ where $\Gamma = (G, \ell)$ is a (extended) tropical curve and (P, s) is a spin structure on G. The spin tropical curve Ψ is *stable* if so is G.

The genus of Ψ is the genus of Γ, and its parity is the parity of (P, s).

Let $\Psi = (\Gamma, P, s)$ and $\Psi' = (\Gamma', P', s')$ be two spin curves, with $\Gamma = (G, \ell)$ and $\Gamma' = (G', \ell')$. We say that Ψ and Ψ' are *isomorphic* if there is an isomorphism of tropical curves $\Gamma \to \Gamma'$ whose induced graph isomorphism $\gamma\colon G \to G'$ satisfies $\gamma_*(P, s) = (P', s')$.

2.5. **The moduli space of spin tropical curves.** We denote by $S^{\text{trop}}_{g,n}$ (respectively, $\overline{S}^{\text{trop}}_{g,n}$) the set of isomorphism classes of spin tropical curves (respectively, extended spin tropical curves). We are going to give them the structure of extended generalized cone complexes.

Let $\Gamma = (G, \ell)$ be an extended tropical curve. Recall that SP_G denotes the set of spin structures on G. The set of spin structures on Γ is written

$$S_\Gamma^{\rm trop} := \{\Psi = (\Gamma, P, s),\ \forall (P, s) \in SP_G\}.$$

Thus $S_\Gamma^{\rm trop}$ is a finite set, partitioned according to the parity of a spin structure as follows: $S_\Gamma^{\rm trop} = (S_\Gamma^{\rm trop})^+ \sqcup (S_\Gamma^{\rm trop})^-$. Notice that $\mathrm{Aut}(\Gamma)$ may act non-trivially on $S_\Gamma^{\rm trop}$.

Next, for a stable spin graph $(G, P, s) \in \mathcal{SP}_{g,n}$ we consider the subsets

$$S_{(G,P,s)}^{\rm trop} \subset S_{g,n}^{\rm trop} \quad \text{and} \quad \overline{S}_{(G,P,s)}^{\rm trop} \subset \overline{S}_{g,n}^{\rm trop}$$

defined as the set of isomorphism classes, respectively, of spin tropical curves and of extended spin tropical curves, whose underlying spin graph is isomorphic to (G, P, s). These sets will define a stratification of $S_{g,n}^{\rm trop}$ and $\overline{S}_{g,n}^{\rm trop}$.

We construct $S_{g,n}^{\rm trop}$ and $\overline{S}_{g,n}^{\rm trop}$ as generalized cone complexes using a procedure analogous to, and compatible with, the one used to construct $M_{g,n}^{\rm trop}$ and $\overline{M}_{g,n}^{\rm trop}$.

We consider the category, $\text{SPIN}_{g,n}$, whose objects are isomorphism classes of stable spin graphs of genus g with n legs and whose morphisms are generated by contractions and by automorphisms of spin graphs. To an isomorphism class of a spin graph (G, P, s) we associate a cone

$$\sigma_{(G,P,s)} = \mathbb{R}_{\geq 0}^E.$$

This cone has the natural integral structure determined by the sub-lattice parametrizing tropical curves whose edges have integral length. We write $\sigma^o_{(G,P,s)} = \mathbb{R}_{>0}^E$ for its interior.

To a contraction $\gamma\colon (G, P, s) \to (G', P', s')$ we associate an injection of cones

$$\iota_\gamma\colon \sigma_{(G',P',s')} \hookrightarrow \sigma_{(G,P,s)}$$

whose image is the face of $\sigma_{(G,P,s)}$ where the coordinates corresponding to $E(G) \smallsetminus E(G')$ are zero. If γ is an automorphism, then ι_γ is the corresponding automorphism of rational cones. By our earlier results, this gives a contravariant functor from $\text{SPIN}_{g,n}$ to the category of rational polyhedral cones. Therefore we can take the colimit of the diagram of cones $\sigma_{(G,P,s)}$ using the inclusions ι_γ, for all morphisms γ in $\text{SPIN}_{g,n}$. We define this colimit to be the moduli space of n-pointed spin tropical curves of genus g:

$$S_{g,n}^{\rm trop} = \varinjlim \left(\sigma_{(G,P,s)}, \iota_\gamma\right).$$

Hence $S_{g,n}^{\rm trop}$ is canonically a generalized cone complex, and the following is easily seen to hold.

Proposition 2.5.1. *The moduli space of spin tropical curves, $S^{\rm trop}_{g,n}$, is a topological space of pure dimension $3g-3+n$. We have a stratification*

$$S^{\rm trop}_{g,n} = \bigsqcup_{[G,P,s]\in[\mathcal{SP}_{g,n}]} S^{\rm trop}_{(G,P,s)}.$$

Moreover, we have

$$S^{\rm trop}_{(G,P,s)} \cong \sigma^o_{(G,P,s)}/\mathrm{Aut}(G,P,s),$$

and $S^{\rm trop}_{(G',P',s')} \subset \overline{S^{\rm trop}_{(G,P,s)}}$ if and only if $[G,P,s] \geq [G',P',s']$.

One constructs the moduli space for extended spin tropical curves in the analogous way. To an isomorphism class of a spin graph (G,P,s) we now associate the extended cone

$$\overline{\sigma}_{(G,P,s)} := \overline{\mathbb{R}}^E_{\geq 0}$$

and its interior $\overline{\sigma}^o_{(G,P,s)} = (\mathbb{R}_{>0} \cup \{\infty\})^E$. The rest of the construction is the same, and yields the moduli space of extended spin tropical curves

$$\overline{S}^{\rm trop}_{g,n} = \varinjlim \left(\overline{\sigma}_{(G,P,s)}, \iota_\gamma\right),$$

whose stratification we denote

$$\overline{S}^{\rm trop}_{g,n} = \bigsqcup_{[G,P,s]\in[\mathcal{SP}_{g,n}]} \overline{S}^{\rm trop}_{(G,P,s)} \cong \bigsqcup_{[G,P,s]\in[\mathcal{SP}_{g,n}]} \overline{\sigma}^o_{(G,P,s)}/\mathrm{Aut}(G,P,s).$$

Proposition 2.5.2. *The moduli space of extended spin tropical curves, $\overline{S}^{\rm trop}_{g,n}$, is a generalized extended cone complex, and a topological space of pure dimension $3g-3+n$ containing $S^{\rm trop}_{g,n}$ as a dense open subset. It has two connected components:*

$$\overline{S}^{\rm trop}_{g,n} = (\overline{S}^{\rm trop}_{g,n})^- \sqcup (\overline{S}^{\rm trop}_{g,n})^+$$

corresponding to odd and even spin tropical curves.

Proof. With respect to what we already said, the only thing that needs to be proved is that $(\overline{S}^{\rm trop}_{g,n})^-$ and $(\overline{S}^{\rm trop}_{g,n})^+$ are connected. This follows by the same argument we used to prove Proposition 2.3.1 (2). ♣

We have a natural map

$$\pi^{\rm trop}\colon \overline{S}^{\rm trop}_{g,n} \longrightarrow \overline{M}^{\rm trop}_{g,n}$$

sending the point parametrizing a spin tropical curve (Γ, P, s), with $\Gamma = (G,\ell)$, to the point parametrizing the tropical curve $(G, \pi^{\rm trop}(\ell))$, where

$$\pi^{\rm trop}(\ell)(e) := \begin{cases} 2\ell(e), & \text{if } e \in E \smallsetminus P; \\ \ell(e), & \text{otherwise.} \end{cases}$$

We consider also the restrictions (with self-explanatory notation)

$$\pi^{\rm trop,-}\colon (\overline{S}^{\rm trop}_{g,n})^- \longrightarrow \overline{M}^{\rm trop}_{g,n} \quad \text{and} \quad \pi^{\rm trop,+}\colon (\overline{S}^{\rm trop}_{g,n})^+ \longrightarrow \overline{M}^{\rm trop}_{g,n}.$$

Next we prove that π^{trop}, and hence $\pi^{\text{trop},-}$ and $\pi^{\text{trop},+}$, are morphisms of extended generalized cone complexes in the sense of [ACP15, Sect. 2].

Proposition 2.5.3. *The map π^{trop} is a morphism of extended generalized cone complexes and for every extended tropical curve Γ we have*

$$(\pi^{\text{trop}})^{-1}([\Gamma]) = S_\Gamma^{\text{trop}}/\text{Aut}(\Gamma).$$

Proof. Obviously π^{trop} is compatible with the cone diagrams defining $\overline{M}_{g,n}^{\text{trop}}$ and $\overline{S}_{g,n}^{\text{trop}}$, and the induced map from $\sigma_{(G,P,s)}$ to σ_G is a morphism of rational polyhedral cones with integral structure (it is the restriction of the integral linear transformation $T\colon \mathbb{R}^E \to \mathbb{R}^E$ defined on the basis $\{e : e \in E\}$ as $T(e) = 2e$ if $e \in E \smallsetminus P$, and $T(e) = e$ if $e \in P$). Hence π^{trop} is a map of extended generalized cone complexes.

Let $\Gamma = (G, \ell)$ be an extended tropical curve. There is a natural map $\rho\colon S_\Gamma^{\text{trop}} \to \overline{S}_{g,n}^{\text{trop}}$ taking (Γ, P, s), with $(P, s) \in SP_G$, to the class of (Γ, P, s), and we have $(\pi^{\text{trop}})^{-1}([\Gamma]) = \text{Im}(\rho)$. Given (P, s) and (P', s') in SP_G, we have $\rho(\Gamma, P, s) = \rho(\Gamma, P', s')$ if and only if there is an automorphism of Γ inducing an isomorphism between (Γ, P, s) and (Γ, P', s'). Therefore $\text{Im}(\rho) = S_\Gamma^{\text{trop}}/\text{Aut}(\Gamma)$. ♣

3. Algebraic stable spin curves

3.1. **Stable spin curves and their moduli space.** We let $\overline{\mathcal{S}}_{g,n}$ be the moduli stack of stable n-pointed spin curves of genus g and $\mathcal{S}_{g,n} \subset \overline{\mathcal{S}}_{g,n}$ be the substack parametrizing theta characteristics on smooth curves. Given a k-scheme B, a section in $\overline{\mathcal{S}}_{g,n}(B)$ is a pair $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$, described as follows. We have a flat morphism $f\colon \widehat{\mathcal{X}} \to B$ such that for every $b \in B$ the fiber $\widehat{X}_b$ is a genus-g quasistable n-pointed curve. Next, $\widehat{\mathcal{L}}$ is a line bundle on $\widehat{\mathcal{X}}$ endowed with a homomorphism $\alpha\colon \widehat{\mathcal{L}}^{\otimes 2} \to \omega_f$ such that for every $b \in B$ the map $\alpha_{|\widehat{X}_b}\colon \widehat{\mathcal{L}}^{\otimes 2}_{|\widehat{X}_b} \to \omega_{\widehat{X}_b}$ is an isomorphism away from the exceptional components, and $\widehat{\mathcal{L}}_{|E} \cong \mathcal{O}_E(1)$ for every exceptional component E of $\widehat{X}_b$.

We have a natural, representable morphism

$$\pi\colon \overline{\mathcal{S}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

sending $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ to the stable model of $\widehat{\mathcal{X}}$. The morphism π is finite of degree 2^{2g}, hence the fiber of π over a point parametrizing a curve X, written $\mathcal{S}_X$, has dimension 0 and length 2^{2g}. With the notation (2), we have $\mathcal{S}_X^0 \subset \mathcal{S}_X$ with equality if and ony if X is nonsingular (see below).

Recall the notation in (1). A point in $\mathcal{S}_X$ parametrizes the isomorphism class of a pair $(\widehat{X}_R, \widehat{L}_R)$, where $\widehat{X}_R$ is the quasistable curve associated to a set of nodes R of X and $\widehat{L}_R$ is a line bundle on $\widehat{X}_R$ such that

(1) the restriction, L_R, of $\widehat{L}_R$ to X_R^ν satisfies $L_R^2 \cong \omega_{X_R^\nu}$;

(2) the restriction of $\widehat{L}_R$ to E_r is $\mathcal{O}_{E_r}(1)$ for every $r \in R$.

Notice that $(\widehat{X}_R, \widehat{L}_R)$ depends only on L_R (i.e. different gluings over the nodes lying on exceptional components of $\widehat{X}_R$ give the same point in $\mathcal{S}_X$). The first requirement implies that the dual graph of X_R^ν is cyclic. Therefore we can alternatively describe the points of $\mathcal{S}_X$ as parametrizing pairs (R, L_R) defined as follows:

(1) $R \subset E$ such that $E \smallsetminus R$ is cyclic;
(2) $L_R \in \operatorname{Pic}(X_R^\nu)$ such that $L_R^2 \cong \omega_{X_R^\nu}$.

We earlier used the notation $(\widehat{X}_R, \widehat{L}_R)$ for a point in $\overline{\mathcal{S}}_{g,n}$, but we can, and will, denote the same point by a pair (X_R^ν, L_R).

Now, $\overline{\mathcal{S}}_{g,n}$ has two connected and irreducible components, denoted

$$\overline{\mathcal{S}}_{g,n}^- \longrightarrow \overline{\mathcal{M}}_{g,n} \qquad \text{and} \qquad \overline{\mathcal{S}}_{g,n}^+ \longrightarrow \overline{\mathcal{M}}_{g,n}$$

parametrizing, respectively, odd and even spin curves, where the parity refers to the parity of $h^0(\widehat{X}_R, \widehat{L}_R)$ (equivalently, the parity of $h^0(X_R^\nu, L_R)$) for every $(\widehat{X}_R, \widehat{L}_R)$ parametrized by $\overline{\mathcal{S}}_{g,n}$.

We write, with self-explanatory notation, $\mathcal{S}_X = \mathcal{S}_X^- \sqcup \mathcal{S}_X^+$. One refers to $\mathcal{S}_X$ as the space of (stable) spin structures on X, with $\mathcal{S}_X^-$ and $\mathcal{S}_X^+$ parametrizing odd and even ones.

3.2. **The dual graph of a stable spin curve.** Let X be a stable curve, G its dual graph, and $R \subset E$ a set of nodes of X; write $P = E \smallsetminus R$. Then the connected components of X_R^ν are in natural bijection with the vertices of the graph G/P. We thus write $X_R^\nu = \sqcup_{v \in V(G/P)} Z_v$ with Z_v connected. Recall, from 1.5, that we view X_R^ν as a pointed curve so that the dual graph of Z_v is the graph $\overline{P}_v$ defined in Section 2, and the weight of v in G/P is equal to the genus of Z_v.

Definition 3.2.1. The *dual spin graph* of a spin curve (X_R^ν, L_R) is the spin graph (G, P, s) defined as follows:

- G is the dual graph of X;
- $P = E \smallsetminus R$;
- $s(v)$ is the parity of $h^0(Z_v, (L_R)_{|Z_v})$, for every $v \in V(G/P)$.

We need to check that we defined an actual spin graph. As we said above, if (X_R^ν, L_R) is a spin curve, then $E \smallsetminus R \in \mathcal{C}_G$. Next, if $v \in V(G/P)$ has weight zero, then Z_v is smooth and has genus zero, and $(L_R)_{|Z_v}$ has degree -1, and so $s(v) = h^0(Z_v, (L_R)_{|Z_v}) = 0$.

Remark 3.2.2. A spin curve and its dual spin graph have the same parity. Indeed, the parity of (X_R^ν, L_R) is the parity of $h^0(X_R^\nu, L_R)$, and we have

$$h^0(X_R^\nu, L_R) = \sum_{v \in V(G/P)} h^0(Z_v, (L_R)|_{Z_v}) \equiv \sum_{v \in V(G/P)} s(v) \mod(2).$$

Proposition 3.2.3. *We have a commutative diagram of surjective maps*

$$\begin{array}{ccc} \overline{\mathcal{S}}_{g,n} & \xrightarrow{\ \psi\ } & [\mathcal{SP}_{g,n}] \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \longrightarrow & \mathcal{G}_{g,n} \end{array}$$

where the two horizontal arrows map a stable (respectively spin) curve to the class of its dual (respectively dual spin) graph, and the right vertical arrow maps $[G, P, s]$ to G. The same holds by replacing $\overline{\mathcal{S}}_{g,n}$ and $[\mathcal{SP}_{g,n}]$ with $\overline{\mathcal{S}}^{+}_{g,n}$ and $[\mathcal{SP}^{+}_{g,n}]$ or, if $g > 0$, with $\overline{\mathcal{S}}^{-}_{g,n}$ and $[\mathcal{SP}^{-}_{g,n}]$.

Proof. The only non-evident claim is that the top horizontal arrow is surjective. Let (G, P, s) be a spin graph. Pick any pointed curve X dual to G. Set $R = E \smallsetminus P$ and let X^{ν}_{R} be the normalization of X at R. As above, we write $\{Z_v,\ v \in V(G/P)\}$ for the connected components of X^{ν}_{R}. To conclude, we need to show that for every $v \in V(G/P)$ we can pick $L_v \in \operatorname{Pic}(Z_v)$ such that $L_v^2 \cong \omega_{Z_v}$ and whose parity is that of $s(v)$. For that, consider $\mathcal{S}^0_{Z_v} = \{L \in \operatorname{Pic}(Z_v) :\ L^2 \cong \omega_{Z_v}\}$.

If Z_v has positive genus, then $\mathcal{S}^0_{Z_v}$ contains elements of even and odd parity. Hence we can choose an element whose parity is the same as $s(v)$. If Z_v has genus zero, then $\mathcal{S}^0_{Z_v}$ consists of one element which, having degree -1, is necessarily even. But in this case $s(v) = 0$, so we are done. ♣

Let X be a stable curve and let us study the structure of $\mathcal{S}_X$. Consider the diagram in Proposition 3.2.3, and let us restrict the map ψ to $\mathcal{S}_X$. If $(G, P, s) \in \mathcal{SP}_{g,n}$ we denote by

$$\mathcal{S}_{(X,P,s)} := \psi^{-1}([G, P, s]) \cap \mathcal{S}_X,$$

the subscheme of spin structures on X whose dual graph is isomorphic to (G, P, s). We thus have a decomposition $\mathcal{S}_X = \sqcup_{(P,s)\in[SP_G]}\mathcal{S}_{(X,P,s)}$.

As we shall state below, the structure of $\mathcal{S}_X$ at loci of the form $\mathcal{S}_{(X,P,s)}$ does not depend on the parity function. It is thus convenient to consider, for any $P \in \mathcal{C}_G$, the following subscheme of $\mathcal{S}_X$

$$\mathcal{S}_{(X,P)} = \sqcup_s \mathcal{S}_{(X,P,s)}$$

where s varies over all parity functions.

Remark 3.2.4. If G is cyclic we have $\mathcal{S}_{(X,G)} = \mathcal{S}^0_X$ (recall (2)). For an arbitrary G there is a bijection between the points of $\mathcal{S}_{(X,P)}$ and the set of theta characteristics, $\mathcal{S}^0_{X^{\nu}_{R}}$, on X^{ν}_{R}.

We have $\mathcal{S}_{(X,P)} = \mathcal{S}^{+}_{(X,P)} \sqcup \mathcal{S}^{-}_{(X,P)}$, self-explanatorily. We now combine some results of [CC03, Sect. 1] with [Har82, Cor. 2.13].

Fact 3.2.5. *Let X be a stable curve of genus g and G its dual graph. Set $b = b_1(G)$ and $|w| = \sum_{v\in V} w(v)$. Let $P \in \mathcal{C}_G$. Then*

(1) $|\mathcal{S}_{(X,P)}| = 2^{b_1(P)+2|w|}$ *and the length of* $\mathcal{S}_X$ *at every point of* $\mathcal{S}_{(X,P)}$ *is equal to* $2^{b-b_1(P)}$*, so that* $\operatorname{length}\mathcal{S}_{(X,P)} = 2^{b+2|w|}$.

(2) If P *is connected and* $b_1(P) \neq 0$*, then*

$$|\mathcal{S}^-_{(X,P)}| = |\mathcal{S}^+_{(X,P)}| = 2^{b_1(P)+2|w|-1}.$$

Now, using Remark 3.2.4 we have the following stratification, highlighting the recursive structure of $\mathcal{S}_X$ by expressing its boundary, $\mathcal{S}_X \smallsetminus \mathcal{S}^0_X$, in terms of the theta-characteristics on the partial normalizations of X

$$\mathcal{S}^{\mathrm{red}}_X = \bigsqcup_{P \in \mathcal{C}_G} \mathcal{S}^{\mathrm{red}}_{(X,P)} \cong \bigsqcup_{E \smallsetminus R \in \mathcal{C}_G} \mathcal{S}^0_{X^\nu_R}.$$

3.3. **Families of stable spin curves.** A *one-parameter family of curves* is a family of (pointed) nodal curves $f\colon \mathcal{X} \to B$ over a regular, connected curve B with a marked point, $b_0 \in B$; we denote by X the fiber over b_0 and by G its dual graph. We will always assume that every fiber over $B \smallsetminus \{b_0\}$ has the same dual graph, denoted by H; we shall denote by Y any such fiber, which we call the "generic" fiber. To such a family f we associate its *dual contraction*:

$$\gamma_f\colon G \longrightarrow H = G/S_0$$

where $S_0 \subset E(G)$ are the nodes of X which are not specializations of nodes of Y. We shall write $\gamma = \gamma_f$ when no confusion can occur.

Assume the above family is polarized, i.e., we have a line bundle, M, on Y specializing to a line bundle, L, on X. Then by [CC18, Prop. 4.3.2] we have an identity of divisors on H

$$\underline{\deg} M = \gamma_* \underline{\deg} L. \tag{6}$$

We shall write $(\mathcal{X}, \mathcal{L}) \to B$ for such a polarized family, with $\mathcal{L}$ a family of line bundles on the fibers of f restricting to L on X and to M on Y.

A *one-parameter family of stable spin curves* is a polarized family, written $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}}) \to B$, such that $\widehat{f}\colon \widehat{\mathcal{X}} \to B$ is a one-parameter family of quasistable curves as defined above (we write $\widehat{X}$, respectively $\widehat{Y}$, for its special, resp. generic, fiber) and such that for every $b \in B$ the fiber, $(\widehat{X}_b, \widehat{\mathcal{L}}_{|\widehat{X}_b})$, is a stable spin curve. Let $f\colon \mathcal{X} \to B$ be the stable model of $\widehat{f}$. Then $\widehat{X}$ is the blow-up of X at a set of nodes R of X, and its dual graph is written $\widehat{G}_R$.

Similarly, let T be the set of nodes of Y that are blown-up in $\widehat{Y}$ and denote by $\widehat{H}_T$ the dual graph of $\widehat{Y}$. Since every exceptional component of $\widehat{Y}$ specializes to an exceptional component of $\widehat{X}$, we have (recall that $T \subset E(H) = E(G) \smallsetminus S_0$)

$$T \subset R \smallsetminus S_0. \tag{7}$$

Let $\widehat{\gamma}\colon \widehat{G}_R \to \widehat{H}_T$ be the dual contraction of $\widehat{f}$. We have a commutative diagram of contractions, where the vertical arrows correspond to the stabilizations $\widehat{X} \to X$ and $\widehat{Y} \to Y$

$$\begin{array}{ccc} \widehat{G}_R & \xrightarrow{\widehat{\gamma}} & \widehat{H}_T \\ {\scriptstyle st}\downarrow & & \downarrow{\scriptstyle st} \\ G & \xrightarrow{\gamma} & H \end{array} \tag{8}$$

Lemma 3.3.1. *Let* $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}}) \to B$ *be a one-parameter family of stable spin curves and* $f \colon \mathcal{X} \to B$ *its stable model. Then, with the above notation,* $T = R \smallsetminus S_0$.

Proof. By (7) we need to prove that every $r \in R \smallsetminus S_0$ lies in T. This is equivalent to saying that every exceptional component, E, of the special fiber, $\widehat{X}$, is either the specialization of an exceptional component of the generic fiber, or neither of the two nodes lying in E is specialization of a node of the generic fiber.

This is a special case of the explicit description of the deformation space given in [CCC07, Subsect. 3.2]. At the beginning of that subsection it is shown that, if E is an exceptional component of the special fiber, then either both nodes of E are preserved, or no node of E is preserved. This is precisely what is needed here. ♣

3.4. **The stratification of $\overline{\mathcal{S}}_{g,n}$.** The goal of this subsection is to describe the stratification of $\overline{\mathcal{S}}_{g,n}$. The strata of $\overline{\mathcal{M}}_{g,n}$ are the loci, $\mathcal{M}_G$, parametrizing curves whose dual graph is isomorphic to G, for G varying in $\mathcal{G}_{g,n}$. In [ACP15, Subsect. 3.4] the strata $\mathcal{M}_G$ are given an explicit presentation

$$\widetilde{\mathcal{M}}_G := \prod_{v \in V(G)} \mathcal{M}_{w(v),\deg(v)+\ell(v)} \longrightarrow \mathcal{M}_G = \left[\widetilde{\mathcal{M}}_G/\mathrm{Aut}(G)\right]. \tag{9}$$

In particular, $\mathcal{M}_G$ is irreducible. It is well known that $\overline{\mathcal{M}}_{g,n} = \sqcup_{G \in \mathcal{G}_{g,n}} \mathcal{M}_G$ is a graded stratification. We shall write $\mathcal{M}_v := \mathcal{M}_{w(v),\deg(v)+\ell(v)}$.

Consider now the morphism $\pi \colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and set

$$\mathcal{S}_G := \pi^{-1}(\mathcal{M}_G).$$

Now, given $P \in \mathcal{C}_G$ and $(P,s) \in SP_G$ we let $\mathcal{S}_{(G,P)}$ be the substack of $\overline{\mathcal{S}}_{g,n}$ parametrizing stable spin curves whose dual spin graph has support isomorphic to (G,P), and $\mathcal{S}_{(G,P,s)}$ be the substack where the dual spin graph is isomorphic to (G,P,s). Then we have

$$\mathcal{S}_G = \sqcup_{P \in [\mathcal{C}_G]} \mathcal{S}_{(G,P)} = \sqcup_{(P,s) \in [SP_G]} \mathcal{S}_{(G,P,s)}. \tag{10}$$

These decompositions are the extensions over the moduli spaces of the ones described earlier for a fixed curve. Notice that if we have two spin structures, (P,s) and (P',s'), on G and an automorphism of G mapping one to the other, then, by definition, $\mathcal{S}_{(G,P,s)} = \mathcal{S}_{(G,P',s')}$. We have thus a decomposition

$$\overline{\mathcal{S}}_{g,n} = \bigsqcup_{[G,P,s]\in[\mathcal{SP}_{g,n}]} \mathcal{S}_{(G,P,s)}. \tag{11}$$

Example 3.4.1. Recall that we view G as an element in $\mathcal{E}_G$. The stratum $\mathcal{S}_{(G,G)}$ is not empty if and only if G is cyclic. Let us assume this is the case. Then $\mathcal{S}_{(G,G)}$ parametrizes theta-characteristics on the curves in $\mathcal{M}_G$, i.e.

$$\mathcal{S}_{(G,G)} = \{(X,L) : X \in \mathcal{M}_G,\ L \in \mathcal{S}^0_X\}.$$

Proposition 3.4.2. *Let (G,P,s) and (H,Q,s') be stable spin graphs. Then the following are equivalent.*

(1) $\mathcal{S}_{(G,P,s)} \cap \overline{\mathcal{S}}_{(H,Q,s')} \neq \emptyset$.
(2) $[G,P,s] \geq [H,Q,s']$.
(3) $\mathcal{S}_{(G,P,s)} \subset \overline{\mathcal{S}}_{(H,Q,s')}$.

Proof. We denote, as before, $R = E(G) \smallsetminus P$ and $T = E(H) \smallsetminus Q$.

(1) $\Rightarrow$ (2). By hypothesis there exists a one-parameter family of stable spin curves $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}}) \to B$ whose generic fiber, $(\widehat{Y}, \mathcal{L}_{\widehat{Y}})$, has dual spin graph (H,Q,s') and whose special fiber, $(\widehat{X}, \mathcal{L}_{\widehat{X}})$, has dual spin graph (G,P,s). Recall that, as described in the commutative diagram (8), we have a contraction $\gamma\colon G \to H$. We need to show that $\gamma_*(P,s) = (Q,s')$. From Lemma 3.3.1 we derive that $\gamma_* P = Q$. The fact that $s'(v') = \sum_{v\in\overline{\gamma}^{-1}(v')} s(v)$ is a consequence of the deformation invariance of the parity of spin curves.

(2) $\Rightarrow$ (3). Consider a contraction $\gamma\colon G \to H$ such that $\gamma_*(P,s) = (Q,s')$. Hence γ induces a contraction $\widehat{\gamma}\colon \widehat{G}_R \to \widehat{H}_T$ and we can form the commutative diagram (8). Now, let X be a pointed curve dual to G. The existence of the contraction γ implies the existence of a one-parameter family of stable curves $f\colon \mathcal{X} \to B$ whose generic fiber, Y, is dual to H and whose special fiber is X. Therefore the space, $\mathcal{S}_Y$, of spin structures on Y, specializes to $\mathcal{S}_X$. We know that every $[(\widehat{X}, \widehat{L})] \in \mathcal{S}_{(X,P,s)}$ is the specialization of some spin structure on Y lying in some stratum $\mathcal{S}_{(Y,\overline{Q},\overline{s}')}$; using Lemma 3.3.1 as before, we get $\gamma_*(P,s) = (\overline{Q}, \overline{s}')$. Since $\gamma_*(P,s) = (Q,s')$ we obtain $(\overline{Q}, \overline{s}') = (Q,s')$, as wanted.

As (3) $\Rightarrow$ (1) is obvious, we are done. ♣

We will prove in Theorem 4.2.4 that $\mathcal{S}_{(G,P,s)}$ is irreducible, and hence, combining with the previous proposition, we will obtain the following.

Theorem 3.4.3. *Decomposition (11) is a graded stratification of $\overline{\mathcal{S}}_{g,n}$.*

3.5. **Local analysis.** We here recall the description of the versal deformation space of a pointed spin curve and of the morphism $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. More details can be found in [ACG11] and [CCC07].

Let $x \in \overline{\mathcal{M}}_{g,n}$ and $y \in \overline{\mathcal{S}}_{g,n}$ be closed points such that $x = \pi(y)$. Assume that x represents the curve X with dual graph G and that y represents the spin curve $(\widehat{X}, \widehat{L})$. Let $E = \{e_1, \ldots, e_\delta\}$ be the nodes of X, and assume that the exceptional components of $\widehat{X}$ lie over the nodes $e_1, \dots, e_r$, for some $r \in \{0, \dots, \delta\}$. There are étale local coordinates $t_1, \dots, t_{3g-3+n}$ such that

$$\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},x} \cong k[[t_1, \dots, t_{3g-3+n}]], \tag{12}$$

where the vanishing of t_i is the divisor corresponding to the deformations of X that are trivial locally at the node e_i, for $i = 1, \dots, \delta$.

Similarly, there are étale local coordinates $s_1, \dots, s_{3g-3+n}$ such that

$$\widehat{\mathcal{O}}_{\overline{S}_{g,n},y} \cong k[[s_1, \dots, s_{3g-3+n}]].$$

By Lemma 3.3.1 we can assume that for $i = 1, \dots, r$, respectively, for $i = r+1, \dots, \delta$, the vanishing of s_i is the locus of the deformations of $(\widehat{X}, \widehat{L})$ that are trivial locally at the nodes contained in the exceptional component of $\widehat{X}$ lying over e_i, respectively, are trivial locally at the node e_i seen as a node of $\widehat{X}$.

Locally at x and y, the morphism $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is induced by the ring homomorphism $\pi^{\#}\colon \widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},x} \to \widehat{\mathcal{O}}_{\overline{\mathcal{S}}_{g,n},y}$ given by $\pi^{\#}(t_i) = s_i^2$ for $i \le r$, and $\pi^{\#}(t_i) = s_i$ for $i > r$.

Let now G be a stable graph. The closures in $\overline{\mathcal{M}}_{g,n}$ and in $\overline{\mathcal{S}}_{g,n}$ of the loci $\mathcal{M}_G$, $\mathcal{S}_G$ and $\mathcal{S}_{(G,P,s)}$ will be written, respectively, $\overline{\mathcal{M}}_G$, $\overline{\mathcal{S}}_G$ and $\overline{\mathcal{S}}_{(G,P,s)}$. We have $\overline{\mathcal{S}}_G = \pi^{-1}(\overline{\mathcal{M}}_G)$.

Let G and G' be two stable graphs such that $\mathcal{M}_{G'} \subset \overline{\mathcal{M}}_G$. We need to describe the structure of $\overline{\mathcal{M}}_G$ at a point $x' \in \mathcal{M}_{G'}$.

Let X' be a stable curve parametrized by x'. Write $E(G') = \{e_i\}_{1 \le i \le \delta'}$ so that $\delta' \ge \delta$. Let J be the set of all contractions of G' to G.

As explained in the proof of [ACG11, Prop. XII.10.11], the étale local picture of $\overline{\mathcal{M}}_G$ at the point x' is the one of $|J|$ linear subspaces in k^{3g-3+n} of dimension $3g-3+n-\delta$ meeting transversally at the origin. Now, as in (12), we have $\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},x'} \cong k[[t_1, \dots, t_{3g-3+n}]]$. Given a contraction $\gamma\colon G' \to G$ in J, we consider the ideal $\mathcal{I}_\gamma = (t_{\gamma_1}, \dots, t_{\gamma_\delta})$, where $\{e_{\gamma_1}, \dots, e_{\gamma_\delta}\}$ is the image of $E(G)$ in the injection $E(G) \subset E(G')$ induced by γ. Then we have

$$\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_G,x'} \cong \frac{k[[t_1, \dots, t_{3g-3+n}]]}{\prod_{\gamma \in J} \mathcal{I}_\gamma} = \frac{k[[t_1, \dots, t_{\delta'}]]}{\prod_{\gamma \in J} \mathcal{I}_\gamma} \otimes k[[t_{\delta'+1}, \dots, t_{3g-3+n}]].$$

Let

$$\nu : \overline{\mathcal{M}}_G^\nu \longrightarrow \overline{\mathcal{M}}_G$$

be the normalization of $\overline{\mathcal{M}}_G$. Then $\overline{\mathcal{M}}_G^\nu$ is smooth and, locally over x', is the disjoint union of the $|J|$ branches described by the ideals $\mathcal{I}_\gamma$. Hence, given $\gamma \in J$, if we let x'_γ be the point in $\overline{\mathcal{M}}_G^\nu$ lying on the branch corresponding to $\mathcal{I}_\gamma$, we have

$$\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_G^\nu,x'_\gamma} \cong \frac{k[[t_1, \dots, t_{3g-3+n}]]}{\mathcal{I}_\gamma}. \tag{13}$$

We define

$$\overline{\widetilde{\mathcal{M}}}_G := \prod_{v \in V(G)} \overline{\mathcal{M}}_{w(v),\deg(v)+\ell(v)}.$$

The normalization $\overline{\mathcal{M}}_G^\nu$ of $\overline{\mathcal{M}}_G$ is given an explicit presentation

$$\chi \colon \overline{\widetilde{\mathcal{M}}}_G \longrightarrow \overline{\mathcal{M}}_G^\nu \tag{14}$$

such that $\overline{\mathcal{M}}_G^\nu = [\overline{\widetilde{\mathcal{M}}}_G/\mathrm{Aut}(G)]$ (see [ACG11, Prop. XII.10.11]).

3.6. **Presentations.** Let (G, P, s) be a stable spin graph. Let $\mathcal{X}_G \to \mathcal{M}_G$ be the universal curve and $\widetilde{\mathcal{X}}_G$ its base change by the map $\widetilde{\mathcal{M}}_G \to \mathcal{M}_G$. Set $\widetilde{\mathcal{S}}_{(G,P,s)} := \mathcal{S}_{(G,P,s)} \times_{\mathcal{M}_G} \widetilde{\mathcal{M}}_G$.

We consider first the case $P = G$, and look at the Cartesian diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{X}}_G & \longrightarrow & \widetilde{\mathcal{M}}_G & \longleftarrow & \widetilde{\mathcal{S}}_{(G,G,s)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_G & \longrightarrow & \mathcal{M}_G = [\widetilde{\mathcal{M}}_G/\mathrm{Aut}(G)] & \longleftarrow & \mathcal{S}_{(G,G,s)} \end{array} \tag{15}$$

We can view $\widetilde{\mathcal{S}}_{(G,G,s)}$ as the moduli space of theta characteristics of the fibers of $\widetilde{\mathcal{X}}_G \to \widetilde{\mathcal{M}}_G$, whose parity function is prescribed by $s(v)$, where v is the unique vertex of $V(G/G)$.

In case $P \neq G$ we need to introduce a new space to have a suitable presentation of $S_{(G,P,s)}$. Set $R = E \smallsetminus P$. Recall that the set of connected components of (G, P, s) is written $\{(\overline{P}_v, s_v),\ v \in V(G/P)\}$. Since $\overline{P}_v$ is a stable graph (with legs), we can consider, for every v, the substacks

$$\mathcal{M}_{\overline{P}_v} \subset \overline{\mathcal{M}}_{g_v,\ell_v+d_v} \text{ and } \mathcal{S}_{(\overline{P}_v,\overline{P}_v,s_v)} \subset \overline{\mathcal{S}}_{g_v,\ell_v+d_v}$$

where g_v, ℓ_v and d_v are as follows. Set $g_v = g(\overline{P}_v)$ and $\ell_v = \sum_{u \in V(\overline{P}_v)} \ell_G(u)$, and $d_v = \sum_{u \in V(\overline{P}_v)} \deg_R(u)$ where $\ell_G(u)$ is the number of legs of G ending at u, and $\deg_R(u)$ is the degree of u as a vertex of the subgraph $\langle R \rangle$ of G. As $G \in \mathcal{G}_{g,n}$, we have $\sum_{v \in V(G/P)} \ell_v = n$ and $\sum_{v \in V(G/P)} d_v = 2|R|$ and $\sum_{v \in V(G/P)} (g_v - 1) + |R| + 1 = g$. We define

$$\widehat{\mathcal{S}}_{(G,P,s)} := \prod_{v \in V(G/P)} \widetilde{\mathcal{S}}_{(\overline{P}_v,\overline{P}_v,s_v)}. \tag{16}$$

Of course, we have $\widehat{\mathcal{S}}_{(G,G,s)} = \widetilde{\mathcal{S}}_{(G,G,s)}$.

Consider the group $\mathrm{Aut}(\overline{P}) = \prod_{v \in V(G/P)} \mathrm{Aut}(\overline{P}_v)$ and recall that its elements fix every leg of $\overline{P}$. Therefore each of them induces an automorphism of G which fixes every edge in R by fixing its half-edges, and maps every component, $\overline{P}_v$, to itself. Hence we have an injection $\mathrm{Aut}(\overline{P}) \hookrightarrow \mathrm{Aut}(G, P, s)$.

Remark 3.6.1. The natural map

$$\widehat{\mathcal{S}}_{(G,P,s)} = \prod_{v \in V(G/P)} \widetilde{\mathcal{S}}_{(\overline{P}_v, \overline{P}_v, s_v)} \longrightarrow \prod_{v \in V(G/P)} \mathcal{S}_{(\overline{P}_v, \overline{P}_v, s_v)}$$

is the quotient of the natural action of $\operatorname{Aut}(\overline{P})$ on $\widehat{\mathcal{S}}_{(G,P,s)}$. Indeed, we know already that $\mathcal{S}_{(\overline{P}_v, \overline{P}_v, s_v)} = [\widetilde{\mathcal{S}}_{(\overline{P}_v, \overline{P}_v, s_v)}/\operatorname{Aut}(\overline{P}_v)]$. As we said above, the elements of $\operatorname{Aut}(\overline{P})$ leave every factor of $\widehat{\mathcal{S}}_{(G,P,s)}$ invariant.

We denote by $\operatorname{Aut}(G/P, s)$ the group of automorphisms of G/P which preserve the sign function, s, on $V(G/P)$; in symbols

$$\operatorname{Aut}(G/P, s) = \{\alpha \in \operatorname{Aut}(G/P) : \alpha(v) = u \Rightarrow s(v) = s(u), \forall v, u \in V(G/P)\}.$$

Now, there is a natural homomorphism

$$\operatorname{Aut}(G, P, s) \longrightarrow \operatorname{Aut}(G/P, s). \tag{17}$$

Indeed, assume, for simplicity G free from legs, then any $\alpha \in \operatorname{Aut}(G, P, s)$ maps R to itself, hence α maps the set of half-edges of R to itself. Now the set of half-edges of R is the set of half-edges of G/P, hence we have a bijection $\overline{\alpha} : H(G/P) \to H(G/P)$ mapping h to $\alpha(h)$. It is trivial to check that $\overline{\alpha}$ gives an automorphism of G/P as wanted.

We denote by $\operatorname{Aut}_G(G/P, s) \subset \operatorname{Aut}(G/P, s)$ the image of the map (17).

Lemma 3.6.2. *We have an exact sequence of groups*

$$0 \longrightarrow \operatorname{Aut}(\overline{P}) \longrightarrow \operatorname{Aut}(G, P, s) \longrightarrow \operatorname{Aut}_G(G/P, s) \longrightarrow 0$$

Proof. Let $\alpha \in \operatorname{Aut}(G, P, s)$. If $\overline{\alpha}$ is the identity of G/P, then α fixes every half-edge of R and maps every $\overline{P}_v$ to itself. Therefore $\alpha \in \operatorname{Aut}(\overline{P})$. Next, every $\alpha \in \operatorname{Aut}(\overline{P})$ clearly induces the identity of G/P. ♣

Proposition 3.6.3. *For every stable spin graph (G, P, s) we have*

$$\mathcal{S}_{(G,P,s)} = \left[\frac{\widehat{\mathcal{S}}_{(G,P,s)}}{\operatorname{Aut}(G, P, s)}\right].$$

Proof. By Remark 3.6.1 we have

$$\widehat{\mathcal{S}}_{(G,P,s)} \longrightarrow \left[\frac{\widehat{\mathcal{S}}_{(G,P,s)}}{\operatorname{Aut}(\overline{P})}\right] = \prod_{v \in V(G/P)} \mathcal{S}_{(\overline{P}_v, \overline{P}_v, s_v)}.$$

We begin by defining a map

$$\theta\colon \prod_{v \in V(G/P)} \mathcal{S}_{(\overline{P}_v, \overline{P}_v, s_v)} \longrightarrow \mathcal{S}_{(G,P,s)}.$$

Let B be a k-scheme. For any $v \in V(G/P)$, let $(\mathcal{Z}_v, \sigma_v, \mathcal{L}_v)$ be a section in $\mathcal{S}_{(\overline{P}_v, \overline{P}_v, s_v)}(B)$, so that the dual graph of the fibers of $(\mathcal{Z}_v, \sigma_v) \to B$ is $\overline{P}_v$.

Set $R = E \smallsetminus P$. Let $\sigma_v^R \subset \sigma_v$ be the set of sections that do not correspond to the legs of G, and set $\sigma^R = \cup_{v \in V(G/P)} \sigma_v^R$. Note that the elements in σ^R

come naturally in pairs indexed by R, with each pair giving the two branches of the corresponding node in R.

For each $e \in R$, consider the 2-pointed, polarized family of rational curves, $(\mathcal{E}_e, \sigma_e, \mathcal{L}_e)$, over B such that $\mathcal{E}_e = \mathbb{P}^1_k \times B$, the two sections of σ_e are constant (say equal to $\{0, \infty\}$), and the polarization is $\mathcal{L}_e = \pi^*_{\mathbb{P}^1_k}\mathcal{O}(1)$. Now, each pair in σ^R can be glued to the corresponding pair σ_e for all $e \in R$. This gives a gluing of $\sqcup_{v \in V(G/P)} \mathcal{Z}_v$ with $\sqcup_{e \in R} \mathcal{E}_e$, and hence a family of connected curves over B, written $\widehat{\mathcal{X}} \to B$. This family is endowed with marked points $\sigma := \cup_{v \in V(G/P)}(\sigma_v \smallsetminus \sigma_v^R)$, so that the fibers have dual graph $\widehat{G}_R$. We have a line bundle $\widehat{\mathcal{L}}$ on $\widehat{\mathcal{X}}$ by gluing $\mathcal{L}_v$ with $\mathcal{L}_e$, for every $v \in V(G/P)$ and $e \in R$ in the obvious way (the choice of gluing will not matter). Then $(\widehat{\mathcal{X}}, \sigma, \widehat{\mathcal{L}}) \in \mathcal{S}_{(G,P,s)}(B)$, and we set

$$\theta\Big(\prod_{v \in V(G/P)} (\mathcal{Z}_v, \sigma_v, \mathcal{L}_v)\Big) := (\widehat{\mathcal{X}}, \sigma, \widehat{\mathcal{L}}).$$

We now let η be the composition of θ with the quotient map described above,

$$\eta : \widehat{\mathcal{S}}_{(G,P,s)} \longrightarrow \left[\frac{\widehat{\mathcal{S}}_{(G,P,s)}}{\mathrm{Aut}(\overline{P})}\right] \xrightarrow{\theta} \mathcal{S}_{(G,P,s)}.$$

We want to show that η is the quotient of the natural action of $\mathrm{Aut}(G, P, s)$ on $\widehat{\mathcal{S}}_{(G,P,s)}$. By Lemma 3.6.2 we have that $\mathrm{Aut}_G(G/P, s)$ acts on $[\frac{\widehat{\mathcal{S}}_{(G,P,s)}}{\mathrm{Aut}(\overline{P})}]$.

From the description in the previous part of the proof, we have that two points have the same image under θ if and only if they are conjugate by $\mathrm{Aut}_G(G/P, s)$. This shows that θ is the quotient by $\mathrm{Aut}_G(G/P, s)$, and hence η is the quotient by $\mathrm{Aut}(G, P, s)$. ♣

We have morphisms $\widetilde{\mathcal{S}}_{(\overline{P}_v, \overline{P}_v, s_v)} \to \widetilde{\mathcal{M}}_{\overline{P}_v}$ whose product gives a morphism

$$\widehat{\mathcal{S}}_{(G,P,s)} \longrightarrow \prod_{v \in V(G/P)} \widetilde{\mathcal{M}}_{\overline{P}_v} \cong \widetilde{\mathcal{M}}_G.$$

By the proposition and the universal property of the fiber product we have a morphism $\lambda\colon \widehat{\mathcal{S}}_{(G,P,s)} \to \widetilde{\mathcal{S}}_{(G,P,s)}$ and a commutative diagram

(18)
$$\begin{array}{ccccc} & & \widetilde{\mathcal{S}}_{(G,P,s)} & \longrightarrow & \widetilde{\mathcal{M}}_G \\ & \overset{\lambda}{\nearrow} & \downarrow & & \downarrow \\ \widehat{\mathcal{S}}_{(G,P,s)} & & & & \\ & \underset{\eta}{\searrow} & & & \\ & & \mathcal{S}_{(G,P,s)} & \longrightarrow & \mathcal{M}_G \end{array}$$

(η is the quotient morphism defined in the proposition).

Lemma 3.6.4. *The map λ in diagram (18) is injective, and it is an isomorphism if and only if* $\mathrm{Aut}(G, P, s) = \mathrm{Aut}(G)$.

Proof. By definition, $\widetilde{\mathcal{S}}_{(G,P,s)}$ parametrizes spin curves whose underlying graph is identified with G and whose spin stucture is of type $\alpha_*(P,s)$ for some $\alpha \in \mathrm{Aut}(G)$.

Similarly, $\widehat{\mathcal{S}}_{(G,P,s)}$ parametrizes disjoint unions, indexed by $v \in V(G/P)$, of spin curves whose underlying graph is identified with $\overline{P}_v$ and whose spin stucture is of type $\alpha^v_*(\overline{P}_v, s_v)$ for some $\alpha^v \in \mathrm{Aut}(\overline{P}_v)$. Since $\prod(\mathrm{Aut}(\overline{P}_v)) = \mathrm{Aut}(\overline{P}) \subset \mathrm{Aut}(G)$, every such union determines a unique element of $\widetilde{\mathcal{S}}_{(G,P,s)}$, and is uniquely determined by it. Hence λ is injective.

For $\alpha \in \mathrm{Aut}(G)$, write $(P_\alpha, s_\alpha) = \alpha_*(P,s)$. We have $\mathcal{S}_{(G,P_\alpha,s_\alpha)} = \mathcal{S}_{(G,P,s)}$ (in $\overline{\mathcal{S}}_{g,n}$), hence $\widetilde{\mathcal{S}}_{(G,P,s)} = \widetilde{\mathcal{S}}_{(G,P_\alpha,s_\alpha)}$. We define $\lambda_\alpha \colon \widehat{\mathcal{S}}_{(G,P_\alpha,s_\alpha)} \longrightarrow \widetilde{\mathcal{S}}_{(G,P,s)}$ exactly as we did for λ (which is now $\lambda = \lambda_{id}$), so that we have

$$\widetilde{\mathcal{S}}_{(G,P,s)} = \cup_{\alpha \in \mathrm{Aut}(G)} \lambda_\alpha(\widehat{\mathcal{S}}_{(G,P_\alpha,s_\alpha)}).$$

Now observe that $\lambda_{\alpha_1}(\widehat{\mathcal{S}}_{(G,P_{\alpha_1},s_{\alpha_1})}) = \lambda_{\alpha_2}(\widehat{\mathcal{S}}_{(G,P_{\alpha_2},s_{\alpha_2})})$ for $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ if and only if $\alpha_2\alpha_1^{-1} \in \mathrm{Aut}(G,P,s)$, and so we are done. ♣

4. Irreducibility of the strata

In this section (G,P,s) will always be a stable spin graph; our goal is to show that $\widehat{\mathcal{S}}_{(G,P,s)}$ and $\mathcal{S}_{(G,P,s)}$ are irreducible.

4.1. Some special cases.

We here show, in Propositions 4.1.1 and 4.1.6, that $\widehat{\mathcal{S}}_{(G,P,s)}$ is irreducible in some cases needed in the proof of Theorem 4.2.3.

Proposition 4.1.1. *Let (G,P,s) be a stable spin graph of genus g. Then $\widehat{\mathcal{S}}_{(G,P,s)}$ is irreducible in the following cases:*

(i) $E = \emptyset$;
(ii) $g \leq 1$;
(iii) G is 3-regular (hence weightless).

Proof. If $E = \emptyset$, then $\widehat{\mathcal{S}}_{(G,P,s)} = \mathcal{S}_{(G,G,s)}$ and a spin structure on G is determined by the value of s at the vertex of G. Hence (i) is a consequence of [Cor89, Lm. 6.3] if $n = 0$, and of [J00, Thm. 3.3.1] in general.

Let $\{(\overline{P}_v, s_v), v \in V(G/P)\}$ be the set of connected components of (G,P,s). By Definition (16), it is enough to show that $\widetilde{\mathcal{S}}_{(\overline{P}_v,\overline{P}_v,s_v)}$ is irreducible.

Assume $g \leq 1$. For $v \in V(G/P)$, either $E(\overline{P}_v) = \emptyset$ and hence $\widetilde{\mathcal{S}}_{(\overline{P}_v,\overline{P}_v,s_v)}$ is irreducible by (i), or $\overline{P}_v$ is a cycle with all vertices of weight 0. So, we can assume that G is a cycle with all vertices of weight 0 and $P = G$. Now, $\widehat{\mathcal{S}}_{(G,G,s)} = \mathcal{S}_{(G,G,s)} \times_{\mathcal{M}_G} \widetilde{\mathcal{M}}_G$. Let X be a cycle of smooth rational curves with G as dual graph. After fixing the sign $s(v)$ of the unique vertex $v \in V(G/G)$, there is exactly one theta characteristic L on X of type (G,s), given by taking the trivial bundle on each component of X and choosing the unique gluing such that $h^0(X,L) \equiv s(v) \bmod (2)$ (see Fact 3.2.5). We get $\mathcal{S}_{(G,G,s)} \cong \mathcal{M}_G$, and hence $\widehat{\mathcal{S}}_{(G,G,s)} \cong \widetilde{\mathcal{M}}_G$, which is irreducible.

If G is 3-regular, then each $\overline{P}_v$ has genus $g_v \leq 1$, and (ii) implies that $\widetilde{\mathcal{S}}_{(\overline{P}_v,\overline{P}_v,s_v)}$ is irreducible for every $v \in V(G/P)$, so $\widehat{\mathcal{S}}_{(G,P,s)}$ is irreducible. ♣

For a vertex v of G we let $\text{loop}(v)$ be the number of loops based at v and

$$g(v) := \text{loop}(v) + w(v),$$

so that for any curve X having G as dual graph the irreducible component C_v of X has arithmetic genus equal to $g(v)$.

Definition 4.1.2. A stable graph G of genus $g \geq 2$ with $E \neq \emptyset$ is *basic* if it is cyclic, and if for every $v \in V$ we have $w(v) \leq 1$ and the following holds

$$w(v) + \deg(v) + \ell(v) \leq 4, \tag{19}$$

with equality only if $\text{loop}(v) \geq 1$.

If G is basic, we say that (G, G, s) is a *basic spin graph* for any s.

Remark 4.1.3. Let G be a basic graph and $v \in V$. Then

(1) $2 \leq \deg(v) \leq 4$ and if $\deg(v) = 4$ then $w(v) = \ell(v) = 0$.
(2) If $|V| = 1$ then $g = 2$.
(3) If $|V| \geq 2$ then the graph obtained by removing from G every loop is a cycle.
(4) $g(v) \leq 2$ and if equality holds then $|V| = 1$.

For $0 \leq i \leq 1$ and $0 \leq j \leq 2$ we define the following subsets of V

$$V_{i,j} := \{v \in V : w(v) = i \text{ and } g(v) = j\}. \tag{20}$$

By the above remark, every vertex of G is contained in some $V_{i,j}$. Set $V^+ := V \smallsetminus V_{0,0}$. The complete list of types of vertices is in Figure 1.

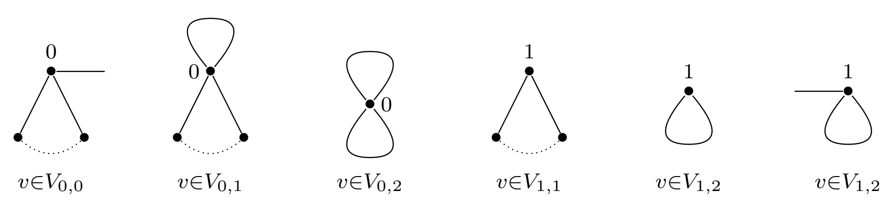


FIGURE 1. Possible types of vertices of a basic graph.

Let X have G as dual graph. For $v \in V^+$, we let $\nu\colon C_v^\nu \to C_v$ be the normalization of C_v. We let $\rho_{C_v}\colon C_v^\nu \to \mathbb{P}^1_k$ be the degree-two map induced by the g^1_2 of $X = C_v$ if $v \in V_{0,2} \cup V_{1,2}$, or by the linear system $|p_v + p'_v|$ of C_v if $v \in V_{0,1} \cup V_{1,1}$, where $\{p_v, p'_v\} := C_v \cap \overline{X \setminus C_v}$. Let $R(\rho_{C_v})$ be the image via ν of the ramification divisor of ρ_{C_v} and write

$$R(\rho_{C_v}) = \begin{cases} r_{v,1} + r_{v,2}, & \text{if } w(v) = 0; \\ r_{v,1} + r_{v,2} + r_{v,3} + r_{v,4}, & \text{if } w(v) = 1. \end{cases}$$

Each $r_{v,i}$ is a smooth point of X, hence $R(\rho_{C_v})$ is a Cartier divisor on X.

A *G-collection* is a collection of indices $I = (i_v)_{v\in V^+}$, with $i_v \in I_v$ for every $v \in V^+$, where

$$I_v := \begin{cases} \{1,2\}, & \text{if } w(v) = 0; \\ \{1,2,3,4\}, & \text{if } w(v) = 1. \end{cases} \tag{21}$$

Lemma 4.1.4. *Let G be a basic graph and X a curve with G as dual graph. Fix $u \in V^+$ and set $\epsilon_u = (-1)^{w(u)}$. Then the odd and even theta characteristics on X are, respectively, the line bundles $L_{X,I}$ and $M_{X,I}$ defined as follows*

$$L_{X,I} = \mathcal{O}_X\Big(\sum_{v\in V^+} r_{v,i_v}\Big), \quad M_{X,I} = \mathcal{O}_X\Big(\epsilon_u 3r_{u,i_u} - \epsilon_u R(\rho_{C_u}) + \sum_{v\in V^+\smallsetminus\{u\}} r_{v,i_v}\Big)$$

for every G-collection $I = (i_v)_{v\in V^+}$.

Proof. Assume $V = \{u\}$, hence G has genus 2. We have to check that the odd and even theta characteristics of X are, respectively, the line bundles $\mathcal{O}_X(r_{u,i})$ and $\mathcal{O}_X(\epsilon_u 3r_{u,i} - \epsilon_u \nu_* R(\rho_u))$ for $i \in I_u$. Since $\omega_X \cong \mathcal{O}_X(2r_{u,i})$ for every i, this is an easy checking.

Assume now $|V| \geq 2$, hence $V_{0,2} = V_{1,2} = \emptyset$. Since the curve X is fixed, we write $L_I = L_{X,I}$ and $M_I = M_{X,I}$. For different G-collections I, J, the line bundles $L_I \otimes L_J^{-1}$ and $M_I \otimes M_J^{-1}$ are not trivial, since so are their restrictions to at least one component of X. As the number of G-collections is $2^{|V_{0,1}|}4^{|V_{1,1}|}$, this is also the number of line bundles of type L_I, and of line bundles of type M_I, which is exactly the number of odd and even theta characteristics on X, by Fact 3.2.5.

Let $E' \subset E$ be the set edges which are not loops. By Remark 4.1.3 there exists an $e \in E'$, and $G - e$ becomes a tree after all loops are removed. Therefore the partial normalization, $\nu\colon X_e^\nu \to X$, of X at e is tree-like.

For every G-collection $I = (i_v)_{v\in V^+}$, the dualizing sheaf of X is

$$\omega_X \cong \mathcal{O}_X\left(\sum_{v\in V^+} 2r_{v,i_v}\right), \tag{22}$$

as $h^0(X_e^\nu, \nu^*\mathcal{O}_X(\sum_{v\in V^+} 2r_{v,i_v})) = g$. Hence $h^0(X, \mathcal{O}_X(\sum_{v\in V^+} 2r_{v,i_v})) = g$, by [Cap09, Lm. 2.2.4(2)], and hence L_I is a theta characteristic for every I.

Let N be a theta characteristic of X. We have $\deg N_{|C_v} \leq 1$ and hence $h^0(C_v, N_{|C_v}) \leq 1$ for every $v \in V$. Therefore

$$h^0(X,N) \leq h^0(X_e^\nu, \nu^*N) \leq \sum_{v\in V(G)} h^0(C_v, N_{|C_v}) - |E' \smallsetminus \{e\}| \leq 1.$$

(We use $|V| = |E'|$.) So $h^0(X,N) = 1$ if N is odd, and $h^0(X,N) = 0$ if N is even. In particular, since of course $h^0(X, L_I) > 0$, the odd theta characteristics on X are as stated. There remains to show every M_I is an even theta characteristic. By [Har82, Thm. 2.14] this is true if we find a G-collection J such that $M_I \otimes L_J^{-1}$ is a nontrivial square root of $\mathcal{O}_X$. Define

$J = (j_v)_{v \in V^+}$ by choosing $j_u = i_u$ if $u \in V_{0,1}$, and $j_u \neq i_u$ if $u \in V_{1,1}$, and setting $j_v = i_v$ for every $v \in V^+ \smallsetminus \{u\}$. Then $M_I \otimes L_J^{-1}$ is non trivial, as one easily checks

$$(M_I \otimes L_J^{-1})|_{C_u} \cong \begin{cases} \mathcal{O}_{C_u}(r_{u,i_u} - r_{u,k}), & \text{if } \{i_u, k\} = \{1,2\}, \\ \mathcal{O}_{C_u}(r_{u,k} - r_{u,k'}), & \text{if } \{i_u, j_u, k, k'\} = \{1,2,3,4\}, \end{cases}$$

which is nontrivial. Finally we show that $T := (M_I \otimes L_J^{-1})^{\otimes 2}$ is trivial. By the definition of M_I and L_J, we have

$$T \cong \begin{cases} \mathcal{O}_X(2r_{u,i_u} - 2r_{u,k}), & \text{if } \{i_u, k\} = \{1,2\}; \\ \mathcal{O}_X(-4r_{u,i_u} + 2r_{u,k} + 2r_{u,k'}), & \text{if } \{i_u, j_u, k, k'\} = \{1,2,3,4\}. \end{cases}$$

An easy calculation shows that $h^0(X_e^\nu, \nu^*\mathcal{O}_X(2r_{u,i_u})) = 2$ which, again by [Cap09, Lm. 2.2.4(2)], implies that $h^0(X, \mathcal{O}_X(2r_{u,i_u})) = 2$. It follows that $\mathcal{O}_X(2r_{u,i_u}) \cong \mathcal{O}_X(2r_{u,h})$ for $h \in \{k, k'\}$, and hence T is trivial. ♣

Remark 4.1.5. Let $z \in \mathcal{M}_{0,4}$ represent the pointed curve $(\mathbb{P}^1; p_1, p_1', p_2, p_2')$. Then z determines a degree-2 map, $\rho_z \colon \mathbb{P}^1 \to \mathbb{P}^1$, such that $\rho_z(p_i) = \rho_z(p_i')$ for $i = 1, 2$. We denote by $R(\rho_z) = r_{z,1} + r_{z,2}$ the ramification divisor of ρ_z. We have an isomorphism $(\mathbb{P}^1; p_1, p_1', p_2, p_2') \cong (\mathbb{P}^1; p_1', p_1, p_2', p_2)$ induced by the involution associated to ρ_z. Therefore the involution on $\mathcal{M}_{0,4}$ exchanging the first two marked points acts like the involution exchanging the last two points. We denote this involution by α, so that the quotient map, $\mathcal{M}_v \to \mathcal{M}_{0,4}/\langle \alpha \rangle$, is two-to-one and ramifies only at $(\mathbb{P}^1; 0, \infty, 1, -1)$.

Proposition 4.1.6. *Let (G, G, s) be a basic stable spin graph. Then $\widetilde{\mathcal{S}}_{(G,G,s)}$ is irreducible.*

Proof. For any $v \in V_{0,0}$ we have $\mathcal{M}_v = \mathcal{M}_{0,3}$ which is a point. Hence the natural étale map from $\widetilde{\mathcal{S}}_{(G,G,s)}$ to $\widetilde{\mathcal{M}}_G$ can be written as follows

$$\xi \colon \widetilde{\mathcal{S}}_{(G,G,s)} \longrightarrow \prod_{v \in V^+} \mathcal{M}_v = \widetilde{\mathcal{M}}_G$$

Since $\widetilde{\mathcal{S}}_{(G,G,s)}$ is smooth it suffices to prove it is connected. We assume throughout the proof that $\mathrm{char}(k) = 0$: the proof in positive characteristic will follow using [DM69, Thm. 4.17], as in [J00, Thm 3.3.11]. We also assume that (G, G, s) is odd; the proof for the other case is the same. By Lemma 4.1.4 we can write

$$\xi^{-1}(z) = \{L_{X,I} : \ \forall\ G\text{-collections } I\}$$

where X is the curve parametrized by the image of z in $\mathcal{M}_G$. To show that $\widetilde{\mathcal{S}}_{(G,G,s)}$ is connected, it suffices to find a point $z \in \widetilde{\mathcal{M}}_G$ such that for every G-collections I, J there is a path in $\widetilde{\mathcal{S}}_{(G,G,s)}$ connecting $L_{X,I}$ to $L_{X,J}$.

Write $I = (i_v)_{v \in V^+}$ and $J = (j_v)_{v \in V^+}$, where $i_v, j_v \in I_v$. Let α_v be the involution of I_v switching i_v and j_v for every $v \in V^+$.

First consider a vertex $v \in V_{0,1} \cup V_{0,2}$, so that $\mathcal{M}_v = \mathcal{M}_{0,4}$.

Let $z_v \in \mathcal{M}_v$ correspond to $(\mathbb{P}^1; 0, \infty, 1, -1)$, let $\rho = \rho_{z_v}$ be the degree-2 map of Remark 4.1.5, and let $r_{z_v,1}$ and $r_{z_v,2}$ be its two ramification points. This enables us to define the following two points in $\mathcal{M}_{0,4}$

$$(\mathbb{P}^1; \rho(r_{z_v,1}), \rho(r_{z_v,2}), \rho(0), \rho(1)), \qquad (\mathbb{P}^1; \rho(r_{z_v,\alpha_v(1)}), \rho(r_{z_v,\alpha_v(2)}), \rho(0), \rho(1)).$$

Fix a path, ψ_v, in $\mathcal{M}_{0,4}$ between these two points.

Now, to any point $(\mathbb{P}^1; q_1, q_2, q_3, q_4)$ of $\mathcal{M}_{0,4}$ we associate the degree-2 covering of $\mathbb{P}^1$ branched at q_1 and q_2, and marked by the (four) pre-images of q_3 and q_4. This does not quite give a point in $\mathcal{M}_{0,4}$, because the four marked points are not naturally ordered. But, using Remark 4.1.5, it clearly determines a point in $\mathcal{M}_{0,4}/\langle\alpha\rangle$. In this way the path ψ_v gives rise to a closed path, ψ'_v, in $\mathcal{M}_{0,4}/\langle\alpha\rangle$ based at $[(\mathbb{P}^1; 0, \infty, 1, -1)]$. Now, there exists a lifting of ψ'_v to $\mathcal{M}_v$, written κ_v, and it is clear that κ_v is a closed path based at z_v.

Now consider a vertex $v \in V_{1,1} \cup V_{1,2}$. Let $z_v \in \mathcal{M}_v = \mathcal{M}_{1,2}$ be the point parametrizing a pointed curve $(Z_v; p_v, p'_v)$. Let $\rho_{z_v} : Z \to \mathbb{P}^1$ be the degree-2 map induced by the linear system $|p_v + p_v|$, and $R(\rho_{z_v}) = r_{z_v,1} + r_{z_v,2} + r_{z_v,3} + r_{z_v,4}$ be its ramification divisor. Write $\rho = \rho_{z_v}$ for simplicity. Consider a path ψ_v in $\mathcal{M}_{0,5}$ starting from the point

$$(\mathbb{P}^1; \rho(r_{z_v,1}), \rho(r_{z_v,2}), \rho(r_{z_v,3}), \rho(r_{z_v,4}), \rho(p_v))$$

and ending at the point

$$(\mathbb{P}^1; \rho(r_{z_v,\alpha_v(1)}), \rho(r_{z_v,\alpha_v(2)}), \rho(r_{z_v,\alpha_v(3)}), \rho(r_{z_v,\alpha_v(4)}), \rho(p_v)).$$

Now, we have a morphism $\mathcal{M}_{0,5} \to \mathcal{M}_{1,2}$, mapping $(\mathbb{P}^1; q_1, q_2, q_3, q_4, q_5)$ to $(Z_v; p, p')$, defined as the pointed curve endowed with the degree-2 covering $Z_v \to \mathbb{P}^1$ branched at the first four points and marked by the two pre-images, p, p', of q_5, taken in any order (we have an isomorphism $(Z_v; p, p') \cong (Z_v; p', p)$). The image of ψ_v under this morphism gives rise to a closed path, κ_v, in $\mathcal{M}_{1,2}$ based at z_v.

Now take a point $z = \prod_{v \in V^+} z_v \in \widetilde{\mathcal{M}}_G$ with $z_v \in \mathcal{M}_v$ as above. Take the closed path κ in $\widetilde{\mathcal{M}}_G$ based at z given by the product of the closed paths κ_v in each $\mathcal{M}_v$. For every $v \in V^+$, the map ρ_{z_v} coincides with the map $\rho_{C_v} : C_v^\nu \to \mathbb{P}^1$ defined before Lemma 4.1.4, and hence we have $R(\rho_{z_v}) = R(\rho_{C_{z,v}})$. By the description of the theta characteristics of X in Lemma 4.1.4 in terms of the ramification points of the maps $\rho_{C_{z,v}}$, we see that, by construction, there is a lifting to $\widetilde{\mathcal{S}}_{(G,G,s)}$ of the path κ to a path starting from $L_{X,I}$ and ending at $L_{X,J}$, as wanted. ♣

4.2. **Proof of the irreducibility.** The proof of Theorem 4.2.3 will will use the following Lemma.

Lemma 4.2.1. *Let G be a non basic Eulerian stable graph of genus $g \geq 2$ with $E(G) \neq \emptyset$. Consider a spin structure of type (G, s) on G. Then there is a stable graph G' contracting to G, and a connected spin structure (P', s') on G' such that:*

(1) $|E(G')| = |E(G)| + 1$ and $b_1(G') = b_1(G)$;

(2) $\mathrm{Aut}(G', P', s') = \mathrm{Aut}(G')$*;*
(3) $\gamma_*(P', s') = (G, s)$ *for every contraction* $\gamma\colon G' \to G$*.*
(4) Any spin structure on G' *contracting to* (G, G, s) *is equal to* (G', P', s')*.*

Proof. Since G is not basic, there exists a vertex v of G such that

$$w(v) + \deg(v) + \ell(v) \geq 4, \tag{23}$$

and if equality holds then G has no loop based at v.

Let $C \subset G$ be a cycle containg v. Denote by e_1, e_2 the edges of C adjacent at v and denote by E_v^* the set of edges of G adjacent to v and different from e_1 and e_2. Then E_v^* has even (possibly zero) cardinality. The graph G' will be defined as a blow-up of v separating e_1, e_2 in such a way that G' is stable. We shall denote by $e' \in E(G')$ the edge to be contracted to v, and by $u_1, u_2 \in V(G')$ the end points of e'. Note that u_1 and u_2 have degree at least 2. The weight function, w', of G' must be such that $w'(u_1) + w'(u_2) = w(v)$. Similarly, the number of legs at each vertex must satisfy $\ell'(u_1) + \ell'(u_2) = \ell(v)$.

We need to distribute the edges in E_v^* between u_1 and u_2, and define w' and ℓ' so that G' is stable. It is clear that (23) implies that this is always possible. On the other hand it is not always possible to have G' also Eulerian. Indeed, G' is Eulerian if and only if u_1 and u_2 have even degree. For that to be possible we need either $\deg(v) \geq 6$ (i.e. $|E_v^*| \geq 4$ and we can attach two edges at u_1 and the remaing ones at u_2) or $\deg(v) = 4$ and $w(v) + \ell(v) \geq 1$ (so that we can attach the two edges of E_v^* at u_1 and define w' and ℓ' so that $w'(u_2) + \ell'(u_2) = 1$), or else $\deg(v) = 2$ (so that $w(v) + \ell(v) \geq 2$ and we define w' and ℓ' so that $w'(u_i) + \ell'(u_i) \geq 1$ for $i = 1, 2$).

The only case left is $\deg(v) = 4$ and $w(v) + \ell(v) = 0$, which is different in that G' is not Eulerian, so we treat it at the end.

In all other cases we have a Eulerian stable graph G' such that $G'/e' = G$; see Figure 2. Let (G', G', s') be such that the value of s' (on the unique vertex of $V(G'/G')$) is equal to the value of s (on the unique vertex of $V(G/G)$). Then it is clear that (G', G', s') satisfies the statement.

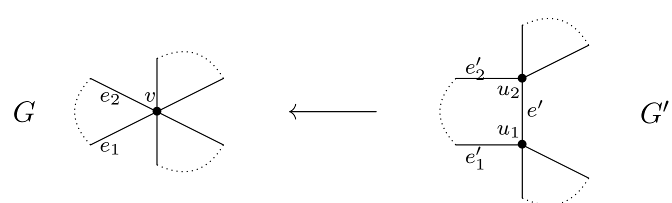


FIGURE 2.

We are left with the case $\deg(v) = 4$ and $w(v) + \ell(v) = 0$; then G has no loops at v. Now we define G' and the corresponding contraction, γ, as in Figure 3. Notice that G' is stable, and has exactly two vertices of odd degree, namely u_1 and u_2. Hence G' is not Eulerian, but its subgraph $P' := G' - e'$, is Eulerian. Consider the spin structure (P', s') where s' has the same value as s. Of course, $\gamma_*(P', s') = (G, s)$ and it obvious that P' is the only cyclic subgraph of G' such that $\gamma_* P' = G$.

To prove that $\mathrm{Aut}(G', P', s') = \mathrm{Aut}(G')$ it suffices to observe that every automorphism of G' must leave $\{u_1, u_2\}$ invariant, and hence it must fix e' which is the only edge containing u_1 and u_2 (G has no loop at v).

Now, any contraction of G' to G must take u_1 and u_2 to vertices of even degree. Therefore the contraction γ is the only possible one. ♣

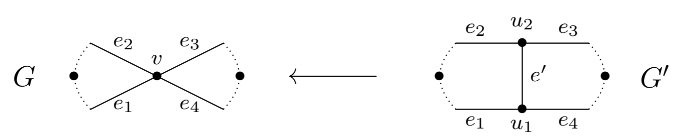


FIGURE 3.

We are ready to prove that $\widehat{\mathcal{S}}_{(G,P,s)}$ is irreducible for every stable spin graph (G, P, s). We will apply the discussion of Subsection 3.5. First, we illustrate the strategy of the proof in the following example, where we show that $\mathcal{S}_{(G,G,s)}$ is irreducible in case $\overline{\mathcal{M}}_G$ is smooth.

Example 4.2.2. Let $G' \to G$ be the contraction of a stable graph of genus 2 with 2 legs as in Figure 4. Notice that the contraction of the edge e' of G' is the unique contraction of G' to G. As G' is the unique stable graph contracting to G, we have that $\overline{\mathcal{M}}_G$ is smooth. Consider a spin graph of type (G, G, s), and let (P', s') be the spin structure on G' such that $P' = [G' - e']$ and that $s'(v') = s(v)$, where $V(G/G) = \{v\}$ and $V(G'/P') = \{v'\}$.

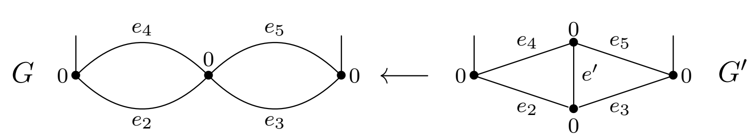


FIGURE 4. The contraction $G' \to G$.

Since G' is 3-regular, Proposition 4.1.1 yields that $\mathcal{S}_{(G',P',s')}$ is irreducible. We claim

(1) $\mathcal{S}_{(G',P',s')} = \overline{\mathcal{S}}_{(G,G,s)} \cap \pi^{-1}(\mathcal{M}_{G'})$;
(2) $\overline{\mathcal{S}}_{(G,G,s)}$ is smooth.

Indeed (1) follows by Proposition 3.4.2 and Lemma 4.2.1. For (2), recall the notations of Subsection 3.5 and consider a point $y \in \mathcal{S}_{(G',P',s')}$. The homomorphism $\pi^\# \colon \widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},\pi(y)} \to \widehat{\mathcal{O}}_{\overline{\mathcal{S}}_{g,n},y}$ is given by $\pi^\#(t_1) = s_1^2$ and $\pi^\#(t_i) = s_i$

for $i \geq 2$, hence we have

$$\widehat{\mathcal{O}}_{\overline{\mathcal{S}}_G,y} \cong \frac{k[[t_1,\dots,t_5]]}{(t_2,t_3,t_4,t_5)} \otimes_{k[[t_1,\dots,t_5]]} k[[s_1,\dots,s_5]] \cong \frac{k[[s_1,\dots,s_5]]}{(s_2,s_3,s_4,s_5)}.$$

Therefore $\overline{\mathcal{S}}_G$, and hence $\overline{\mathcal{S}}_{(G,G,s)}$, is smooth at y.

We prove that $\mathcal{S}_{(G,G,s)}$ is irreducible by showing that so is $\overline{\mathcal{S}}_{(G,G,s)}$. Let $\mathcal{W}_1,\dots,\mathcal{W}_p$ be the irreducible components of $\overline{\mathcal{S}}_{(G,G,s)}$. Then $\mathcal{S}_{(G',P',s')} = \cup_{i=1}^{p}(\mathcal{W}_i \cap \mathcal{S}_{(G',P',s')})$. Since the map $\mathcal{S}_{(G,G,s)} \to \mathcal{M}_G$ is finite and étale each irreducible component of $\mathcal{S}_{(G,G,s)}$ dominates $\mathcal{M}_G$. Hence the restriction of the map $\overline{\mathcal{S}}_{(G,G,s)} \to \overline{\mathcal{M}}_G$ to each $\mathcal{W}_i$ is surjective. Therefore, by (1), every intersection $\mathcal{W}_i \cap \mathcal{S}_{(G',P',s')}$ is not empty. Since $\mathcal{S}_{(G',P',s')}$ is irreducible and different $\mathcal{W}_i$ cannot intersect (as $\overline{\mathcal{S}}_{(G,G,s)}$ is smooth), we get $p = 1$, as wanted.

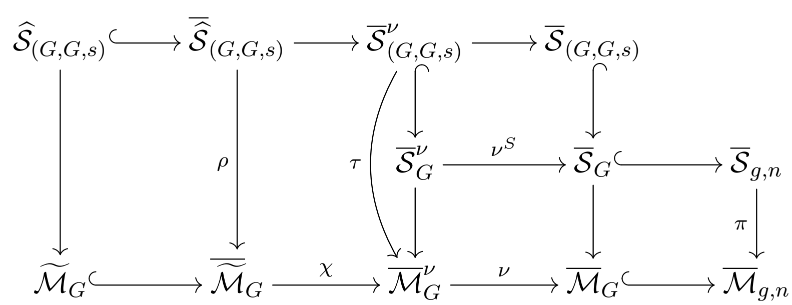
Theorem 4.2.3. *$\widehat{\mathcal{S}}_{(G,P,s)}$ is irreducible for any stable spin graph (G,P,s).*

Proof. We proceed by induction on $d_G := \dim \mathcal{M}_G$. If $d_G = 0$, the statement follows from Proposition 4.1.1 (iii). So we assume $d_G \geq 1$.

By (16) we can assume $G = P$. By Proposition 4.1.6 we can assume that (G,G,s) is not basic and, by Proposition 4.1.1 (i) and (ii), that G has genus $g \geq 2$ with $E(G) \neq \emptyset$. Using Lemma 4.2.1, we fix a stable graph G' contracting to G with $d_{G'} = d_G - 1$, and a spin structure (P',s') on G' such that $\mathrm{Aut}(G',P',s') = \mathrm{Aut}(G')$ and $\gamma_*(P',s') = (G,s)$ for every contraction $\gamma\colon G' \to G$; recall that γ contracts exactly one edge. By Proposition 3.4.2 and Lemma 4.2.1, we have

$$\mathcal{S}_{(G',P',s')} = \overline{\mathcal{S}}_{(G,G,s)} \cap \pi^{-1}(\mathcal{M}_{G'}). \tag{24}$$

Let $\nu\colon \overline{\mathcal{M}}^{\nu}_G \to \overline{\mathcal{M}}_G$ be the normalization and, with the notation in Subsection 3.5, consider the following diagram, where the squares are all Cartesian.



Since $G = P$ we have $\widehat{\mathcal{S}}_{(G,G,s)} = \widetilde{\mathcal{S}}_{(G,G,s)} = \mathcal{S}_{(G,G,s)} \times_{\mathcal{M}_G} \widetilde{\mathcal{M}}_G$.

Step 1. $\widehat{\overline{\mathcal{S}}}_{(G,G,s)}$ is smooth at every point lying over $\mathcal{M}_{G'}$.

Notice that $\widehat{\overline{\mathcal{S}}}_{(G,G,s)} \to \overline{\mathcal{S}}^{\nu}_{(G,G,s)}$ is étale because so is χ (recall (14)) and $\overline{\mathcal{S}}^{\nu}_{(G,G,s)}$ is union of irreducible components of $\overline{\mathcal{S}}^{\nu}_G$. Hence it suffices to show that $\overline{\mathcal{S}}^{\nu}_G$ is smooth at any point y^{ν} lying over $\mathcal{M}_{G'}$. We will use Subsection 3.5 and notation (13). Let $x'_{\gamma} \in \nu^{-1}(\mathcal{M}_{G'})$ be the point over which y^{ν}

lies; let $x' = \nu(x'_\gamma) \in \overline{\mathcal{M}}_{g,n}$ and $y = \nu^S(y^\nu) \in \overline{\mathcal{S}}_{g,n}$. Locally at y^ν we have

$$\widehat{\mathcal{O}}_{\overline{\mathcal{S}}^\nu_G, y^\nu} \cong \frac{k[[t_1, \ldots, t_{3g-3+n}]]}{\mathcal{I}_\gamma} \otimes_{k[[t_1,\ldots,t_{3g-3+n}]]} k[[s_1, \ldots, s_{3g-3+n}]],$$

where the vanishing of t_1 corresponds to locally trivial deformations at the node contracted by γ. Let $\mathcal{I}$ be the ideal of $k[[t_1, \ldots, t_{3g-3+n}]]$ generated by $t_2, \ldots, t_{3g-3+n}$, then $\mathcal{I}_\gamma \subset \mathcal{I}$. Locally at x' and y, the map π is induced by $\pi^\#\colon \widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},x'} \to \widehat{\mathcal{O}}_{\overline{\mathcal{S}}_{g,n},y}$, given $\pi^\#(t_i) = s_i$ for $i \geq 2$, and by $\pi^\#(t_1) = s_1^h$ with $h = 1, 2$ depending on, respectively, whether $P' = G'$ or $P' = G' - e'$ (recall the proof of Lemma 4.2.1). Therefore $\overline{\mathcal{S}}^\nu_G$ is smooth at y^ν.

Step 2. There is a stratum, $\mathcal{N}$, of $\overline{\widetilde{\mathcal{M}}}_G$ such that $\rho^{-1}(\mathcal{N})$ is irreducible.

Consider $\gamma\colon G' \to G$ and let e' be the contracted edge. Let u_1, u_2 be the vertices incident to e', and v_0 be the vertex of G to which e' is contracted. We introduce a connected graph, H, having one edge and two vertices, u_1^H, u_2^H, such that, for $1 = 1, 2$, we have

$$w_H(u_i^H) = w_{G'}(u_i)$$
$$\ell_H(u_i^H) = \deg_{G'}(u_i) + \ell_{G'}(u_i) - 1.$$

Then H is a stable graph of genus $g_H = w_H(u_1^H) + w_H(u_2^H)$ with $n_H = \ell_H(u_1^H) + \ell_H(u_2^H)$ legs. We consider the corresponding codimension-one stratum

$$\mathcal{M}_H \subset \overline{\mathcal{M}}_{g_H, n_H} = \overline{\mathcal{M}}_{w_G(v_0), \deg(v_0) + \ell_G(v_0)}$$

and define

$$\mathcal{N} := \mathcal{M}_H \times \Big(\prod_{v \in V(G) \smallsetminus \{v_0\}} \mathcal{M}_{w_G(v), \deg_G(v) + \ell_G(v)} \Big).$$

Then $\mathcal{N}$ can be identified with a codimension-one stratum of $\overline{\widetilde{\mathcal{M}}}_G$, and the map $\chi\colon \overline{\widetilde{\mathcal{M}}}_G \to \overline{\mathcal{M}}^\nu_G$ sends this stratum onto one of the copies, written $\mathcal{M}^\gamma_{G'}$, of $\mathcal{M}_{G'}$ contained in the pre-image $\nu^{-1}(\mathcal{M}_{G'})$ (recall from Subsection 3.5 that $\nu^{-1}(\mathcal{M}_{G'})$ is the disjoint union of copies of $\mathcal{M}_{G'}$, one for each contraction $G' \to G$). There is a natural map $\widetilde{\mathcal{M}}_{G'} \to \mathcal{N}$ which presents $\mathcal{N}$ as a quotient $\mathcal{N} = [\widetilde{\mathcal{M}}_{G'}/\mathrm{Aut}(H)]$. Since $\mathrm{Aut}(H)$ is trivial (as $n_H > 0$) we have

$$\mathcal{N} \cong \widetilde{\mathcal{M}}_{G'}.$$

Moreover, by (24), we have an isomorphism $\tau^{-1}(\mathcal{M}^\gamma_{G'}) \cong \mathcal{S}_{(G',P',s')}$. Hence

$$\rho^{-1}(\mathcal{N}) \cong \mathcal{S}_{(G',P',s')} \times_{\mathcal{M}_{G'}} \widetilde{\mathcal{M}}_{G'} \cong \widehat{\mathcal{S}}_{(G',P',s')},$$

where the second isomorphism follows from Lemma 3.6.4 (as $\mathrm{Aut}(G', P', s') = \mathrm{Aut}(G')$). By induction $\widehat{\mathcal{S}}_{(G',P',s')}$ is irreducible, hence so is $\rho^{-1}(\mathcal{N})$.

Step 3. $\widehat{\mathcal{S}}_{(G,G,s)}$ is irreducible.

It suffices to prove that $\overline{\widehat{\mathcal{S}}}_{(G,G,s)}$ is irreducible. By contradiction, assume that $\overline{\widehat{\mathcal{S}}}_{(G,G,s)}$ has $p \geq 2$ irreducible components, written $\mathcal{W}_1, \ldots, \mathcal{W}_p$. By

Step 1, two different components, $\mathcal{W}_i$ and $\mathcal{W}_j$, do not intersect in $\rho^{-1}(\mathcal{N})$, hence we have a disjoint union

$$\rho^{-1}(\mathcal{N}) = \sqcup_{i=1}^{p} \left(\mathcal{W}_i \cap \rho^{-1}(\mathcal{N})\right).$$

Now, every irreducible component of $\widehat{\mathcal{S}}_{(G,G,s)}$ surjects onto $\widetilde{\mathcal{M}}_G$, hence every $\mathcal{W}_i$ surjects (via ρ) onto $\widetilde{\widetilde{\mathcal{M}}}_G$. Therefore each intersection $\mathcal{W}_i \cap \rho^{-1}(\mathcal{N})$ is not empty. This contradicts the irreducibility of $\rho^{-1}(\mathcal{N})$. ♣

The following is a consequence of Theorem 4.2.3 and Proposition 3.6.3.

Theorem 4.2.4. *$\mathcal{S}_{(G,P,s)}$ is irreducible for any stable spin graph (G, P, s).*

5. The tropicalization of $\overline{\mathcal{S}}_{g,n}$

5.1. **The tropicalization map.** Let $\mathcal{Y}$ be a proper Deligne-Mumford stack over k and let Y be its coarse moduli space. We write Y^{an} for the Berkovich analytification, defined in [B90]. Since $\mathcal{Y}$ is proper, for any non-Archimedean field extension K of the trivially valued field k, a K-point of Y^{an} is represented by a morphism $\operatorname{Spec} R \to \mathcal{Y}$, where R is the valuation ring of K. We denote by val_K the valuation of K and assume K complete.

We shall consider the analytifications $\overline{M}_{g,n}^{\text{an}}$ and $\overline{S}_{g,n}^{\text{an}}$. Let x^{an} be a point in $\overline{M}_{g,n}^{\text{an}}$. Then, up to field extension, x^{an} can be represented by a stable n-pointed curve $\mathcal{X} \to \operatorname{Spec} R$, where R is as above. Let X be the reduction over the closed point of $\operatorname{Spec} R$, and G the dual graph of X. The (well known) tropicalization map

$$\operatorname{Trop}_{\overline{\mathcal{M}}_{g,n}} : \overline{M}_{g,n}^{\text{an}} \longrightarrow \overline{M}_{g,n}^{\text{trop}}$$

takes x^{an} to the class of the tropical curve $\Gamma = (G, \ell)$, where, given a node, e, of X and an étale neighborhood where the local equation of $\mathcal{X}$ at e is $xy = f_e$, for $f_e \in R$, we have $\ell(e) = \operatorname{val}_K(f_e)$.

Similarly, a point y^{an} in $\overline{S}_{g,n}^{\text{an}}$ is represented by a stable n-pointed spin curve $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}}) \to \operatorname{Spec} R$. We let $(\widehat{X}, \widehat{L})$ be the reduction over k and write X for the stable model of $\widehat{X}$. Keeping this notation, we state the following

Definition/Lemma 5.1.1. *We have a* tropicalization *map*

$$\operatorname{Trop}_{\overline{\mathcal{S}}_{g,n}} : \overline{S}_{g,n}^{\text{an}} \longrightarrow \overline{S}_{g,n}^{\text{trop}}$$

taking $y^{\text{an}} \in \overline{S}_{g,n}^{\text{an}}$ to the spin tropical curve (Γ, P, s), where $\Gamma = (G, \ell)$ with (G, P, s) the dual spin graph of $(\widehat{X}, \widehat{L})$, and ℓ is defined as follows. For every node e of X, let e' be a node of $\widehat{X}$ lying over e, and let $xy = h_e$ be an étale-local equation for $\widehat{\mathcal{X}}$ at e', with $h_e \in R$, then

$$\ell(e) = val_K(h_e).$$

Proof. The map $\operatorname{Trop}_{\overline{\mathcal{S}}_{g,n}}$ is obviously well defined if there is only one node, e', of $\widehat{X}$ lying over the node e of X. Suppose we have two nodes, e', e'' lying over e. By the description of the universal deformation of a spin curve given in [CCC07, Sect. 3.2] (in particular Eq. (4)), there are étale neighborhoods of $\widehat{\mathcal{X}}$ around e' and e'' in which the local equations of $\widehat{\mathcal{X}}$ are respectively $xy = h_e$ and $x'y' = h_e$, for the same $h_e \in R$. Hence $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}$ is well defined. One can show as in [Viv13, Lem-Def. 2.4.1.] that $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}$ is independent of the choice of R and of the local equation. ♣

The map $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ induces a map of Berkovich analytic spaces,

$$\pi^{\mathrm{an}}\colon \overline{S}^{\mathrm{an}}_{g,n} \longrightarrow \overline{M}^{\mathrm{an}}_{g,n},$$

defined as follows. Let $y^{\mathrm{an}} \in \overline{S}^{\mathrm{an}}_{g,n}$ be represented by $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}}) \to \operatorname{Spec} R$, and let $\psi\colon \operatorname{Spec} R \to \overline{\mathcal{S}}_{g,n}$ be the corresponding map. Then $x^{\mathrm{an}} := \pi^{\mathrm{an}}(y^{\mathrm{an}})$ is represented by the morphism $\pi \circ \psi\colon \operatorname{Spec} R \to \overline{\mathcal{M}}_{g,n}$. It is clear that x^{an} is represented by the stable model, $\mathcal{X} \to \operatorname{Spec} R$, of $\widehat{\mathcal{X}} \to \operatorname{Spec} R$.

We shall prove in Theorem 5.2.2 that the tropicalization maps are compatible with π^{trop} and π^{an} i.e. that $\pi^{\mathrm{trop}} \circ \mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}} = \mathrm{Trop}_{\overline{\mathcal{M}}_{g,n}} \circ \pi^{\mathrm{an}}$.

5.2. **The skeleton of $\overline{\mathcal{S}}_{g,n}$.** We recall some results for toroidal embeddings of Deligne-Mumford stacks from [ACP15, Sect. 6].

Let $\mathcal{Y}$ be as above and assume that $U \subset \mathcal{Y}$ is a toroidal embedding of Deligne-Mumford stacks. Given a scheme V and an étale morphism $V \to \mathcal{Y}$, one considers the group, D_V, of Cartier divisors on V supported on $V \smallsetminus U_V$, where $U_V = U \times_{\mathcal{Y}} V$, and the submonoid $\mathrm{E}_V \subset \mathrm{D}_V$ of effective divisors. Let $\mathrm{D}_{\mathcal{Y}}$ and $\mathrm{E}_{\mathcal{Y}}$ be the étale sheaves on $\mathcal{Y}$ associated to these presheaves.

Given a stratum $W \subset \mathcal{Y}$ and a geometric point y of W, there is a natural action of the étale fundamental group $\pi_1^{et}(W, y)$ on the stalk $\mathrm{D}_{\mathcal{Y},y}$ that preserves the stalk $\mathrm{E}_{\mathcal{Y},y} \subset \mathrm{D}_{\mathcal{Y},y}$. The *monodromy group* H_W is the image of $\pi_1^{et}(W, y)$ in $\mathrm{Aut}(\mathrm{D}_{\mathcal{Y},y})$. We define the extended cone

$$\overline{\sigma}_W := \mathrm{Hom}(\mathrm{E}_{\mathcal{Y},y}, \overline{\mathbb{R}}_{\geq 0}). \tag{25}$$

The *skeleton* $\overline{\Sigma}(\mathcal{Y})$ of $\mathcal{Y}$ is the extended generalized cone complex

$$\overline{\Sigma}(\mathcal{Y}) := \varinjlim \overline{\sigma}_W, \tag{26}$$

where $\overline{\sigma}_W \hookrightarrow \overline{\sigma}_{W'}$ for $W' \subset \overline{W}$, and the automorphisms of $\overline{\sigma}_W$ are induced by the elements of the monodromy group H_W.

There is a remarkable retraction, $\mathbf{p}_{\mathcal{Y}} : Y^{\mathrm{an}} \to \overline{\Sigma}(\mathcal{Y})$, described as follows. Assume that the point y^{an} of Y^{an} is represented by a map $\psi\colon \operatorname{Spec} R \to \mathcal{Y}$. Let $y \in \mathcal{Y}$ be the image of the closed point of $\operatorname{Spec} R$ and let W be the stratum of $\mathcal{Y}$ containing y. We have a chain of maps

$$\mathrm{E}_{\mathcal{Y},y} \xrightarrow{\ \epsilon\ } \widehat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\ \psi^{\#}\ } R \xrightarrow{\ \mathrm{val}_K\ } \overline{\mathbb{R}}_{\geq 0} \tag{27}$$

where ϵ is the map that takes an effective divisor to its local equation. Then $\mathbf{p}_{\mathcal{Y}}(y^{\mathrm{an}}) \in \overline{\Sigma}(\mathcal{Y})$ is the equivalence class of the homomorphism (27), which is an element of $\overline{\sigma}_W$.

The inclusion $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is a toroidal embedding of Deligne-Mumford stacks (see [ACP15, Sect. 3.3]), so we can consider the corresponding skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$. As for $\overline{\mathcal{S}}_{g,n}$, we have a similar situation.

Proposition 5.2.1. *The inclusion $\mathcal{S}_{g,n} \subset \overline{\mathcal{S}}_{g,n}$ is a toroidal embedding of Deligne-Mumford stacks.*

The map $\pi \colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is a toroidal morphism of (toroidal) Deligne-Mumford stacks.

The map $\pi^{\mathrm{an}} \colon \overline{S}^{\mathrm{an}}_{g,n} \to \overline{M}^{\mathrm{an}}_{g,n}$ restricts to a map of extended generalized cone complexes $\overline{\Sigma}(\pi) \colon \overline{\Sigma}(\overline{\mathcal{S}}_{g,n}) \longrightarrow \overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$.

Proof. Let x and y be closed points of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{S}}_{g,n}$ respectively, such that $x = \pi(y)$. Let $S_{(G,P,s)}$ be the stratum containing y. With the notation of Subsection 3.5, we write $E = \{e_1, \ldots, e_\delta\}$ so that $P = \{e_{r+1}, \ldots, e_\delta\}$.

Locally at y, the boundary divisor $\overline{\mathcal{S}}_{g,n} \smallsetminus \mathcal{S}_{g,n}$ is given by $\prod_{i=1}^{\delta} s_i = 0$, so $\mathcal{S}_{g,n} \subset \overline{\mathcal{S}}_{g,n}$ is a toroidal embedding of Deligne-Mumford stacks.

Next, π is toroidal, because, locally at x and y, it is induced by the ring homomorphism $\pi^{\#} \colon \widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},x} \to \widehat{\mathcal{O}}_{\overline{\mathcal{S}}_{g,n},y}$ given by $\pi^{\#}(t_i) = s_i^2$ for $i \leq r$, and $\pi^{\#}(t_i) = s_i$ for $i > r$.

The last part follows from [ACP15, Prop. 6.1.8] ♣

We denote by $\Sigma(\overline{\mathcal{M}}_{g,n})$, respectively $\Sigma(\overline{\mathcal{S}}_{g,n})$, the generalized cone complex obtained by restricting $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$, resp. $\overline{\Sigma}(\overline{\mathcal{S}}_{g,n})$, to its "finite part".

By [ACP15, Thm. 1.2.1], there is an isomorphism of extended generalized cone complexes $\overline{\Phi}_{\overline{\mathcal{M}}_{g,n}} \colon \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \longrightarrow \overline{M}^{\mathrm{trop}}_{g,n}$ which restricts to an isomorphism $\Sigma(\overline{\mathcal{M}}_{g,n}) \cong M^{\mathrm{trop}}_{g,n}$, and such that $\mathrm{Trop}_{\overline{\mathcal{M}}_{g,n}} = \overline{\Phi}_{\overline{\mathcal{M}}_{g,n}} \circ \mathbf{p}_{\overline{\mathcal{M}}_{g,n}}$.

Theorem 5.2.2. *There is an isomorphism of extended generalized cone complexes $\overline{\Phi}_{\overline{\mathcal{S}}_{g,n}} \colon \overline{\Sigma}(\overline{\mathcal{S}}_{g,n}) \longrightarrow \overline{S}^{\mathrm{trop}}_{g,n}$, restricting to an isomorphism $\Sigma(\overline{\mathcal{S}}_{g,n}) \cong S^{\mathrm{trop}}_{g,n}$. Moreover, the following diagram is commutative*

$$\begin{array}{ccccc} \overline{S}^{\mathrm{an}}_{g,n} & \xrightarrow{\mathbf{p}_{\overline{\mathcal{S}}_{g,n}}} & \overline{\Sigma}(\overline{\mathcal{S}}_{g,n}) & \xrightarrow{\overline{\Phi}_{\overline{\mathcal{S}}_{g,n}}} & \overline{S}^{\mathrm{trop}}_{g,n} \\ {\scriptstyle \pi^{\mathrm{an}}}\downarrow & & {\scriptstyle \overline{\Sigma}(\pi)}\downarrow & & {\scriptstyle \pi^{\mathrm{trop}}}\downarrow \\ \overline{M}^{\mathrm{an}}_{g,n} & \xrightarrow{\mathbf{p}_{\overline{\mathcal{M}}_{g,n}}} & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) & \xrightarrow{\overline{\Phi}_{\overline{\mathcal{M}}_{g,n}}} & \overline{M}^{\mathrm{trop}}_{g,n} \end{array}$$

and $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}} = \overline{\Phi}_{\overline{\mathcal{S}}_{g,n}} \circ \mathbf{p}_{\overline{\mathcal{S}}_{g,n}}$.

Proof. Recall our definition

$$\overline{S}^{\mathrm{trop}}_{g,n} = \varinjlim \overline{\sigma}_{(G,P,s)},$$

where the right-hand side is the colimit of the diagram of extended cones $\overline{\sigma}_{(G,P,s)}$, with $\overline{\sigma}_{(H,Q,s')} \hookrightarrow \overline{\sigma}_{(G,P,s)}$ for $(G,P,s) \geq (H,Q,s')$, and the automorphisms of $\overline{\sigma}_{(G,P,s)}$ are induced by $\mathrm{Aut}(G,P,s)$.

Consider a stratum $W = \mathcal{S}_{(G,P,s)}$ of $\overline{\mathcal{S}}_{g,n}$. Given a point $y \in W$, by Subsection 3.5 we have an isomorphism of monoids $\mathrm{E}_{\overline{\mathcal{S}}_{g,n},y} \to \mathbb{Z}^E_{\geq 0}$, so by (25), the extended cone $\overline{\sigma}_W$ is naturally isomorphic to $\overline{\sigma}_{(G,P,s)}$. We can thus identify them and rewrite (26) as follows

$$\overline{\Sigma}(\overline{\mathcal{S}}_{g,n}) = \varinjlim \overline{\sigma}_{(G,P,s)},$$

where $\overline{\sigma}_{(H,Q,s')} \hookrightarrow \overline{\sigma}_{(G,P,s)}$ for $\overline{\mathcal{S}}_{(G,P,s)} \subset \overline{\mathcal{S}}_{(H,Q,s')}$, and the automorphisms of $\overline{\sigma}_{(G,P,s)}$ are induced by the elements of the monodromy group $H_{\mathcal{S}_{(G,P,s)}}$.

By Proposition 3.4.2, the existence of an isomorphism $\overline{\Phi}_{\overline{\mathcal{S}}_{g,n}}$ as in the statement follows once we prove that $H_{\mathcal{S}_{(G,P,s)}} = \mathrm{Aut}(G,P,s)$ for every (G,P,s). To show this, let $y \in \mathcal{S}_{(G,P,s)}$ and recall diagram (18). By subsection 3.5, the set E determines a group basis for $\mathrm{D}_{\overline{\mathcal{S}}_{g,n},y}$ and a monoid basis for $\mathrm{E}_{\overline{\mathcal{S}}_{g,n},y}$. The locally constant sheaf of sets on $\mathcal{S}_{(G,P,s)}$ whose stalk at every point is the set of nodes of the underlying stable curve becomes trivial when pulled back to $\widehat{\mathcal{S}}_{(G,P,s)}$. Hence the pull-backs of $\mathrm{D}_{\overline{\mathcal{S}}_{g,n}}$ and $\mathrm{E}_{\overline{\mathcal{S}}_{g,n}}$ to $\widehat{\mathcal{S}}_{(G,P,s)}$ are trivial. By Proposition 3.6.3 and Theorem 4.2.3, the action of $\pi_1^{et}(\mathcal{S}_{(G,P,s)}, y)$ on $\mathrm{D}_{\overline{\mathcal{S}}_{g,n},y}$ factors through its quotient $\mathrm{Aut}(G,P,s)$, and hence $H_{\mathcal{S}_{(G,P,s)}} = \mathrm{Aut}(G,P,s)$.

We now prove that $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}} = \overline{\Phi}_{\overline{\mathcal{S}}_{g,n}} \circ \mathbf{p}_{\overline{\mathcal{S}}_{g,n}}$ and that the diagram in the statement is commutative; its left square is so by [ACP15].

Consider a point y^{an} in $\overline{S}^{\mathrm{an}}_{g,n}$ given by $\psi\colon \operatorname{Spec} R \to \overline{\mathcal{S}}_{g,n}$. Assume that the image, y, of the closed point of $\operatorname{Spec} R$ lies in the stratum $\mathcal{S}_{(G,P,s)}$. Let $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}}) = [(\Gamma, P, s)] \in \overline{S}^{\mathrm{trop}}_{g,n}$, where $\Gamma = (G, \ell)$. The set E can be seen as a monoid basis of the free monoid $\mathrm{E}_{\overline{\mathcal{S}}_{g,n},y}$, and ℓ is given by the composition

$$\ell: \quad E \longrightarrow \mathrm{E}_{\overline{\mathcal{S}}_{g,n},y} \xrightarrow{\mathbf{p}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}})} \overline{\mathbb{R}}_{\geq 0}$$

Recalling the definition of $\mathbf{p}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}})$ given in (27), we obtain

$$\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}}) = \overline{\Phi}_{\overline{\mathcal{S}}_{g,n}} \circ \mathbf{p}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}}).$$

Let $x^{\mathrm{an}} = \pi^{\mathrm{an}}(y^{\mathrm{an}})$, then x^{an} is represented by $\pi \circ \psi\colon \operatorname{Spec} R \to \overline{\mathcal{M}}_{g,n}$. With the notations of Subsection 3.5, $\mathrm{Trop}_{\overline{\mathcal{M}}_{g,n}}(x^{\mathrm{an}})$ is the tropical curve (G,ℓ) such that

$$\ell(e_i) = \mathrm{val}_K(\psi^{\#}(\pi^{\#}(t_i))) = \begin{cases} 2\mathrm{val}_K(\psi^{\#}(s_i)) & \text{if } 1 \leq i \leq r \\ \mathrm{val}_K(\psi^{\#}(s_i)) & r < i \leq \delta \end{cases}$$

since we have $\pi^{\#}(t_i) = s_i^2$ for $i \leq r$ and $\pi^{\#}(t_i) = s_i$ for $i > r$.

On the other hand, $\mathrm{Trop}_{\overline{\mathcal{S}}_{g,n}}(y^{\mathrm{an}})$ is the tropical spin curve in $\overline{S}^{\mathrm{trop}}_{(G,P,s)}$ such that the edge e_i of G has length $\mathrm{val}_K(\psi^{\#}(s_i))$. By the definition of

π^{trop}, the length of e_i on the curve $\pi^{\text{trop}} \circ \text{Trop}_{\overline{\mathcal{S}}_{g,n}}(y^{\text{an}})$ is $2\text{val}_K(\psi^{\#}(s_i))$ if $i \leq r$ and $\text{val}_K(\psi^{\#}(s_i))$ otherwise; so it is equal to $\ell(e_i)$.

We thus proved $\pi^{\text{trop}} \circ \text{Trop}_{\overline{\mathcal{S}}_{g,n}}(y^{\text{an}}) = \text{Trop}_{\overline{\mathcal{M}}_{g,n}} \circ \pi^{\text{an}}(y^{\text{an}})$. ♣

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