COMBINATORICS OF COMPACTIFIED UNIVERSAL JACOBIANS

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Abstract. We use orientations on stable graphs to express the combinatorial structure of the compactified universal Jacobians in degrees $g-1$ and g over $\overline{M}_g$, and construct for them graded stratifications compatible with the one of $\overline{M}_g$. In particular, for a stable curve we exhibit graded stratifications of the compactified Jacobians in terms of totally cyclic, respectively rooted, orientations on subgraphs of its dual graph.

1. Introduction and Preliminaries

1.1. **Introduction.** The boundary of the compactifications of various moduli spaces often exhibits a stratification in terms of increasingly degenerate objects. A basic example of this phenomenon is $\overline{M}_g$, the compactification of the moduli space of smooth curves of genus $g \geq 2$ by stable curves, where the boundary strata parametrize curves with an increasing number of nodes.

This widespread behaviour has received new attention lately thanks to recent progress in tropical and non-Archimedean geometry. In fact, a thorough study of the boundary of $\overline{M}_g$ and of its combinatorial incarnation has led to a remarkable discovery. In loose words, the Berkovich skeleton of $\overline{M}_g$ (or, the tropicalization of $\overline{M}_g$) is the moduli space for the skeleta of stable curves over complete valued fields (or, the moduli space of tropical curves, $\overline{M_g^{\mathrm{trop}}}$); an analogous result holds for other moduli spaces, like $\overline{M}_{g,n}$ or the space of admissible covers. These facts are proved, building upon results of [9], [11] and [26], in [1] for $\overline{M}_{g,n}$ and in [17] for admissible covers; see also [6], [27], [28], [7] for related progress. Apart from being of interest in its own right, this discovery explicitly connects different areas of geometry, enabling one to transfer ideas and techniques from one area to the other.

As we said, the starting point was the study of the boundary from the combinatorial point of view. First, one shows it admits a stratification of a specific combinatorial type, which we call a *graded stratification by a poset* $\mathcal{P}$; see Definition 1.3.2. Second, one identifies the stratifying poset, $\mathcal{P}$, with a combinatorial object interesting on its own. For example, for $\overline{M}_g$ the stratifying poset is $\mathcal{SG}_g$, the set of all stable graphs of genus g partially ordered with respect to edge-contraction. The graded stratification map (again Definition 1.3.2), $\overline{M}_g \to \mathcal{SG}_g$, sends a curve X to its dual graph, G.

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The purpose of this paper is to study, from this perspective, the compactification of the universal degree-d Jacobian (or degree-d Picard variety) over $\overline{M}_g$, constructed in [12]. Recall that for any $d \in \mathbb{Z}$ the compactification of the universal degree-d Jacobian is a projective morphism $\psi : \overline{P}^d_g \to \overline{M}_g$ whose fiber over an automorphism-free curve X is $\operatorname{Pic}^d(X)$ if X is smooth, and a compactified degree-d Jacobian, $\overline{P}^d_X$, if X is singular. In general, the fiber of ψ over X is the quotient of $\overline{P}^d_X$ by $\operatorname{Aut}(X)$; see Subsection 4.5. As d varies, so does the structure of $\overline{P}^d_g \to \overline{M}_g$, but it is well known that there are only finitely many non-isomorphic types, each of which can be realized by a value of d such that $0 \leq d \leq g$.

In this paper we concentrate on the cases $d = g-1$ and $d = g$, which are of special interest. Indeed, the case $d = g-1$ has been studied extensively in the past because of its natural connection with Prym varieties, the Theta divisor and the Torelli problem; see [8], [2], [16]. The case $d = g$ is notable because $\overline{P}^g_g$ is the coarse moduli scheme of a Deligne-Mumford stack, and its fiber over the curve X is a compactified Jacobian of Néron type, i.e. it compactifies the Néron model of the Jacobian of a regular one-parameter smoothing of X.

The space $\overline{P}^d_X$ parametrizes line bundles on partial normalizations of X having a special multidegree; as multidegrees on X coincide with divisors on the dual graph, G, of X, we call such special multidegrees *stable divisors*. This leads to a stratification of $\overline{P}^d_X$ given by the sets of nodes that are normalized, and by the sets of stable divisors on the partial normalization. For a fixed curve X the existence of such a stratification was essentially known, but a combinatorially interesting incarnation for it was not, with the exception of the case $d = g-1$. Indeed, it was known that a divisor of degree $g-1$ is stable if and only if it is the divisor associated to a *totally cyclic* orientation on G. Preceeding the notion of stable divisor, this observation was made in [8, Lemma 2.1] while studying Prym varieties. Independently, using the basic inequality of [12], this is a consequence of a theorem in graph theory, known as Hakimi's Theorem (originally in [21], for a formulation in our framework see [3, Theorem 4.8]). The graded stratification of $\overline{P}^{g-1}_X$ by totally cyclic orientations was established in [16] to study the Torelli map of stable curves.

We will prove results of a similar type in case $d = g$, and show that $\overline{P}^g_X$ has a graded stratification by the poset of *rooted* (generalized) orientations on G; see Definition 1.4.1. In particular, we show that a divisor is stable if and only if it is the divisor associated to a rooted orientation. We note that from this and [3, Lemma 3.3] it easily follows that the notions of break divisor, as introduced in [22], and of stable divisor coincide.

To be more precise, we will introduce for a stable graph G two graded posets: the poset $\overline{\mathcal{OP}}^0_G$ of totally cyclic orientation classes on spanning subgraphs of G, and the poset $\overline{\mathcal{OP}}^1_G$ of rooted orientation classes on spanning subgraphs of G. The partial order will be given by edge removal. We treat the cases $d = g-1$ and $d = g$ simultaneously, so we write $b = 0, 1$ and $d = g - 1 + b$. By mapping a point to its stratum we get a graded stratification map $\overline{P_X^{g-1+b}} \to \overline{\mathcal{OP}}_G^b$; see Theorem 4.3.4.

Next, we extend our analysis over $\overline{M}_g$, which we know is stratified by $\mathcal{SG}_g$ ordered by edge contraction. Our goal is to endow $\overline{P}_g^{g-1}$ and $\overline{P}_g^g$ with a graded stratification compatible with the one of $\overline{M}_g$ and with the fiberwise stratifications established earlier. In order to do that we first study the behaviour of $\overline{\mathcal{OP}}_G^b$ under edge-contractions and prove, in Theorem 3.3.1, that to every edge-contraction from G to G' there corresponds a quotient of posets $\overline{\mathcal{OP}}_G^b \to \overline{\mathcal{OP}}_{G'}^b$. Second, we study the natural action of $\mathrm{Aut}(G)$ on $\overline{\mathcal{OP}}_G^b$ and introduce a poset, $[\overline{\mathcal{OP}}_g^b]$, with a quotient of posets, $[\overline{\mathcal{OP}}_g^b] \to \mathcal{SG}_g$, whose fiber over G is $\overline{\mathcal{OP}}_G^b/\mathrm{Aut}(G)$; see Proposition 3.5.3.

The theorem below summarizes the main results of the paper.

Theorem 1.1.1. *Let $b = 0, 1$. The following diagram is commutative. The four horizontal maps, denoted by σ, are graded stratification maps, and the vertical map μ is a quotient of posets.*

$$
\begin{tikzcd}
\overline{P_g^{g-1+b}} \arrow[rrrrr, two heads, "\sigma"] \arrow[dddd, "\psi"'] & & & & & {[\overline{\mathcal{OP}}_g^b]} \arrow[dddd, "\mu"] \\
& \overline{P}_X^{g-1+b}/\mathrm{Aut}(X) \arrow[ul, hook] \arrow[rrr, two heads, "\sigma"] \arrow[dd] & & & {[\overline{\mathcal{OP}}_G^b]} \arrow[ur, hook] \arrow[dd] & \\
& & \overline{P}_X^{g-1+b} \arrow[ul] \arrow[r, two heads, "\sigma"] \arrow[dl] & \overline{\mathcal{OP}}_G^b \arrow[ur] \arrow[dr] & & \\
& {[X]} \arrow[rrr, mapsto] \arrow[dl, mapsto] & & & {[G]} \arrow[dr, mapsto] & \\
\overline{M}_g \arrow[rrrrr, two heads, "\sigma"] & & & & & \mathcal{SG}_g
\end{tikzcd}
$$

The theorem gives the sought-for combinatorial presentation of the compactified Jacobian of a curve, and of the compactified universal Jacobian over $\overline{M}_g$, for $d = g - 1, g$. The next question now is to provide the tropical version of the theorem, starting from the fact that the left-bottom corner of the diagram should be occupied by the moduli space of tropical curves, $\overline{M_g^{\mathrm{trop}}}$, while the right side should be the same, up to isomorphism. This will involve constructing skeleta of $\overline{P}_X^d$ and $\overline{P}_g^d$ as moduli spaces of suitable polyhedral objects. This research direction relates to results of [7], where the skeleton of the Jacobian of a curve over a valuation ring is shown to be the Jacobian of the skeleton of the curve. Results of [24] show that the compactification considered there agrees with the one constructed in [12], the one we are concerned with in this paper in case $d = g$. Results of [18] indicate that one can extend this description to the universal setting on the combinatorial side.

Another natural question remaining is about the extention of our results to $\overline{P}_g^d$ with $d \neq g, g-1$. Apart from the case $d = g-2$, which is essentially immediate by taking the residual of the degree g case, we so far have no candidate for a stratifying poset.

After a section containing preliminaries, Sections 2 to 3 establish the framework for graphs and generalized orientations, together with the behaviour under edge-contractions of totally cyclic and rooted orientations on subgraphs of the full graph. Although everything will be presented in a self-contained manner, our way of thinking about these objects has been influenced significantly by [20] and [5], which study the interplay between orientations and the divisors they define (or as they are sometimes called, the "indegree sequences of orientations"). In Section 4 we apply the combinatorial results to compactified Jacobians of degree $g-1$ and g and prove the claims contained in Theorem 1.1.1.

1.2. **Graphs.** Throughout the paper G denotes a vertex-weighted finite graph of genus g; we allow loops and multiple edges. We denote by $V = V(G)$ the set of vertices of G, by $E = E(G)$ the set of edges of G and by

$$w : V \to \mathbb{N}; \quad v \mapsto w(v)$$

the weight function of G. We write $c(G)$ for the number of connected components of G. The *genus*, $g = g(G)$, of G, is defined as follows

$$g(G) := \sum_{v \in V} w(v) - |V| + |E| + c(G).$$

We think of an edge of G as the union of two *half-edges*, each of which has a vertex of G as end, so that the ends of an edge e are the ends of its half-edges and e is a *loop* if the two ends coincide. We write $H = H(G)$ for the set of half-edges of G. We have a natural two-to-one surjection $H \to E$, and we write $\{h_e^+, h_e^-\}$ for the preimage of $e \in E$.

The *degree*, or valency, of a vertex v, written $\deg v$, is the number of half-edges whose end is v.

A *subgraph* of G will be endowed with the weight function obtained by restricting the weight function of G.

For a non empty $Z \subset V$, we write $Z^c := V \smallsetminus Z$. The *induced subgraph*, $G[Z] \subset G$, is the subgraph whose vertex-set is Z, whose edge-set is the set of all edges of G having both ends in Z, and whose weight function is the restriction to Z of the one of G. We set

$$g(Z) := g(G[Z]) = |E(G[Z])| - |Z| + c(G[Z]) + \sum_{v \in Z} w(v).$$

If $S \subset E$ is a set of edges of G, we write $G - S$ for the graph obtained from G by removing S; notice that G and $G - S$ have the same vertices, in other words $G - S$ is a so-called *spanning* subgraph of G. We denote by $\langle S \rangle$ the subgraph of G spanned by S, so that $E(\langle S \rangle) = S$ and the vertices of $\langle S \rangle$ are the vertices adjacent to the edges in S.

A *cut* of G is a set of edges, $S \subset E$, such that for a partition $V = Z \sqcup Z^c$, with $\emptyset \subsetneq Z \subsetneq V$, our S is the set of all edges adjacent to both Z and Z^c. We also write $S = E(Z, Z^c)$ for such a cut. For a non empty cut S we have $c(G) < c(G - S)$. We shall use the following elementary

Remark 1.2.1. Let $S \subset E$ be a cut of G and let $H \subset G$ be a subgraph. Then either $S \cap E(H) = \emptyset$ or $S \cap E(H)$ is a cut for H.

If $e \in E$ is such that $\{e\}$ is a cut, then e is called a *bridge*. We denote by $G_{br} \subset E$ the set of bridges of G.

Remark 1.2.2. For any $S \subset E$ we have $g(G) \geq g(G - S)$, with equality if and only if $S \subset G_{br}$.

A *morphism* between two graphs, $\eta : G \to G'$, is given by two maps, $\eta_V : V(G) \to V(G')$ and $\eta_E : E(G) \to E(G') \cup V(G')$ such that $\eta_E(e)$ has ends $\eta_V(v)$ and $\eta_V(w)$ for any $e \in E(G)$ whose ends are v and w. We sometime write just $\eta = \eta_E$ and $\eta = \eta_V$.

An *isomorphism* between two graphs, $\alpha : G \to G'$, is a morphism such that α_V is a bijection, $\alpha_E : E(G) \to E(G')$ is a bijection, and such that for every $v \in V(G)$ the weight of $\alpha_V(v)$ equals the weight of v. An isomorphism induces also a bijection between the half-edges of G and G'.

An *automorphism* is an isomorphism of G with itself. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G.

G is *semistable* if it is connected, $g(G) \geq 2$, and has no vertex of weight 0 and degree less than 2.

G is *stable* if it is semistable and has no vertex of weight 0 and degree less than 3. The set of all stable graphs of genus g is denoted by $\mathcal{SG}_g$.

Notice that $\mathcal{SG}_g$ is finite.

1.3. **Posets.** A *poset*, $(\mathcal{P}, \leq)$, or just $\mathcal{P}$, is a set partially ordered with respect to "$\leq$". Let $p_1, p_2 \in \mathcal{P}$. We say that p_2 *covers* p_1 if $p_1 < p_2$ and if there is no $p' \in \mathcal{P}$ such that $p_1 < p' < p_2$.

Let $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{Q}, \leq_{\mathcal{Q}})$ be two posets. We say that a map $\mu : \mathcal{P} \to \mathcal{Q}$. is a *morphism of posets* if $p_1 \leq_{\mathcal{P}} p_2$ implies $\mu(p_1) \leq_{\mathcal{Q}} \mu(p_2)$. We say that μ is a *quotient (of posets)* if for any $q_1, q_2 \in \mathcal{Q}$ such that $q_1 \leq_{\mathcal{Q}} q_2$ there exist $p_1 \in \mu^{-1}(q_1)$ and $p_2 \in \mu^{-1}(q_2)$ such that $p_1 \leq_{\mathcal{P}} p_2$. In particular, a quotient is a surjective morphism of posets.

We will apply the following trivial lemma a few times.

Lemma 1.3.1. *Let $\mathcal{P}$ be a finite poset and $\sim$ an equivalence relation on $\mathcal{P}$. Let $\pi : \mathcal{P} \to \overline{\mathcal{P}} = \mathcal{P}/_{\sim}$ be the quotient. Assume the following holds*

For every $x, y \in \mathcal{P}$ with $y \geq x$ and for every $y \sim y'$ there exists $x' \sim x$ such that $y' \geq x'$.

Then $\overline{\mathcal{P}}$ is a poset as follows: for $\overline{x}, \overline{y} \in \overline{\mathcal{P}}$ set $\overline{y} \geq \overline{x}$ if there exist $x' \sim x$ and $y' \sim y$ such that $y' \geq x'$. Moreover π is a quotient of posets.

The lemma holds if we switch roles between x and y, i.e. if we assume that for every $x \sim x'$ there exists $y' \sim y$ such that $y' \geq x'$.

A *rank* on a poset $\mathcal{P}$ is a map $\rho : \mathcal{P} \to \mathbb{N}$ such that if p_2 covers p_1 then $\rho(p_2) = \rho(p_1) + 1$. Of course, $\mathbb{N}$ is a poset and a rank is a morphism of posets. A poset endowed with a rank is called a *graded poset*.

Definition 1.3.2. Let M be an algebraic variety and let $\mathcal{P}$ be a poset. A *stratification of M by $\mathcal{P}$* is a partition of M

$$M = \sqcup_{p\in\mathcal{P}} M_p$$

such that the following hold for every $p, p' \in \mathcal{P}$.

(1) the *stratum* M_p is irreducible and quasi-projective;
(2) if $M_p \cap \overline{M_{p'}}$ is not empty, then $M_p \subset \overline{M_{p'}}$;
(3) $M_p \subset \overline{M_{p'}}$ if and only if $p \leq p'$.

A stratification of M by $\mathcal{P}$ is called *graded* if the following is a rank on $\mathcal{P}$

$$\mathcal{P} \longrightarrow \mathbb{N}; \qquad p \mapsto \dim M_p.$$

Let $\sigma : M \twoheadrightarrow \mathcal{P}$ be a surjective map. We call σ a *(graded) stratification map* if the fibers of σ form a (graded) stratification of M by $\mathcal{P}$.

1.4. **Generalized orientations.** Let G be a graph and e an edge of G. An *orientation* on e is the assignment of a direction so that one half-edge of e is the starting half-edge and the other is the ending half-edge. Accordingly, the vertex adjacent to the starting half-edge will be called the *source* of e, and the vertex adjacent to the ending half-edge will be called the *target* of e. If e is a loop then its base vertex is both source and target.

An *orientation*, O, on G is the assignment of an orientation on every edge of G. If $x \in V$ is the source (respectively, the target) of $e \in E$ we say that e is *O-outcoming* from x (resp. *O-incoming* at x); we say simply "outcoming" or "incoming" when no confusion seems possible.

A *generalized orientation* on G is the assignment, for every $e \in E$, of either an orientation on e, or of both orientations on e; in the latter case we say that e is *bioriented*. So, a bioriented edge has both its ends as targets and sources.

For $b \in \mathbb{N}$ a *b-orientation* is a generalized orientation having exactly b bioriented edges. We thus recover usual orientations as 0-orientations (which we shall continue to call "orientations" to ease the terminology)

In this paper, we shall mostly be interested in the cases $b = 0, 1$.

Definition 1.4.1. Let G be a graph.

An orientation (i.e. a 0-orientation) on G is *totally cyclic* if it has no *directed cut* i.e. if every non empty cut $E(Z, Z^c)$ has an edge with target in Z and one edge with target in Z^c.

A 1-orientation on G with bioriented edge e is *rooted*, or *e-rooted*, if for every $Z \subsetneq V$ such that $e \in G[Z]$, the cut $E(Z, Z^c)$ contains an edge with target in Z^c.

We denote

$$\mathcal{O}^0(G) := \{O : \ O \text{ is a totally cyclic orientation on } G\}$$

and

$$\mathcal{O}^1(G) := \{O : \ O \text{ is a rooted 1-orientation on } G\}.$$

The terminology "totally cyclic" and "rooted" is motivated by 1.4.2 (b), and 1.6.4, respectively.

Let G be a cycle. We say that G is *cyclically oriented* if it is given a totally cyclic orientation (of course, a cycle admits exactly two totally cyclic orientations). From [15, Lemma 2.4.3] we have:

Fact 1.4.2. *(a) $\mathcal{O}^0(G)$ is not empty if and only if G is free from bridges.*
(b) Let G be connected. An orientation on G is totally cyclic if and only if every pair of vertices is contained in a cyclically oriented cycle.

Convention 1.4.3. Assume G has no edges. Then the empty orientation will be considered as totally cyclic, so that $\mathcal{O}^0(G)$ consists of exactly that orientation.

If G consists of a single vertex, then the empty orientation will be considered rooted, so that $\mathcal{O}^0(G) = \mathcal{O}^1(G)$.

Notice that, by definition, an orientation on a graph is totally cyclic if and only if its restriction to every connected component of G is totally cyclic.

Remark 1.4.4. Let O be a totally cyclic orientation on a connected graph G. For any e of G, let O_e be the 1-orientation having e as bioriented edge and such that every remaining edge is oriented according to O. The definition implies that O_e is rooted. This gives an injection (not a surjection)

$$\mathcal{O}^0(G) \times E \longrightarrow \mathcal{O}^1(G); \qquad (O, e) \mapsto O_e.$$

Lemma 1.4.5. *$\mathcal{O}^1(G)$ is not empty if and only if G is connected.*

Proof. If G admits a rooted 1-orientation then, by definition, every cut $E(Z, Z^c)$ is non empty, hence G must be connected.

Conversely, let G be connected and set

$$G - G_{br} = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_n$$

with G_i connected for $i = 1, \ldots n$. Of course, G_i is bridgeless for every i, hence we can fix on G_i a totally cyclic orientation, O_i.

We pick an edge e of G_1 and consider the 1-orientation on G_1 having e as bioriented edge and such that every other edge is oriented according to O_1. This is a rooted 1-orientation, as noted in Remark 1.4.4. We fix this orientation on G_1 from now on, and we fix the orientations $O_2, \ldots, O_n$ on the remaining G_i.

Let us show how to orient G_{br} to obtain a rooted 1-orientation. Let $B_1 \subset G_{br}$ be the set of bridges adjacent to G_1 and, up to reordering $G_2, \ldots, G_n$, let $G_2, \ldots, G_{n_1}$ be adjacent to B_1, so that the following subgraph of G

$$H_2 = G_1 \cup B_1 \cup G_2 \cup \ldots \cup G_{n_1}$$

is connected. Since G is connected, $n_1 \geq 2$. Orient every edge in B_1 pointing away from G_1. It is easy to check that the so obtained 1-orientation on H_2 is rooted. If $H_2 = G$ we are done. If not we iterate as follows. Let $B_2 \subset G_{br}$ be the set of bridges adjacent to H_2 and let $G_{n_1+1}, \ldots G_{n_2}$ be the components not contained in H_2 and adjacent to B_2, so that the following

$$H_3 = H_2 \cup B_2 \cup G_{n_1+1} \cup \ldots \cup G_{n_2}$$

is connected. Orient every edge in B_2 away from H_2 so that the so-obtained 1-orientation is rooted. If $H_2 = G$ we stop, otherwise we iterate. Since G is connected, after a finite number, say m, of iterations we get $H_m = G$. ♣

1.5. **Divisors of generalized orientations.** The group of divisors on G, written $\mathrm{Div}(G)$, is the free abelian group generated by V. We shall identify $\mathrm{Div}(G) = \mathbb{Z}^V$ and denote a divisor on G by $\underline{d} = \{\underline{d}_v\}_{v\in V}$.

The degree of a divisor $\underline{d}$ is defined as $|\underline{d}| = \sum_{v\in V} \underline{d}_v$ and we write $\mathrm{Div}^k(G)$ for the set of divisors of degree k.

If $\underline{d}, \underline{d}' \in \mathrm{Div}(G)$ are such that $\underline{d}_v \leq \underline{d}'_v$ for every $v \in V$, we write $\underline{d} \leq \underline{d}'$.

If $S \subset E$, then G and $G-S$ have the same vertices, hence we shall identify $\mathrm{Div}(G) = \mathrm{Div}(G - S)$.

If $Z \subset V$ we write $\underline{d}_Z$ for the restriction of $\underline{d}$ to Z and $|\underline{d}_Z| = \sum_{v\in Z} \underline{d}_v$.

To a generalized orientation $O \in \mathcal{O}^b(G)$ (recall that if $E(G)$ is not empty b is the number of bioriented edges) we associate a divisor, $\underline{d}^O \in \mathrm{Div}(G)$, whose v coordinate, for every $v \in V$, is defined as follows

$$\underline{d}^O_v := \begin{cases} w(v) - 1 + \underline{t}^O_v & \text{if } E(G) \neq \emptyset \\ w(v) - 1 + b & \text{if } E(G) = \emptyset \end{cases}$$

where $\underline{t}^O_v$ denotes the number of half-edges having v as target, so that $\underline{t}^O := \{\underline{t}^O_v\}$ is also in $\mathrm{Div}(G)$.

Next, if G is connected and $O \in \mathcal{O}^b(G)$ one easily checks

$$|\underline{d}^O| = g(G) - 1 + b. \tag{1}$$

For any $Z \subset V$ we denote by $t^O(Z)$ the number of edges not contained in $G[Z]$ having target in Z, and by $b(Z)$ the number of bioriented edges contained in $G[Z]$. Notice the following

$$t^O(Z) = \sum_{z\in Z} \underline{t}^O_z - |E(G[Z])| - b(Z). \tag{2}$$

The following trivial lemma generalizes (1).

Lemma 1.5.1. *Let O be a b-orientation on G and let $Z \subset V$ be such that $G[Z]$ is connected. Then*

$$|\underline{d}^O_Z| = g(Z) - 1 + b(Z) + t^O(Z). \tag{3}$$

Proof. We have

$$|\underline{d}^O_Z| = \sum_{z\in Z} \underline{d}^O_v = \sum_{z\in Z} (w(z) - 1 + \underline{t}^O_z) = \sum_{z\in Z} w(z) - |Z| + \sum_{z\in Z} \underline{t}^O_z.$$

Now, $g(Z) = \sum_{z\in Z} w(z) - |Z| + |E(G[Z])| + 1$ hence, by (2),

$|\underline{d}^O_Z| = g(Z) - 1 - |E(G[Z])| + \sum_{z\in Z} \underline{t}^O_z = g(Z) - 1 + b(Z) + t^O(Z).$ ♣

The following lemmas characterize totally cyclic and rooted orientations. They are slight generalizations of [10, Lemma 1] and the remark thereafter.

Lemma 1.5.2. *Let O be a 0-orientation on a connected graph G. The following are equivalent.*

(a) O is totally cyclic.
(b) $t^O(Z) > 0$ for every non empty $Z \subsetneq V$.
(c) $t^O(Z) > 0$ for every non empty $Z \subsetneq V$ with $G[Z]$ connected.
(d) $|\underline{d}^O_Z| > g(Z) - 1$ for every non empty $Z \subsetneq V$ with $G[Z]$ connected.

Proof. (a) $\Rightarrow$ (b). By hypothesis the cut $E(Z, Z^c)$ must have some edge with target in Z, hence $t^O(Z) > 0$.

(b) $\Rightarrow$(c) is obvious.

(c) $\Rightarrow$(d). By (3) (with $b(Z) = 0$) and by hypothesis we have

$$|\underline{d}^O_Z| = g(Z) - 1 + t^O(Z) > g(Z) - 1.$$

(d) $\Rightarrow$ (a). Let $E(U, U^c)$ be a cut in G, we must prove that $E(U, U^c)$ is not a directed cut. Let $Z \subset U$ such that $G[Z]$ is a connected component of $G[U]$. Of course, $E(Z, Z^c) \subset E(U, U^c)$. By (3) we have

$$t^O(Z) = |\underline{d}^O_Z| - (g(Z) - 1) > 0$$

where the inequality follows by hypothesis. Hence $E(U, U^c)$ has an edge with target in Z, hence in U. The same argument applied to U^c shows that $E(U, U^c)$ has an edge with target in U^c. ♣

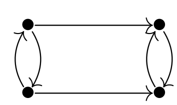

FIGURE 1. A non totally-cyclic orientation O with $\underline{t}^O > 0$.

Lemma 1.5.3. *Let O be a non empty 1-orientation on G and let e be its bioriented edge. The following are equivalent.*

(a) O is e-rooted.
(b) $t^O(Z) > 0$ for every non empty $Z \subset V$ with $e \notin G[Z]$.
(c) $t^O(Z) > 0$ for every non empty $Z \subset V$ such that $G[Z]$ is connected and $e \notin G[Z]$.
(d) $|\underline{d}^O_Z| > g(Z) - 1$ for every $Z \subsetneq V$ such that $G[Z]$ is connected.

Proof. (a) $\Rightarrow$ (b). By hypothesis $e \in G[Z^c]$. As O is rooted the cut $E(Z, Z^c)$ must have some edge with target in Z, hence $t^O(Z) > 0$.

(b) $\Rightarrow$(c) is obvious.

(c) $\Rightarrow$(d). If $e \notin G[Z]$ the proof is the same as for Lemma 1.5.2. If $e \in G[Z]$ we apply (3); as $b(Z) = 1$ we get

$$|\underline{d}^O_Z| = g(Z) + t^O(Z) \geq g(Z) > g(Z) - 1.$$

(d) $\Rightarrow$ (a). Let $E(U, U^c)$ be a cut in G with $e \in G[U]$. Let W be a connected component of $G[U^c]$, it suffices to show that $E(U, U^c)$ contains an edge with target in W. Now (3) applied to W yields

$$g(W) - 1 + t^O(W) = |\underline{d}^O_W| > g(W) - 1,$$

by hypothesis. Hence $t^O(W) > 0$, as wanted. ♣

1.6. **Equivalence of generalized orientations.**

Definition 1.6.1. We define two generalized orientations, O and O', on a graph G to be *equivalent*, and write $O \sim O'$, if $\underline{d}^O = \underline{d}^{O'}$.

We denote by $\overline{O}$ the equivalence class of O.

Remark 1.6.2. Let O and O' be two b-orientations, with $b = 0, 1$. By Lemmas 1.5.2 and 1.5.3, if $O \sim O'$ then O is totally cyclic (resp. rooted) if and only if so is O'.

We now introduce the sets of equivalence classes of totally cyclic orientations, and of rooted 1-orientations, on G written

$$\overline{\mathcal{O}}^0(G) := \mathcal{O}^0(G)/\sim \qquad \text{and} \qquad \overline{\mathcal{O}}^1(G) := \mathcal{O}^1(G)/\sim . \tag{4}$$

Remark 1.6.3. *Equivalence of 1-orientations through reversal of directed paths.* Let O be a 1-orientation whose bioriented edge e has ends v_0, v_1. We say that a path $P \subset G$ is O-directed from e to v, with $v \neq v_0, v_1$, if the first edge of P is e and if the component of $P - e$ containing v is a directed path with v as target.

Let $P \subset G$ be an O-directed path from e to v_{n+1} as in the Figure 2. Let $e' \subset P$ be the last edge of the path, so that the ends of e' are v_n and v_{n+1}. Define a new 1-orientation, O' on G as follows. Let e' be the bioriented edge, reverse the orientation on every remaining edge of P, and fix on e the orientation from v_1 to v_0. Notice that P is an O'-directed path from e' to v_0. Let O' coincide with O on the remaining edges of G. It is clear that O and O' are equivalent.

FIGURE 2. Equivalence of 1-orientations through the reversal of a directed path.

Lemma 1.6.4. *Let O be a non empty 1-orientation on a connected graph G and let e be its bioriented edge. The following are equivalent.*

(a) O is e-rooted.

(b) For every $v \in V$ there exists an O-directed path from e to v.

(c) For every $e' \in E$ there exists a 1-orientation O' whose bioriented edge is e' and such that $O \sim O'$.

Proof. (a) $\Rightarrow$ (b). Let x, y be the ends of e and let $Z_1 = \{x, y\}$. Since O is e-rooted and $e \in G[Z_1]$ the set, W_1, of vertices in Z_1^c that are targets of edges with source in Z_1 is not empty. Set $Z_2 = Z_1 \cup W_1$. If W_1 contains v we are done. If not, we iterate as follows. As O is rooted the set, W_2, of vertices in Z_2^c that are targets of edges with source in Z_2 is not empty. By construction, every vertex w in W_2 is the target of an edge with source in W_1, and hence w is the last vertex of a directed path starting with e. If W_2 contains v we are done, otherwise we iterate. Since G is connected, after finitely many steps this process includes all vertices of G, so we are done.

(b) $\Rightarrow$ (c). Let e' be oriented from v to w and let P be an O-directed path from e to v. We define O' as the 1-orientation obtained by reversing the orientation of P, as defined in 1.6.3.

(c) $\Rightarrow$ (a). By contradiction, suppose O is not rooted. Hence there exists a cut $E(Z, Z^c)$ directed away from Z and such that $e \in G[Z^c]$. Up to replacing Z with a subset, we can assume that $G[Z]$ is connected. We thus have $t^O(Z) = 0$ and, as $e \notin G[Z]$,

$$|\underline{d}_Z^O| = g(Z) - 1 + t^O(Z) = g(Z) - 1. \tag{5}$$

Pick $e' \in G[Z]$ and let O' be a 1-orientation with e' as bioriented edge such that $O \sim O'$, which exists by hypothesis. As $e' \in G[Z]$ we have

$$|\underline{d}_Z^O| = |\underline{d}_Z^{O'}| = g(Z) + t^{O'}(Z) \geq g(Z)$$

a contradiction with (5). ♣

2. Posets associated to graphs.

2.1. **Edge contractions.** Let G be a graph and $S \subset E$ a set of edges of G. The *(weighted) contraction* of S is a map of weighted graphs, $\gamma : G \to G/S$ (introduced in [11]). Informally γ is given by contracting to a vertex every edge in S, and such that the weight of a vertex v of G/S equals the genus of the subgraph of G which gets contracted into v. Rigorously, consider the subgraph, $\langle S \rangle \subset G$, spanned by the edges in S and let $\langle S \rangle = H_1 \sqcup \ldots \sqcup H_m$ be its decomposition in connected components. Now set

$$V(G/S) := V(G) \smallsetminus V(\langle S \rangle) \sqcup \{v_1, \ldots, v_m\}, \qquad E(G/S) := E(G) \smallsetminus S.$$

We have two maps,

$$\gamma_V : V(G) \longrightarrow V(G/S) \quad \text{and} \quad \gamma_E : E(G) \longrightarrow E(G/S) \cup V(G/S), \tag{6}$$

where γ_V is the identity on $V(G) \smallsetminus V(\langle S \rangle)$ and maps every vertex of H_i to v_i, and γ_E is the identity on $E(G) \smallsetminus S$ and maps every $e \in S$ to v_i such that e lies in H_i. It is clear that γ_V and γ_E determine a morphism of graphs $\gamma : G \to G/S$, as wanted. Finally, the weight function $w_{/S} : V(G/S) \to \mathbb{N}$ is defined as follows:

$$w_{/S}(v) = g(\gamma^{-1}(v)).$$

Indeed, $\gamma^{-1}(v)$ is the subgraph of G induced by the subset $\gamma_V^{-1}(v) \subset V(G)$, hence its genus is well defined.

For convenience we view the identity $G \to G$ as the trivial contraction.

Remark 2.1.1. We list some useful consequences of the definition.

(a) G is connected if and only if G/S is connected.
(b) $g(G) = g(G/S)$.
(c) If G is stable, or semistable, so is G/S.

Let $S \subset E(G)$ be a subset of edges of a graph G, we set

$$G(S) = G/(E \smallsetminus S). \tag{7}$$

Lemma 2.1.2. *Let $S \subset E(G)$ and $H := G/S$. Let $T \subset E(H)$. Then*

(a) $H - T = (G - T)/S$.
(b) $H(T) = G(T)/S = G(T)$.
(c) T is a cut of H if and only T is a cut of G.
(d) $H_{br} = \emptyset$ if and only if $G_{br} \subset S$.

Proof. It suffices to assume $S = \{e\}$; let $x, y \in V$ be the ends of e. Denote by $v_e \in H$ the vertex to which e is contracted; we have natural identifications

$$E(H) = E(G) \smallsetminus \{e\} \quad \text{and} \quad V(H) = V(G) \cup \{v_e\} \smallsetminus \{x, y\}.$$

Let us prove (a). Using the above identities and the fact that $e \notin T$, we have natural identifications (viewed as equalities):

$$E(H-T) = E(H) \smallsetminus T = E(G) \smallsetminus (T \cup \{e\}) = E(G-T) \smallsetminus \{e\} = E\Big(\frac{G-T}{e}\Big)$$

and, since $V(H-T) = V(H)$

$$V(H-T) = V(G) \cup \{v_e\} \smallsetminus \{x,y\} = V(G-T) \cup \{v_e\} \smallsetminus \{x,y\} = V\Big(\frac{G-T}{e}\Big).$$

It is clear that the above identifications induce a natural isomorphism between $H - T$ and $(G - T)/e$. (a) is proved.

(b). We have

$$H(T) = \frac{H}{E(H) \smallsetminus T} = \frac{G/e}{E(G) \smallsetminus (e \cup T)} = \frac{G}{(E(G) \smallsetminus T) \cup e} = \frac{G(T)}{e} = G(T).$$

(c). By (a) we have $H - T = (G - T)/S$, which is connected if and only if $G - T$ is connected.

(d) follows trivially from the preceeding parts. ♣

For any two graphs, G and G', we define the following *edge-contraction* relation

$$G' \geq G \quad \text{if} \quad G' = G/S \text{ for some } S \subset E(G). \tag{8}$$

Edge-contraction is easily seen to be a partial order on the set of all graphs.

Proposition 2.1.3. *The set $\mathcal{SG}_g$, endowed with the edge-contraction relation defined in* (8)*, is a graded poset with respect to the following rank*

$$\mathcal{SG}_g \longrightarrow \mathbb{N}: \qquad G \mapsto 3g-3-|E(G)|.$$

Proof. It is well known that for every $G \in \mathcal{SG}_g$ we have $|E(G)| \leq 3g-3$.

Let us prove that $\mathcal{SG}_g$ is graded. Let $G, H \in \mathcal{SG}_g$ such that H covers G. Hence $H = G/S$ for some non empty $S \subset E(G)$. We claim $|S| = 1$. Indeed, if $|S| \geq 2$ there exists a non empty $S' \subsetneq S$. But then by Remark 2.1.1 $G/S' \in \mathcal{SG}_g$ and

$$H > G/S' > G,$$

a contradiction. Therefore $|S| = 1$ and $|E(H)| = |E(G)| - 1$ as wanted. ♣

2.2. **The posets of bridgeless and connected subgraphs.** Let G be a graph and E its edge-set. The set of all subsets of E, written $\mathcal{P}(E)$, will be considered as a poset with respect to reverse inclusion, i.e. we set

$$S \leq S' \quad \text{if} \quad S' \subset S \tag{9}$$

for any $S, S' \subset E$.

We are interested in two special sub-posets of $\mathcal{P}(E)$, written $\mathcal{A}^0_G$ and $\mathcal{A}^1_G$, related to totally cyclic orientations, respectively rooted, orientations. We saw that $\mathcal{O}^0(G) \neq \emptyset$ (i.e. G admits a totally cyclic orientation) only if G is free from bridges. We need to study all totally cyclic orientations on all spanning subgraphs of G, so we consider the following set

$$\mathcal{A}^0_G := \{S \subset E : (G-S)_{br} = \emptyset\}.$$

Next, we know $\mathcal{O}^1(G) \neq \emptyset$ (i.e. G admits a rooted 1-orientation) only if G is connected, hence we set

$$\mathcal{A}^1_G := \{S \subset E : \ G-S \text{ is connected}\}.$$

Of course, $\mathcal{A}^1_G$ is empty if G is not connected.

We have the following simple fact.

Lemma 2.2.1. *Let $b = 0, 1$ and assume G connected if $b = 1$. Then $\mathcal{A}^b_G$ is a graded poset with respect to* (9)*, with rank function mapping S to $g(G-S)$.*

In particular, $\mathcal{A}^0_G$ has E as unique minimal element and G_{br} as unique maximal element, with $g(G-E) = \sum_{v \in V} w(v)$ and $g(G-G_{br}) = g(G)$. If G is connected, then $\mathcal{A}^1_G$ has $\emptyset$ as unique maximal element, and its minimal elements are the $S \subset E$ such that $G-S$ is a spanning tree.

Remark 2.2.2. For any $S \subset E$ we have $\mathcal{A}^0_{G-S} \hookrightarrow \mathcal{A}^0_G$. If $S = G_{br}$ the injection induces an identification $\mathcal{A}^0_G = \mathcal{A}^0_{G-G_{br}}$. Indeed, for every $S \in \mathcal{A}^0_G$ we have $G_{br} \subset S$, hence S is also an element of $\mathcal{A}^0_{G-G_{br}}$.

2.3. **Posets of orientations.** We shall be considering generalized orientations defined on various spanning subgraphs of a fixed graph G. To keep track of these subgraphs we shall use subscripts, as follows. Given $S \subset E$, we shall denote by O_S a generalized orientation on $G - S$. A generalized orientation with no subscript will be defined on the whole graph.

Definition 2.3.1. Let G be a graph and let $S, T \subset E(G)$. Given two generalized orientations O_S on $G - S$ and O_T on $G - T$ we set

$$O_S \leq O_T \quad \text{if} \quad S \leq T \quad \text{and} \quad (O_T)_{|G-S} = O_S.$$

It is easy to check that the above is a partial order.

We introduce, for a fixed graph G, the set of all totally cyclic orientations on all spanning subgraphs of G.

$$\mathcal{OP}^0_G := \bigsqcup_{S \in \mathcal{A}^0_G} \mathcal{O}^0(G - S). \tag{10}$$

Similarly, for rooted 1-orientations

$$\mathcal{OP}^1_G := \bigsqcup_{S \in \mathcal{A}^1_G} \mathcal{O}^1(G - S). \tag{11}$$

The notation "$\mathcal{OP}$" indicates that $\mathcal{OP}^0_G$ and $\mathcal{OP}^1_G$ are endowed with the poset structure introduced by Definition 2.3.1.

Finally, we consider orientations up to equivalence:

$$\overline{\mathcal{OP}}^0_G := \bigsqcup_{S \in \mathcal{A}^0_G} \overline{\mathcal{O}}^0(G - S) \quad \text{and} \quad \overline{\mathcal{OP}}^1_G := \bigsqcup_{S \in \mathcal{A}^1_G} \overline{\mathcal{O}}^1(G - S). \tag{12}$$

We will define a poset structure on $\overline{\mathcal{OP}}^0_G$ and $\overline{\mathcal{OP}}^1_G$. We fix the following

Convention 2.3.2. Let $S \subset E(G)$ and consider $G(S) = G/(E - S)$. Fix a b-orientation, $\tilde{O}$, on $G(S)$. We have identifications $E(G(S)) = E(\langle S \rangle) = S$, hence we can define a b-orientation, $\tilde{O}^*$, on $\langle S \rangle$ as follows. Let $e \in S$. If e is $\tilde{O}$-bioriented then e gets $\tilde{O}^*$-bioriented. If e is not a loop of $G(S)$ then e gets $\tilde{O}^*$-oriented according to $\tilde{O}$. If e is a loop of $G(S)$ we choose an arbitrary orientation on e. We refer to $\tilde{O}^*$ as a b-orientation *induced by* $\tilde{O}$.

Lemma 2.3.3. *Let $b = 0, 1$ and $S, T \in \mathcal{A}^b_G$ with $T \subset S$. Then for every $O_S \in \mathcal{O}^b(G - S)$ there exists $O_T \in \mathcal{O}^b(G - T)$ such that $O_T \geq O_S$.*

Moreover, if $O_S \sim O'_S$ for some $O'_S \in \mathcal{O}^b(G - S)$, there exists $O'_T \in \mathcal{O}^b(G - T)$ such that $O'_T \geq O'_S$ and $O'_T \sim O_T$.

Proof. We first assume $b = 0$. Up to replacing G with $G - T$, we can assume $T = \emptyset$ and G bridgeless. Hence $G(S)$ is bridgless and we can fix a totally cyclic orientation, $\tilde{O}$, on it. Using 2.3.2, $\tilde{O}$ induces an orientation, $\tilde{O}^*$, on $\langle S \rangle$. Then $O_T := O_S \cup \tilde{O}^*$ is an orientation on G. We claim O_T is totally cyclic. By contradiction, let $F \subset E(G)$ be an O_T-directed cut of G. Then

$F \cap E(G-S) = \emptyset$, as $G-S$ admits no O_S-directed cuts. Therefore $F \subset S$, hence, using Lemma 2.1.2 (c), F is a directed cut of $G(S)$, which is not possible. Finally, if $O_S \sim O'_S$, we construct O'_T using the same orientations $\tilde{O}$ and $\tilde{O}^*$ used to construct O_T. Obviously, $\underline{d}^{O_T} = \underline{d}^{O'_T}$, hence we are done.

The proof for $b = 1$ follows the same steps. Up to replacing G with $G - T$ we can assume $T = \emptyset$. Now $G(S)$ is bridgeless. Indeed, if $e \in S$ is a bridge of $G(S)$ it has to be a bridge of G, and hence $G - S$ is not connected, which is impossible by hypothesis. We can thus fix a totally cyclic 0-orientation, $\tilde{O}$, on $G(S)$, and let $\tilde{O}^*$ be a 0-orientation on $\langle S \rangle$ induced by $\tilde{O}$. Set $O_T := O_S \cup \tilde{O}^*$; arguing as for $b = 0$ one checks that O_T is a rooted 1-orientation on G. The rest of the proof is the same as for $b = 0$. ♣

Proposition 2.3.4. *Let $b = 0, 1$. Then $\overline{\mathcal{OP}}^b_G$ is partially ordered as follows. For $\overline{O}_S$ and $\overline{O}_T$ we set $\overline{O}_S \leq \overline{O}_T$ if $S \leq T$ and if one of the two equivalent conditions below holds.*

(i) There exist $O'_S \in \overline{O}_S$ and $O'_T \in \overline{O}_T$ such that $(O'_T)_{|G-S} = O'_S$.
(ii) For every $O'_S \in \overline{O}_S$ there exists $O'_T \in \overline{O}_T$ such that $(O'_T)_{|G-S} = O'_S$.

Moreover, the forgetful map, $\overline{\mathcal{OP}}^b_G \to \mathcal{A}^b_G$, sending $\overline{O}_S$ to S, is a quotient of poset, and the map sending $\overline{O}_S$ to $g(G-S)$ is a rank on $\overline{\mathcal{OP}}^b_G$.

Proof. Lemma 2.3.3 yields that (i) implies (ii), and the converse is obvious. Lemma 1.3.1 yields that we have a partial order on $\overline{\mathcal{OP}}^b_G$. The two forgetful maps are onto by Fact 1.4.2 and Lemma 1.4.5, and they are quotients by Lemma 2.3.3. The rest of the statement is clear. ♣

Remark 2.3.5. If $\overline{O}_S \leq \overline{O}_T$ then $\underline{d}^{O_S} \leq \underline{d}^{O_T}$, but the converse is not true. See Figure 3, where all vertices have weight 1, $T = \emptyset$ and S consists of the bottom edge on the right of the first graph.

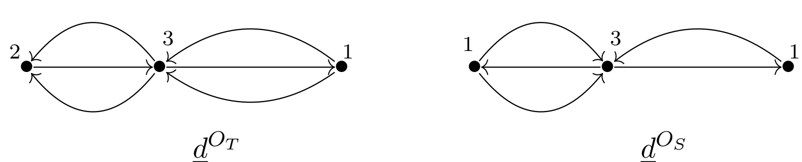

FIGURE 3. $\underline{d}^{O_S} \leq \underline{d}^{O_T}$ but $\overline{O}_S \not\leq \overline{O}_T$

Using Remark 2.2.2 and similarly to it, we have

Remark 2.3.6. For any $S \subset E$ we have $\mathcal{OP}^0_{G-S} \subset \mathcal{OP}^0_G$. If $S = G_{br}$ we have two identifications

$$\mathcal{OP}^0_G = \mathcal{OP}^0_{G-G_{br}} \qquad \overline{\mathcal{OP}}^0_G = \overline{\mathcal{OP}}^0{}_{G-G_{br}}.$$

Remark 2.3.7. Consider the map

$$\overline{\mathcal{OP}}_G^0 \longrightarrow \mathrm{Div}(G); \qquad O_S \mapsto \underline{d}^{O_S}. \tag{13}$$

Its restriction to $\overline{\mathcal{O}}^0(G-S)$ is injective for every $S \in \mathcal{A}_G^0$, yet, the map is not injective. See for example the following picture, where S and T are drawn as the dotted edge.

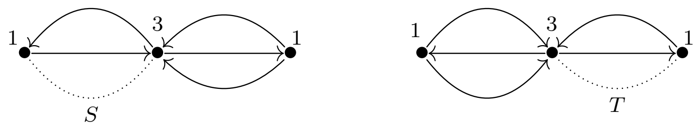

FIGURE 4. $\underline{d}^{O_S} = \underline{d}^{O_T}$ but $\overline{O}_S \not\sim \overline{O}_T$

3. Functoriality under edge contractions

3.1. **Bridgeless and connected subgraphs.** We begin by studying the behaviour of $\mathcal{A}_G^0$ and $\mathcal{A}_G^1$ under edge-contractions. Let GRAPHS be the category whose objects are graphs and whose morphisms are contractions. Let POSETS be the category whose objects are posets and whose morphisms are morphisms of posets. For $b = 0, 1$ we have a map between the objects of these categories,

$$\mathcal{A}^b : \{\text{GRAPHS}\} \longrightarrow \{\text{POSETS}\}; \quad G \mapsto \mathcal{A}_G^b. \tag{14}$$

Using this map, we shall define two functors from GRAPHS to POSETS, a covariant functor, written $(\mathcal{A}^b, \mathcal{A}_*^b)$, and a contravariant functor, written $(\mathcal{A}^b, \mathcal{A}^{b*})$, so that $\mathcal{A}_*^b$ and $\mathcal{A}^{b*}$ are the functor maps defined on morphisms.

Lemma 3.1.1. *Let $b = 0, 1$. For any $\gamma : G \to H = G/S_0$ and any $S \in \mathcal{A}_G^b$ set*

$$\gamma_* S := S \smallsetminus S_0.$$

Then the following hold.

(a) $\gamma_* S \in \mathcal{A}_H^b$.
(b) If $T \in \mathcal{A}_G^b$ is such that $S \leq T$, then $\gamma_* S \leq \gamma_* T$.
(c) Let $\delta : H \to J$ be a contraction of H. Then $(\delta \circ \gamma)_* = \delta_* \circ \gamma_*$.

In other words, the following is a covariant functor

$$(\mathcal{A}^b, \mathcal{A}_*^b) : \text{GRAPHS} \longrightarrow \text{POSETS}$$

where $\mathcal{A}_*^b(\gamma)(S) = \gamma_* S$ for every $\gamma : G \to H$ and $S \in \mathcal{A}_G^b$.

Proof. We have, by Lemma 2.1.2(a)

$$H - \gamma_* S = H - (S \smallsetminus S_0) = \frac{G - (S \smallsetminus S_0)}{S_0}.$$

If $b = 0$ we must check $H - \gamma_* S$ has no bridges. As $G - S$ has no bridges any bridge of $G - (S \smallsetminus S_0)$ must lie in S_0, hence its quotient by S_0 is bridgeless, and we are done. If $b=1$ we must prove $H-\gamma_*S$ is connected. As $G-S$ is connected so is $G-(S\smallsetminus S_0)$, hence so is its quotient. (a) is proved.

(b) and (c) are obvious. ♣

Recall that $\mathcal{A}^0_G$ and $\mathcal{A}^1_G$ are graded posets. Now, the map γ_* does not preserve the gradings. Indeed, let $e\in E(G)\smallsetminus G_{br}$. Set $S=S_0=\{e\}$ so that $\gamma_*S=\emptyset$. We have $g(G-S)=g(G)-1$ and $g(H-\gamma_*S)=g(H)=g(G)$. By contrast, the "pull-back" map, with the associated contravariant functor, defined below does preserve the grading.

Lemma 3.1.2. *Let $b=0,1$. For any $\gamma:G\to H=G/S_0$ and $T\in\mathcal{A}^b_H$ define $\gamma^*T\subset E(G)$ as follows*

$$\gamma^*T:=\begin{cases} T\cup(G-T)_{br} & \text{if } b=0\\ T & \text{if } b=1.\end{cases} \tag{15}$$

Then the following hold.

(a) $\gamma^*T\in\mathcal{A}^b_G$ and $g(H-T)=g(G-\gamma^*T)$.

(b) If $R\in\mathcal{A}^b_H$ is such that $R\leq T$, then $\gamma^*R\leq\gamma^*T$.

(c) Let $\delta:H\to J$ be a contraction of H. Then $(\delta\circ\gamma)^*=\gamma^*\circ\delta^*$.

In short, the following is a grading-preserving, contravariant functor

$$(\mathcal{A}^b,\mathcal{A}^{b*}):\text{GRAPHS}\longrightarrow\text{POSETS}$$

where $\mathcal{A}^{b*}(\gamma)(T)=\gamma^*T$ for every $\gamma:G\to H$ and $T\in\mathcal{A}^b_H$.

Proof. The only nontrivial claim of (a) is the last, i.e. that γ^* preserves the rank. The proof in case $b=0$ trivially gives the proof for $b=1$, so let us concentrate on the former.

$$g(H-T)=g\Big(\frac{G-T}{S_0}\Big)=g(G-T)=g\Big((G-T)-(G-T)_{br}\Big)=g(G-\gamma^*T),$$

where we used Lemma 2.1.2(a) in the first equality, and that contractions and bridge-removals preserve the genus in the second and third equality.

(b) is obvious if $b=1$. Let $R\in\mathcal{A}^0_H$ such that $T\subset R$. We must prove $\gamma^*T\subset\gamma^*R$. It is clearly enough to prove $(G-T)_{br}\subset(G-R)_{br}$.

Since $(H-T)_{br}=\emptyset$ and, by Lemma 2.1.2(a), $H-T=(G-T)/S_0$, we have $(G-T)_{br}\subset S_0$. Hence $(G-T)_{br}\cap R=\emptyset$. Therefore, as $G-R\subset G-T$, we have $(G-T)_{br}\subset(G-R)_{br}$ as wanted.

We omit the direct proof of (c), which follows easily from 3.1.3(c). ♣

We have the following

Proposition 3.1.3. *Let $b=0,1$. Fix a contraction $\gamma:G\to H=G/S_0$. Let $S\in\mathcal{A}^b_G$ and $T\in\mathcal{A}^b_H$. Then*

(a) $\gamma_*\gamma^*T=T$ (equivalently, $\mathcal{A}^b_*(\gamma)\mathcal{A}^{b*}(\gamma)=\mathrm{id}_{\mathcal{A}^b_H}$).

(b) $T\subset\gamma_*S\Leftrightarrow\gamma^*T\subset S$.

(c) γ^*T is the smallest element of $\mathcal{A}^b_G$ whose image under γ_* equals T.

(d) $\mathcal{A}^b_*(\gamma):\mathcal{A}^b_G\to\mathcal{A}^b_H$ is a quotient of posets.

(e) If $S_0 \subset G_{br}$ then $\mathcal{A}^b_*(\gamma): \mathcal{A}^b_G \to \mathcal{A}^b_H$ is an isomorphism.

Proof. (a), (b) and (c) are obvious if $b=1$, so assume $b=0$. We have $\gamma_*\gamma^*T = \gamma_*(T \cup (G-T)_{br}) = \big(T\cup(G-T)_{br}\big) \smallsetminus S_0$. By hypothesis $(H-T)_{br}$ is empty, hence, by Lemma 2.1.2, $(G-T)_{br} \subset S_0$. Therefore

$$\gamma_*\gamma^*T = \big(T\cup(G-T)_{br}\big)\smallsetminus S_0 = T \smallsetminus S_0 = T.$$

(a) is proved. The implication $\Leftarrow$ in (b) follows trivially from (a). For the other implication, the hypothesis is $T \subset S \smallsetminus S_0$, hence $T \subset S$. Since $\gamma^*T = T \cup (G-T)_{br}$ it is enough to prove $(G-T)_{br} \subset S$. We have $G - S \subset G - T$, hence every bridge of $G - T$ is either contained in S, or a bridge of $G-S$. As $G-S$ is bridgeless, we conclude $(G-T)_{br} \subset S$.

(c) follows immediately from (b).

(d). Part (a) implies $\mathcal{A}^b_*(\gamma)$ is surjective and, for any $T, T' \in \mathcal{A}^0_H$, we have $T = \gamma_*\gamma^*T$ and $T' = \gamma_*\gamma^*T'$. By Lemma 3.1.2, if $T \le T'$ then $\gamma^*T \le \gamma^*T'$. Hence we are done.

(e). Notice that $\mathcal{A}^{b*}(\gamma)$ is obviously injective. If S_0 is made of bridges of G then $S_0 \subset S$ for any $S \in \mathcal{A}^0_G$, and $S \cap S_0 = \emptyset$ for any $S \in \mathcal{A}^1_G$. Hence $\mathcal{A}^b_*(\gamma)$ is injective, and we are done. ♣

3.2. **Direct image of divisors and orientations.** In this subsection we will denote by $\gamma : G \to G/S_0 = H$ a contraction, with $S_0 \subset E(G)$. To any contraction γ we associate a map, easily checked to be a surjective group homomorphism, from $\mathrm{Div}(G)$ to $\mathrm{Div}(H)$ mapping $\underline{d}$ to $\gamma_*\underline{d}$ defined as follows

$$(\gamma_*\underline{d})_v := \sum_{z \in \gamma_V^{-1}(v)} \underline{d}_z$$

for any $v \in V(H)$. Let $\delta : H \to J$ be a contraction. Then

$$(\delta\circ\gamma)_*(\underline{d}) = \delta_*(\gamma_*(\underline{d})). \tag{16}$$

In the sequel we shall employ the following notation. Let O be a generalized orientation on G and let $\gamma : G \to H$ be a contraction. As $E(H)$ identifies with a subset of $E(G)$ we can restrict O to $E(H)$, thus defining a generalized orientation on H, denoted by $O_{|H}$.

Let $S \subset E$ and let O_S be a generalized orientation on $G - S$. We have $E(H - \gamma_*S) = E(G - S\cup S_0) \subset E(G-S)$, so we can define (abusing notation again) the following generalized orientation on $H - \gamma_*S$

$$\gamma_*O_S := (O_S)_{|H-\gamma_*S}. \tag{17}$$

As a final piece of notation, to γ and $S \subset E$ we associate the divisor $\underline{c}^{\gamma,S}$ on H such that for any $v \in V(H)$

$$\underline{c}^{\gamma,S}_v := |\{e \in S_0 \cap S : \ \gamma(e) = v\}|. \tag{18}$$

If $S = E(G)$ we write $\underline{c}^{\gamma} = \underline{c}^{\gamma,E(G)}$. Of course, $\underline{c}^{\gamma,S} \ge 0$ and equality holds if and only if $S \cap S_0 = \emptyset$.

Proposition 3.2.1. *Let G be a graph, $S \subset E$, and O_S a b-orientation on $G - S$, with $b = 0, 1$. Let $\gamma : G \to H = G/S_0$ be a contraction such that no edge of S_0 is bioriented. Then $\gamma_* O_S$ is a b-orientation on $H - \gamma_* S$ and the following hold.*

(a) If $O_S \in \mathcal{O}^b(G - S)$ then $\gamma_* O_S \in \mathcal{O}^b(H - \gamma_* S)$.
(b) Let $\delta : H \to J$ be a contraction of H. Then $(\delta \circ \gamma)_* O_S = \delta_* \gamma_* O_S$.
(c) $\gamma_* \underline{d}^{O_S} = \underline{d}^{\gamma_* O_S} - \underline{c}^{\gamma, S}$.
(d) Let O'_S be a b-orientation on $G - S$. If $O'_S \sim O_S$ then $\gamma_* O'_S \sim \gamma_* O_S$.
(e) Let O_T be a b-orientation on $G - T$. If $O_S \leq O_T$ then $\gamma_* O_S \leq \gamma_* O_T$.

Proof. It is clear that $\gamma_* O_S$ is a b-orientation on $H - \gamma_* S$ whose bioriented edge, in case $b = 1$, is the same as that of O_S.

(a). We need to show $\gamma_* O_S$ is totally cyclic if $b = 0$, and rooted if $b = 1$. It suffices to prove that if F is a directed cut of $H - \gamma_* S$ then F is a directed cut of $G - S$. We can assume $S_0 = \{e_0\}$. If $e_0 \notin S$ then $\gamma_* S = S$. By Lemma 2.1.2 (c), every directed cut of $H - S$ is also a directed cut of $G - S$ and we are done. If $e_0 \in S$ set $T = S \smallsetminus \{e_0\}$. We have

$$H - \gamma_* S = H - T = (G - T)/e_0.$$

A directed cut, F, of $H - \gamma_* S$ is thus a directed cut of $G - T$. Now, $G - S \subset G - T$, hence F is a directed cut in $G - S$. (a) is proved.

(b) is trivial.

(c). For any $v \in V(H)$ set $Z_v = \gamma^{-1}(v)$, which is a connected subgraph of G. We have $g(Z_v) = \sum_{z \in V(Z_v)} \big(w(z) - 1\big) + |E(Z_v)| + 1$, hence

$$(\gamma_* \underline{d}^{O_S})_v = \sum_{z \in V(Z_v)} \big(w(z) - 1 + \underline{t}_z^{O_S}\big) = g(Z_v) - 1 - |E(Z_v)| + \sum_{z \in V(Z_v)} \underline{t}_z^{O_S}.$$

Let $t^{O_S}(Z_v)$ be the number of edges with target in Z_v and not contained in it. As every edge of Z_v lies in S_0,

$$|E(Z_v)| = \sum_{z \in V(Z_v)} \underline{t}_z^{O_S} - t^{O_S}(Z_v) + \underline{c}_v^{\gamma, S}.$$

Therefore

$$(\gamma_* \underline{d}^{O_S})_v = g(Z_v) - 1 + t^{O_S}(Z_v) - \underline{c}_v^{\gamma, S}. \tag{19}$$

On the other hand we have

$$(\underline{d}^{\gamma_* O_S})_v = w_{/S_0}(v) - 1 + \underline{t}_v^{\gamma_* O_S} = g(Z_v) - 1 + t^{O_S}(Z_v). \tag{20}$$

Indeed, by definition of contraction, $w_{/S_0}(v) = g(Z_v)$ and, clearly, the number of O_S-incoming edges at Z_v equals the number of $\gamma_* O_S$-incoming edges at v. Comparing (19) and (20) yields (c).

(d). By hypothesis, $\underline{d}^{O_S} = \underline{d}^{O'_S}$, hence $\underline{t}^{O_S} = \underline{t}^{O'_S}$. Hence, by (2), for any $v \in V(H)$ we have $t^{O_S}(Z_v) = t^{O'_S}(Z_v)$ as Z_v does not contain bioriented edges. Combining with (20) we get $\underline{d}^{\gamma_* O_S} = \underline{d}^{\gamma_* O'_S}$, and we are done.

(e). By assumption we have $S \leq T$ and $(O_T)_{|G-S} = O_S$. We obviously have $\gamma_* S \leq \gamma_* T$. Next, as $H - \gamma_* S \subset H - \gamma_* T$

$$(\gamma_* O_T)_{|H-\gamma_* S} = (O_T)_{|H-\gamma_* S} = (O_{T\ |G-S})_{|H-\gamma_* S} = O_{S\ |H-\gamma_* S} = \gamma_* O_S.$$

The proof is complete ♣

Example 3.2.1. In the picture we have $S = S_0 = \{e\}$.

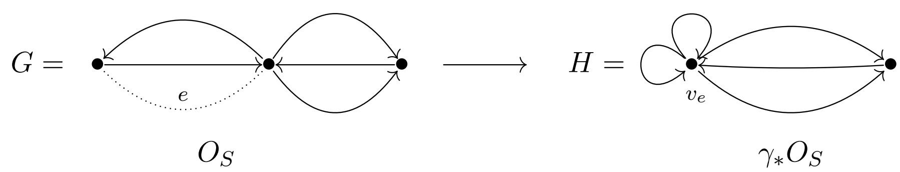

FIGURE 5. Case $S = S_0$

Assume all vertices of G have weight 1, so that v_e has weight 2 in H. We have, ordering the vertices from left to right, $\underline{t}^{O_S} = \underline{d}^{O_S} = (1,2,2)$, $\underline{t}^{\gamma_* O_S} = (3,2)$, $\underline{d}^{\gamma_* O_S} = (4,2)$, and $\gamma_* \underline{d}^{O_S} = (3,2)$. Hence $\underline{d}^{\gamma_* O_S} > \gamma_* \underline{d}^{O_S}$.

From the previous result we derive a few facts.

Proposition 3.2.2. *Fix $\gamma : G \to H = G/S_0$ and let $b = 0, 1$.*

(a) Let $b = 0$. Then we have a morphism of posets

$$\overline{\gamma}_* : \overline{\mathcal{OP}}^0_G \longrightarrow \overline{\mathcal{OP}}^0_H; \quad \overline{O}_S \mapsto \overline{\gamma_* O_S}.$$

(b) Let $b = 1$ and $S_0 \neq E(G)$. Then we have a morphism of posets

$$\overline{\gamma}_* : \overline{\mathcal{OP}}^1_G \longrightarrow \overline{\mathcal{OP}}^1_H; \quad \overline{O}_S \mapsto \overline{\gamma_* O'_S}$$

for any $O'_S \sim O_S$ whose bioriented edge is not in S_0.

(c) Let $b = 0, 1$ and let $\delta : H \to H/T_0$ be a contraction; if $b = 1$ assume $T_0 \neq E(H)$. Then $\overline{(\delta \circ \gamma)}_* = \overline{\delta}_* \circ \overline{\gamma}_*$.

Proof. If $b = 0$ the statement is a trivial consequence of 3.2.1.

For $b = 1$, pick any $\overline{O}_S \in \overline{\mathcal{OP}}^0_G$. By Lemma 1.6.4, there exists $O'_S \sim O_S$ whose bioriented edge does not lie in S_0. Then 3.2.1 yields that $\gamma_* O'_S$ is a well-defined element in $\mathcal{OP}^0_H$, and different choices of O'_S yield equivalent elements in $\mathcal{OP}^0_H$. Hence $\overline{\gamma}_* \overline{O}_S$ is a well defined element of $\overline{\mathcal{OP}}^0_H$. The rest of the proof follows from 3.2.1. ♣

Corollary 3.2.3. *Let $\gamma : G \to H = G/S_0$ be a contraction. Then we have a commutative diagram of posets*

$$\begin{array}{ccc} \mathcal{OP}^0_G & \xrightarrow{\gamma_*} & \mathcal{OP}^0_H \\ \downarrow & & \downarrow \\ \overline{\mathcal{OP}}^0_G & \xrightarrow{\overline{\gamma}_*} & \overline{\mathcal{OP}}^0_H, \end{array}$$

where the vertical arrows are the quotient maps. If $S_0 \subset G_{br}$ then the horizontal arrows are bijections.

Remark 3.2.4. If $S_0 \subset G_{br}$ the lower arrow, $\overline{\gamma}_*$, is a bijection also for $b = 1$. The proof uses a different language so we omit it as we will not need it.

Proof. The commutativity of the diagram follows from Propositions 3.2.1 and 3.2.2. For the remaining part it is enough to prove that γ_* is a bijection.

We have $S_0 \subset S$ for all $S \in \mathcal{A}^0_G$ and we already know we have a bijection $\mathcal{A}^0_G \to \mathcal{A}^0_H$ mapping S to $\gamma_* S$. Now $G - S$ and $H - \gamma_* S$ have exactly the same edges, hence we have an injection $\gamma_* : \mathcal{O}^0(G - S) \hookrightarrow \mathcal{O}^0(H - \gamma_* S)$. We proved that γ_* is injective. Now pick $O_T \in \mathcal{O}(H - T)$. Let $S = \gamma^* T = T \cup (G - T)_{br}$ so that $\gamma_* S = T$. We have $(G - T)_{br} \subset S_0$ hence

$$E(G - \gamma^* T) = E(G) \smallsetminus \big(T \cup (G - T)_{br}\big) \subset E(G) \smallsetminus (T \cup S_0) = E(H - T).$$

Therefore we can restrict O_T to $G - \gamma^* T$, obtaining an orientation easily seen to be totally cyclic and to map to O_T via γ_*. Hence γ_* is surjective. ♣

Corollary 3.2.5. *The inclusion $\iota : G - G_{br} \hookrightarrow G$ and the contraction $\gamma : G \to G/G_{br}$ induce natural isomorphisms (viewed as identifications)*

$$\mathcal{OP}^0_{G - G_{br}} \stackrel{\iota_*}{=} \mathcal{OP}^0_G \stackrel{\gamma_*}{=} \mathcal{OP}^0_{G/G_{br}}$$

and

$$\overline{\mathcal{OP}}^0{}_{G - G_{br}} \stackrel{\overline{\iota}_*}{=} \overline{\mathcal{OP}}^0_G \stackrel{\overline{\gamma}_*}{=} \overline{\mathcal{OP}}^0_{G/G_{br}}.$$

Proof. Combine Remark 2.3.6 with 3.2.3. ♣

3.3. **Quotients of orientation spaces.**

Our next goal is to give a more precise description of the map $\overline{\gamma}_* : \overline{\mathcal{OP}}^b_G \to \overline{\mathcal{OP}}^b_H$ introduced in Proposition 3.2.2.

Theorem 3.3.1. *Let $\gamma : G \to H = G/S_0$ be a contraction with $S_0 \subsetneq E(G)$; let $b = 0, 1$. Then $\overline{\gamma}_* : \overline{\mathcal{OP}}^b_G \to \overline{\mathcal{OP}}^b_H$ is a quotient of posets mapping $\overline{\mathcal{O}}^b(G - \gamma^* T)$ onto $\overline{\mathcal{O}}^b(H - T)$ for every $T \subset E(H)$.*

We begin with the case $b = 0$, for which we have the following.

Proposition 3.3.2. *Let $\gamma : G \to H$ be a contraction. Then the map $\gamma_* : \mathcal{OP}^0_G \to \mathcal{OP}^0_H$ is a quotient of posets mapping $\mathcal{O}^0(G - \gamma^* T)$ onto $\mathcal{O}^0(H - T)$ for every $T \in \mathcal{A}^0_H$.*

Proof. We proceed in three steps. Steps 1 and 2 prove that γ_* is a quotient, Steps 1 and 3 prove that it is onto, as stated.

Step 1. *Suppose $G_{br} = \emptyset$, then the restriction of γ_* to $\mathcal{O}^0(G)$ gives a surjection $\mathcal{O}^0(G) \twoheadrightarrow \mathcal{O}^0(H)$.*

We can assume $S_0 = \{e\}$. As $G_{br} = \emptyset$ we have $H_{br} = \emptyset$. Fix $\tilde{O} \in \mathcal{O}^0(H)$. If e is a loop or if H has only one vertex the statement is trivial, so we exclude this and let $x, y \in V(G)$ be the ends of e. Now, using convention 2.3.2, we have an orientation $\tilde{O}^*$ on $G-e$ induced by $\tilde{O}$. We shall denote $O_e=\tilde{O}^*$ and prove that we can extend O_e to e by a totally cyclic orientation on G, written O. Obviously, we will have $\gamma_* O=\tilde{O}$.

We denote $v_e=\gamma(e)$. Since $\tilde{O}$ is totally cyclic we can fix a cyclically oriented cycle $C\subset H$ containing v_e. Then it is easy to check that the edges of C generate in G a subgraph, $P:=\langle E(C)\rangle$, which is an O_e-directed path having x and y as ends. Of course, P does not contain e, hence $C_e:=P+e$ is a cycle in G. We now orient e in such a way that C_e becomes a cyclically oriented cycle. This gives an orientation, $O\geq O_e$, on G, which we claim is totally cyclic. Indeed, let $F\subset E(G)$ be an O-directed cut. Then $e\in F$ (for otherwise F would be a $\tilde{O}$-directed cut of H). Hence $F\cap E(C_e)\neq\emptyset$, and hence $F\cap E(C_e)$ is a directed cut of the cyclically oriented cycle C_e. This is not possible. Step 1 is proved.

Step 2. *Let $O_T,O_R\in\mathcal{OP}^0_H$ with $O_T\geq O_R$. Then there exist $O_{\gamma^*T},O_{\gamma^*R}\in\mathcal{OP}^0_G$ such that $\gamma_*O_{\gamma^*T}=O_T$, $\gamma_*O_{\gamma^*R}=O_R$ and $O_{\gamma^*T}\geq O_{\gamma^*R}$.*

By hypothesis $T\geq R$, hence $G-\gamma^*T\supset G-\gamma^*R$. We assume $S_0=\{e\}$ and we use the same set-up of Step 1.

We begin by fixing a totally cyclic orientation O_{γ^*R} induced by O_R as described in Step 1. To define O_{γ^*R} on $G-\gamma^*R$ the only choices we make are for non-loop edges corresponding to loops of $H-R$ (the orientation is chosen arbitrarily, see 2.3.2), and for the contracted edge e, if $e\in G-\gamma^*R$ (the orientation is chosen to ensure total cyclicity).

Now, among all orientations induced by O_T on $G-\gamma^*T$ according to 2.3.2, we choose one, written O_{γ^*T}, with the requirement that it agrees with O_{γ^*R} on $G-\gamma^*R$. Hence every non loop-edge corresponding to a loop of $H-R$, is oriented in the same way as in O_{γ^*R} and, more importantly, if the contracted edge e is contained in $G-\gamma^*R$ then it has to be O_{γ^*T}-oriented as in O_{γ^*R}.

Obviously, $O_{\gamma^*T}\geq O_{\gamma^*R}$. We need to check O_{γ^*T} is totally cyclic. By construction, we need to prove it only in case $e\in G-\gamma^*R$ (in the other case the O_{γ^*T}-orientation on e is given as in Step 1, to ensure O_{γ^*T} is totally cyclic). By contradiction, let F be a directed cut of $G-\gamma^*T$. Then $e\in F$, for otherwise F would be a cut of $H-T$. Hence $F\cap E(G-\gamma^*R)$ is not empty. Hence F induces a directed cut of $G-\gamma^*R$, which is not possible.

Step 3. *The restriction of γ_* to $\mathcal{O}^0(G-\gamma^*T)$ is a surjection onto $\mathcal{O}^0(H-T)$.*

We shall reduce this to Step 1, to do which we need to handle the problem that $(G-\gamma^*T)/S_0$ may fail to be equal to $H-T$.

Consider the contraction induced by restricting γ to $G-T$

$$\gamma_{|G-T}:G-T\longrightarrow (G-T)/S_0=H-T$$

(using 2.1.2 (a)). We have $(G-T)_{br}\subset S_0$, hence we can factor $\gamma_{|G-T}$

$$\gamma_{|G-T}:G-T\longrightarrow (G-T)/(G-T)_{br}\xrightarrow{\gamma'}(G-T)/S_0=H-T.$$

Set

$$J := \frac{G-T}{(G-T)_{br}}, \qquad \tilde{J} := \frac{J}{S_0 - (G-T)_{br}} = H - T.$$

As J is bridgeless we can apply the conclusion of Step 1 to the contraction $\gamma' : J \to \tilde{J}$. Hence γ'_* yields a surjection

$$\mathcal{O}^0(J) \longrightarrow \mathcal{O}^0(\tilde{J}) = \mathcal{O}^0(H-T). \tag{21}$$

On the other hand we have natural identifications

$$\mathcal{OP}^0_J = \mathcal{OP}^0_{G-T} = \mathcal{OP}^0_{(G-T)-(G-T)_{br}} = \mathcal{OP}^0_{G-\gamma^* T}$$

using 3.2.5 for the first two equalities. Combining this with (21) we obtain the surjection

$$\mathcal{OP}^0_{G-\gamma^* T} \longrightarrow \mathcal{OP}^0_{H-T},$$

(given by restricting γ_* because of the functoriality of all the constructions involved). Step 3 is proved, and so is the proposition. ♣

Proof of Theorem 3.3.1.

The case $b = 0$ follows from Proposition 3.3.2. Suppose $b = 1$. We argue similarly to the proof of Proposition 3.3.2. We begin by proving that $\overline{\gamma}_*$ induces a surjection $\overline{\mathcal{O}}^1(G) \twoheadrightarrow \overline{\mathcal{O}}^1(H)$. We can assume G and H connected, and $S_0 = \{e\}$; we write $v_e = \gamma(e)$ and $x, y \in V(G)$ for the ends of e ($x \neq y$ otherwise we are done).

Fix $\tilde{O} \in \mathcal{O}^1(H)$, then, by 2.3.2, we have a 1-orientation $O_e = \tilde{O}^*$ on $G - e$ induced by $\tilde{O}$. We shall prove we can extend O_e by a rooted orientation, O, on G, whose bioriented edge is the same as that of $\tilde{O}$, denoted by $\tilde{e}$.

We shall use 1.6.4. As $\tilde{O}$ is rooted, there exists a directed path $\tilde{P} \subset H$ from $\tilde{e}$ to v_e. It is clear that the edges of $\tilde{P}$ span in G a directed path, P, from $\tilde{e}$ to x (say) and not containing e. We set $P_e = P + e$ and orient e so that P_e is a directed path from $\tilde{e}$ to y. Let O be the so-obtained orientation on G; we shall prove it is rooted using 1.6.4 (b).

Let $w \in V(G)$, we must exhibit an O-directed path from $\tilde{e}$ to w. If $w = x, y$ it suffices to take P or P_e. So we can assume w is also a vertex of H different from v_e. Let $\tilde{P}_w \subset H$ be a directed path from $\tilde{e}$ to w. If $\tilde{P}_w$ does not contain v_e then $\tilde{P}_w$ is naturally identified with a directed path in G from $\tilde{e}$ to w and we are done. If v_e is in $\tilde{P}_w$, we can write $\tilde{P}_w = \tilde{Q}_1 + \tilde{Q}_2$ where $\tilde{Q}_1$ is a directed path from $\tilde{e}$ to v_e and $\tilde{Q}_2$ is a directed path from v_e to w not containing v_e. Hence $\tilde{Q}_2$ corresponds to a directed path, Q_2, from either x or y to w. In G, we attach Q_2 to either P (if Q_2 starts at x) or P_e (if Q_2 starts at y) getting a path in G directed from $\tilde{e}$ to w.

We conclude that the restriction of $\overline{\gamma}_*$ to $\overline{\mathcal{O}}^1(G)$ surjects onto $\overline{\mathcal{O}}^1(H)$.

The rest of the proof is the same as for Proposition 3.3.2, Steps 2 and 3, mutatis mutandis. Theorem 3.3.1 is proved. ♣

3.4. **Orientations in genus** g**.** We use notation (14).

Definition 3.4.1. Let $g \geq 2$ and let $b = 0, 1$. Set

$$\mathcal{A}_g^b := \{(G, S) : \ G \in \mathcal{SG}_g, \ S \in \mathcal{A}_G^b\}$$

and endow it with the following partial order relation:

$$(G, S) \leq (H, T) \qquad \text{if} \qquad G \leq H \quad \text{and} \quad S \leq \gamma^* T$$

for some (possibly trivial) contraction $\gamma : G \to H$.

It is easy to check that $\mathcal{A}_g^b$ is indeed a poset inducing, for every $G \in \mathcal{SG}_g$, the poset structure on $\mathcal{A}_G^b$ defined earlier.

Proposition 3.4.2. *Let* $g \geq 2$ *and let* $b = 0, 1$.
(a) The map $\mathcal{A}_g^b \longrightarrow \mathcal{SG}_g$ *mapping* (G, S) *to* G *is a quotient of posets.*
(b) The following is a rank on $\mathcal{A}_g^b$

$$\rho_{\mathcal{A}_g^b} : \mathcal{A}_g^b \longrightarrow \mathbb{N}; \qquad (G, S) \mapsto 3g - 3 - |E(G)| + g(G - S).$$

Proof. The map in (a) is clearly a surjective morphism of posets. To check that it is a quotient, pick $G, H \in \mathcal{SG}_g$ with $G \leq H$. Fix $T \in \mathcal{A}_H^b$, then $\gamma^* T \in \mathcal{A}_G^b$ and, of course, $(G, \gamma^* T) \leq (H, T)$. (a) is proved.

Write $\rho = \rho_{\mathcal{A}_g^b}$. Fix $(G, S), (H, T) \in \mathcal{A}_g^b$ such that (H, T) covers (G, S).

First, suppose $G \neq H$. We claim $S = \gamma^* T$. By contradiction, suppose $S < \gamma^* T$. Then

$$(G, S) < (G, \gamma^* T) < (H, T)$$

a contradiction. Hence $S = \gamma^* T$. But then G covers H in $\mathcal{SG}_g$, indeed if $G < G' < H$ for some $G' \in \mathcal{SG}_g$ then $(G, S) < (G', T') < (H, T)$, where T' is the pull-back of T to G' under the contraction $G' \to H$; this is impossible. As H covers G, Proposition 2.1.3 gives $|E(G)| = |E(H)| + 1$, hence

$$\rho(G, S) - (3g - 3) = g(G - S) - |E(G)| = g(G - \gamma^* T) - |E(H)| - 1.$$

Now, Lemma 3.1.2 (a) yields $g(G - \gamma^* T) = g(H - T)$, hence

$$\rho(H, T) - \rho(G, S) = g(H - T) - |E(H)| - (g(H - T) - |E(H)| - 1) = 1.$$

As wanted. Now, suppose $G = H$. Then $\gamma^* T = T$ and T covers S (for otherwise we would have $(G, S) < (G, S') < (G, T)$ for S' between S and T). By Lemma 2.2.1 we have

$$g(G - S) = g(G - T) - 1 = g(H - T) - 1.$$

Since $|E(G)| = |E(H)|$ we are done. ♣

Definition 3.4.3. Assume $b = 0, 1$. Set

$$\overline{\mathcal{OP}}_g^b := \{(G, \overline{O}_S) : \ G \in \mathcal{SG}_g, \ \overline{O}_S \in \overline{\mathcal{O}}^b(G - S)\}.$$

Let $(H, \overline{O}_T), (G, \overline{O}_S) \in \overline{\mathcal{OP}}_g^b$. We set $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ if $G \leq H$ and if there exists a contraction $\gamma : G \to H$ such that

(1) $S \leq \gamma^* T$, or equivalently (by 3.1.3(b)), $\gamma_* S \leq T$;

(2) $\overline{\gamma}_*\overline{O}_S \leq \overline{O}_T$.

(Of course, (2) implies (1), but we listed both for clarity.)

The definition is illustrated in the picture below. By (1) we have $H - T \supset H - \gamma_* S$. Hence $\overline{O}_T$ can be restricted to $H - \gamma_* S$. By Definition 2.3.1, we require that this restriction be equal to $\overline{\gamma}_*\overline{O}_S$.

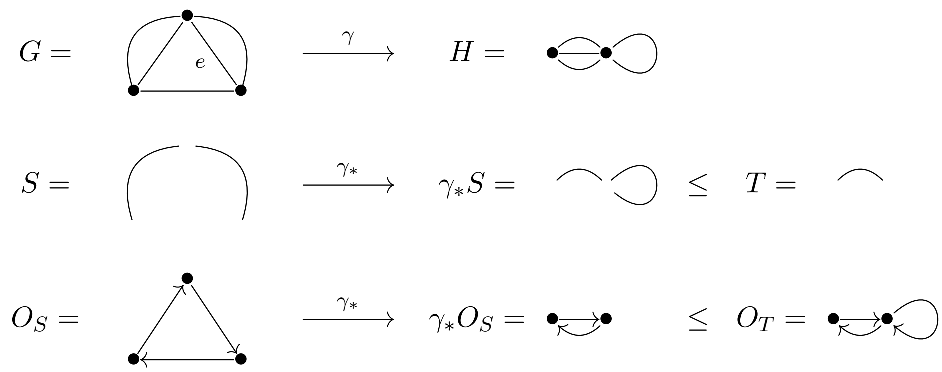

FIGURE 6. An example of the partial order on $\overline{\mathcal{OP}}_g^0$: $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ with $\gamma : G \to H$ contracting e. The orientations O_S, $\gamma_* O_S$ and O_T are living on $G - S$, $H - \gamma_* S$ and $H - T$, respectively.

Proposition 3.4.4. *Assume $b = 0, 1$. Then $\overline{\mathcal{OP}}_g^b$ is a poset such that the inclusion $\overline{\mathcal{OP}}_G^b \hookrightarrow \overline{\mathcal{OP}}_g^b$ is a morphism of posets for every $G \in \mathcal{SG}_g$. Moreover, the following hold.*

(a) The forgetful maps
$$\chi : \overline{\mathcal{OP}}_g^b \longrightarrow \mathcal{SG}_g; \qquad (G, \overline{O}_S) \mapsto G$$
and
$$\tau : \overline{\mathcal{OP}}_g^b \longrightarrow \mathcal{A}_g^b; \qquad (G, \overline{O}_S) \mapsto (G, S)$$
are quotients of posets.

(b) The following is a rank on $\overline{\mathcal{OP}}_g^b$
$$\rho_{\overline{\mathcal{OP}}_g^b} : \overline{\mathcal{OP}}_g^b \longrightarrow \mathbb{N}; \qquad (G, \overline{O}_S) \mapsto 3g - 3 - |E(G)| + g(G - S).$$

Proof. The only property of partial orders which is not an obvious consequence of the definition is transitivity. Suppose $(G, O_S) \leq (H, O_T)$ and $(H, O_T) \leq (J, O_U)$, let $\delta : H \to J$ be a contraction. Then we have the following contraction.
$$\delta \circ \gamma : G \longrightarrow J.$$

Next, by 3.1.1(c) we have $(\delta\circ\gamma)_*=\delta_*\circ\gamma_*$. Hence, as $\gamma_*S\leq T$ and $\delta_*T\leq U$ we have

$$(\delta\circ\gamma)_*S=\delta_*(\gamma_*S)\leq\delta_*(T)\leq U$$

proving the first requirement of Definition 3.4.3. Finally, to show that $\overline{O}_U\geq(\overline{\delta\circ\gamma})_*\overline{O}_S$ we must restrict $\overline{O}_U$ to $J-(\overline{\delta\circ\gamma})_*S$ and check it is equal to $(\overline{\delta\circ\gamma})_*\overline{O}_S$. This is trivial.

(a). The map $\chi:\overline{\mathcal{OP}}_g^b\to\mathcal{SG}_g$ factors as follows

$$\chi:\overline{\mathcal{OP}}_g^b\xrightarrow{\tau}\mathcal{A}_g^b\longrightarrow\mathcal{SG}_g$$

and Proposition 3.4.2 states that $\mathcal{A}_g^b\to\mathcal{SG}_g$ is a quotient. Hence it suffices to prove that τ is a quotient. Now, τ is clearly a surjective morphism of posets. Let $(G,S)\leq(H,T)$ and let $\gamma:G\to H$ be a contraction such that $S\leq\gamma^*T$. Now pick $\overline{O}_S\in\overline{\mathcal{OP}}_G^b$, then $\overline{\gamma}_*\overline{O}_S\in\overline{\mathcal{OP}}_H^b$. By Lemma 2.3.3, there exists $\overline{O}_T\in\overline{\mathcal{OP}}_H^b$ such that $\overline{\gamma}_*\overline{O}_S\leq\overline{O}_T$. As $\tau(G,\overline{O}_S)=(G,S)$ and $\tau(H,\overline{O}_T)=(H,T)$ the proof of (a) is complete.

(b). Notice that $\rho_{\overline{\mathcal{OP}}_g^b}(G,\overline{O}_S)=\rho_{\mathcal{A}_g^b}(G,S)$, the latter being the rank defined in Proposition 3.4.2.

Now, τ is such that if $\tau(G,\overline{O}_S)=\tau(G',\overline{O}_{S'})$ then $G=G'$ and $S=S'$, hence $(G,\overline{O}_S)$ and $(G',\overline{O}_{S'})$ are not comparable. Hence if $(H,\overline{O}_T)$ covers $(G,\overline{O}_S)$ then (H,T) covers (G,S). Therefore $\tau\circ\rho_{\mathcal{A}_g^b}=\rho_{\overline{\mathcal{OP}}_g^b}$ is a rank on $\overline{\mathcal{OP}}_g^b$. The proof is complete. ♣

3.5. **Automorphisms of graphs.** We need to extend the functoriality results proved for edge-contractions in Section 3 to isomorphisms of graphs.

We need the following statement, whose proof is trivial.

Definition/Proposition 3.5.1. *Let $\alpha:G\to G'$ be an isomorphism.*

(1) Let $b=0,1$. Then we have an isomorphism of posets

$$\alpha_*:\mathcal{A}_G^b\longrightarrow\mathcal{A}_{G'}^b;\qquad S\mapsto\alpha_*S=\alpha(S).$$

(2) For $O_S\in\mathcal{O}^b(G-S)$ define $\alpha_*O_S\in\mathcal{O}^b(G'-\alpha_*S)$ so that, for any $e\in E(G)$, the starting half-edge of $\alpha(e)$ is the image under α of the starting half-edge of e. Then we have an isomorphism of posets

$$\alpha_*:\mathcal{OP}_G^b\longrightarrow\mathcal{OP}_{G'}^b;\qquad O_S\mapsto\alpha_*O_S.$$

(3) The isomorphism in (2) descends to an isomorphism of posets

$$\alpha_*:\overline{\mathcal{OP}}_G^b\longrightarrow\overline{\mathcal{OP}}_{G'}^b.$$

Definition 3.5.2. We say that $(H,\overline{O}_T),(G,\overline{O}_S)\in\overline{\mathcal{OP}}_g^b$ are *conjugate*, and write $(H,\overline{O}_T)\equiv(G,\overline{O}_S)$, if $G=H$ and there exists $\alpha\in\operatorname{Aut}(G)$ such that $\alpha_*\overline{O}_T=\overline{O}_S$.

Conjugacy is clearly an equivalence relation on $\overline{\mathcal{OP}}_g^b$. We denote

$$[\overline{\mathcal{OP}}_G^b] = \overline{\mathcal{OP}}_G^b/_{\equiv} \qquad \text{and} \qquad [\overline{\mathcal{OP}}_g^b] = \overline{\mathcal{OP}}_g^b/_{\equiv}$$

and write $[O_S]$ and $(G,[O_S])$ for an element of $[\overline{\mathcal{OP}}_G^b]$ and $[\overline{\mathcal{OP}}_g^b]$ respectively.

Proposition 3.5.3. *Notation as above. We endow $[\overline{\mathcal{OP}}_g^b]$ with the following partial order: $(G,[O_S]) \leq (H,[O_T])$ if there exist $\overline{O}_{T'} \in [O_T]$ and $\overline{O}_{S'} \in [O_S]$ such that $(G,\overline{O}_{S'}) \leq (H,\overline{O}_{T'})$ in $\overline{\mathcal{OP}}_g^b$.*

Then the quotient $\overline{\mathcal{OP}}_g^b \to [\mathcal{OP}_g^b]$ is a quotient of posets, the inclusion $[\mathcal{OP}_G^b] \hookrightarrow [\mathcal{OP}_g^b]$ a morphism of posets, and the forgetful map $[\mathcal{OP}_g^b] \to \mathcal{SG}_g$ is a quotient of posets. Furthermore

$$\rho_{[\mathcal{OP}_g^b]}(G,[O_S]) = 3g-3-|E(G)|+g(G-S)$$

is a rank function.

Proof. Let $\gamma : G \to H$ be a contraction such that $(G,\overline{O}_S) \leq (H,\overline{O}_T)$.

By Lemma 1.3.1, it suffices to prove that for any $\overline{O}_{S'} \equiv \overline{O}_S$ there exists $\overline{O}_{T'} \equiv \overline{O}_T$ such that $\overline{O}_{S'} \leq \overline{O}_{T'}$. We have $\overline{O}_S = \alpha_*\overline{O}_{S'}$ for some $\alpha \in \mathrm{Aut}(G)$. If γ is trivial then $\overline{O}_S \leq \overline{O}_T$ and $\overline{O}_{S'} = \alpha_*^{-1}\overline{O}_S \leq \alpha_*^{-1}\overline{O}_T$, as $\alpha_*^{-1}\overline{O}_T \equiv \overline{O}_T$ we are done.

Suppose γ nontrivial. By hypothesis $(\overline{O}_T)_{|H-\gamma_*S} = \overline{\gamma}_*\overline{O}_S$. Let γ' be the contraction obtained by composing α with γ:

$$\gamma' : G \xrightarrow{\alpha} G \xrightarrow{\gamma} H$$

We have $\overline{O}_{S'} \in \overline{O}(G-\alpha_*^{-1}S)$; set $S' = \alpha_*^{-1}S$. We claim

$$(\overline{O}_T)_{|H-\gamma'_*S'} = \overline{\gamma}'_*\overline{O}_{S'}$$

which of course implies $\overline{O}_{S'} \leq \overline{O}_T$. We have

$$\overline{\gamma}'_*\overline{O}_{S'} = \overline{\gamma}_*\overline{\alpha}_*\overline{O}_{S'} = \overline{\gamma}_*\overline{O}_S = (\overline{O}_T)_{|H-\gamma_*S} = (\overline{O}_T)_{|H-\gamma_*\alpha_*\alpha_*^{-1}S} = (\overline{O}_T)_{|H-\gamma'_*S'}$$

as claimed. Hence $[\mathcal{OP}_g^b]$ is a poset and $\overline{\mathcal{OP}}_g^b \to [\mathcal{OP}_g^b]$ a quotient of posets.

The inclusion $[\mathcal{OP}_G^b] \hookrightarrow [\mathcal{OP}_g^b]$ is obviously a morphism of poset.

By Proposition 3.4.4 the forgetful map $\chi : \overline{\mathcal{OP}}_g^b \to \mathcal{SG}_g$ is a quotient of posets. It is clear that χ factors as follows

$$\chi : \overline{\mathcal{OP}}_g^b \longrightarrow [\mathcal{OP}_g^b] \longrightarrow \mathcal{SG}_g.$$

Since $\overline{\mathcal{OP}}_g^b \to [\mathcal{OP}_g^b]$ is a quotient, $[\mathcal{OP}_g^b] \to \mathcal{SG}_g$ is also a quotient.

The claim about the rank follows from the fact that conjugate elements of $\overline{\mathcal{OP}}_g^b$ have the same rank. ♣

4. Stratifying the compactified universal Picard variety.

4.1. **Dictionary between graphs and nodal curves.** From now on, X will be an algebraic, projective, reduced curve of arithmetic genus g having at most nodes as singularities, and whose (weighted) dual graph is $G = (V, E)$. Recall that V is the set of irreducible components of X and E is the set of nodes of X, with an edge/node joining the two vertices/components on which it lies. The weight of a vertex/component is its geometric genus. We shall use the same symbols for edges and nodes, but we shall write $X = \cup_{v \in V} C_v$ for the irreducible components of X. The genus of X is equal to the genus of G. Sometimes we shall say, somewhat abusively, that "X is dual to G".

Let $S \subset E$ and let $\nu_S : X_S^\nu \to X$ the normalization of X at S. The dual graph of X_S^ν is $G - S$, and $g(X_S^\nu) = g(G - S)$.

We denote by $\hat{X}_S$ the nodal curve obtained by attaching to X_S^ν, for every node $e \in S$, a smooth rational component, named *exceptional component*, to the two branch points in $\nu_S^{-1}(e)$. Of course, X and $\hat{X}_S$ have the same genus.

If X is a stable curve, the curves of the form $\hat{X}_S$ are called *quasistable*. Two exceptional components of a quasistable curve never intersect.

The dual graph of $\hat{X}_S$ will be denoted by $\hat{G}_S$. So, $\hat{G}_S$ is obtained from G by inserting a vertex of weight zero, v_e, in every edge $e \in S$. We refer to v_e as the *exceptional* vertex corresponding to the exceptional component C_{v_e} of $\hat{X}_S$, and we write h_e, j_e for the two edges of $\hat{G}_S$ adjacent to v_e. We have $\hat{X}_S = X_S^\nu \cup (\cup_{e \in S} C_{v_e})$.

The set of non-exceptional vertices of $\hat{G}_S$ is naturally identified with $V(G)$. We denote $\hat{S} = \{h_e, j_e, \ \forall e \in S\} \subset E(\hat{G}_S)$ so that we have a natural inclusion $G - S \subset \hat{G}_S - \hat{S}$.

Let L be a line bundle on X, the multidegree of L is defined as follows: $\underline{\deg}(L) = \{\deg_{C_v} L, \ \ \forall v \in V\}$. We shall identify $\underline{\deg}(L)$ with a divisor on G, whose v-coordinate is $\deg_{C_v} L$, so that we have a map

$$\underline{\deg} : \mathrm{Pic}(X) \longrightarrow \mathrm{Div}(G); \qquad L \mapsto \underline{\deg}(L),$$

where $\mathrm{Pic}(X)$ is the Picard scheme of X. We have

$$\mathrm{Pic}(X) = \bigsqcup_{\underline{d} \in \mathrm{Div}(G)} \mathrm{Pic}^{\underline{d}}(X)$$

where $\mathrm{Pic}^{\underline{d}}(X) := \underline{\deg}^{-1}(\underline{d})$ is the moduli space of line bundles of multidegree $\underline{d}$. Of course, $\mathrm{Pic}^{\underline{d}}(X)$ is isomorphic to the generalized Jacobian, $\mathrm{Pic}^{(0,\ldots,0)}(X)$, of X.

4.2. **Compactified Jacobians of a curve.** Let X be a stable curve of genus g. We introduce $\overline{P}_X^d$, its compactified degree-d Picard variety, or compactified degree-d Jacobian. $\overline{P}_X^d$ is a connected, reduced, possibly reducible, projective variety of pure dimension g whose smooth locus is a disjoint union of (finitely many) g-dimensional varieties parametrizing line bundles of degree d on X.

Several constructions of $\overline{P}^d_X$ exist in the literature, [23], [12], [25], [19], and they have been proved to be isomorphic to one another even though their modular interpretations are different. We here adopt the modular interpretation given in [12], according to which $\overline{P}^d_X$ parametrizes "stably balanced" line bundles of degree d on certain quasistable curves having stable model X. To give the precise description we need some definitions.

Definition 4.2.1. Let $G = \sqcup_{i=1}^c G_i$ have c connected components.

(a) A divisor $\underline{d} \in \mathrm{Div}^g(G)$ is *stable* if $c = 1$ and if for every $Z \subset V(G)$ we have $|\underline{d}_Z| > g(Z) - 1$.

(b) Suppose $c = 1$. A divisor $\underline{d} \in \mathrm{Div}^{g-1}(G)$ is *stable* if for every $Z \subsetneq V(G)$ we have $|\underline{d}_Z| > g(Z) - 1$.
For arbitrary c, a divisor $\underline{d} \in \mathrm{Div}^{g-c}(G)$ is stable if its restriction to every G_i is stable of degree $g(G_i) - 1$.

The somewhat artificial requirement, in (a), that stable divisors of degree g exist only on connected graphs, serves our goals and simplifies terminology.

As we are interested in the cases $d = g$ and $d = g - c$, we shall often unify our statements by writing

$$d = g - c + b \qquad \text{with} \qquad b = 0, 1.$$

If G is a graph of genus g with c connected components, for $b = 0, 1$ we set

$$\Sigma^b(G) := \{\underline{d} \in \mathrm{Div}^{g-c+b}(G) : \ \underline{d} \text{ is stable}\}.$$

Definition 4.2.2. Let X be a stable curve of genus g and G its dual graph. Let $S \subset E(G)$ and $b = 0, 1$. A line bundle $\hat{L}_S \in \mathrm{Pic}^{g-1+b}\hat{X}_S$, and its multidegree $\underline{\deg}\hat{L}_S$, are said to be *stably balanced* if

(a) $\hat{L}_S$ has degree 1 on each exceptional component;

(b) $\underline{\deg}_{X^\nu_S}\hat{L}_S$ is a stable divisor on $G - S$ of degree $g(G-S) - c(G-S) + b$.

Line bundles $\hat{L}_S \in \mathrm{Pic}^{g-1+b}\hat{X}_S$ as above are referred to as "stably balanced line bundles of X". Two stably balanced line bundles, $\hat{L}_S$ and $\hat{M}_T$, of X are *equivalent* if $S = T$ and if their restrictions to X^ν_S are isomorphic.

By definition, $\hat{L}_S$ has total degree $g - 1 + b$ and degree 1 on every exceptional component, hence the restriction of $\hat{L}_S$ to X^ν_S satisfies

$$\deg_{X^\nu_S} \hat{L}_S = g - 1 + b - |S|.$$

Remark 4.2.3. For $S \subset G$ we have

$$\Sigma^b(G - S) = \{\underline{d} \in \mathrm{Div}^{g(G-S)-c(G-S)+b}(G) : \ \underline{d} \text{ is stable.}\},$$

and a divisor in $\Sigma^b(G - S)$ has total degree $g - 1 + b - |S|$. Indeed, let us check that

$$g(G - S) - c(G - S) + b = g - 1 + b - |S|.$$

If $b=0$ we have $g(G-S)=\sum_{i=1}^{c} g(G_i)$ and $g=\sum_{i=1}^{c} g(G_i)+|S|-c+1$ so that

$$\deg_{X_S^\nu} \hat{L}_S = g-1-|S| = \sum_{i=1}^{c} g(G_i) - c = g(G-S)-c,$$

as claimed. If $b=1$, to admit stable divisors the graph $G-S$ must be connected, hence $g(G-S)=g-|S|$. Hence $\deg_{X_S^\nu} \hat{L}_S = g-|S| = g(G-S)$.

From [12] we have

Fact 4.2.4. *Let X be a stable curve of genus g and let $b=0,1$. Then $\overline{P}_X^{g-1+b}$ is a coarse moduli space for equivalence classes of stably balanced line bundles of degree $g-1+b$ of X.*

The above statement uses a different terminology from the original one ([12, Prop. 8.2]) so we need a few words to explain that it is indeed the same. If $b=0$ this is already known (see [16] for example), so let us concentrate on the case $b=1$, i.e. degree g. For degree g the results of [13], such as Thm. 5.9, apply in their strongest form. Moreover, from Sect. 7 (in particular Lemma 7.6), we get that our definition 4.2.2 coincides with the definition of stably balanced line bundles given there.

We need to establish an explicit connection between Definitions 4.2.1 and 4.2.2. For any quasistable curve $\hat{X}_S$ we have a (not unique) contraction

$$\delta : \hat{G}_S \longrightarrow G = \hat{G}_S/S_0,$$

with $S_0=\{j_e,\ \forall e\in S\}$ where j_e is an edge of $\hat{G}_S$ adjacent to the exceptional vertex v_e. Clearly, δ depends on the choice of j_e for each $e\in S$.

Now, let $\underline{d}\in \mathrm{Div}(G)$. We denote by $\widehat{\underline{d}}\in \mathrm{Div}(\hat{G}_S)$ the following divisor

$$\widehat{\underline{d}}_v := \begin{cases} \underline{d}_v & \text{if } v\in V(G) \\ 1 & \text{if } v=v_e,\ e\in S. \end{cases}$$

In short, $\widehat{\underline{d}}$ extends $\underline{d}$ with degree 1 on all exceptional vertices.

We have the following simple fact, for which we use notation (18).

Lemma 4.2.5. *Let X be stable and G its dual graph. Let $\underline{d}_S$ be a stable divisor on $G-S$. Then $\widehat{\underline{d}_S}$ is stably balanced and we have a surjective map*

$$\mathrm{Pic}^{\widehat{\underline{d}_S}}(\hat{X}_S) \longrightarrow \mathrm{Pic}^{\underline{d}_S}(X_S^\nu); \qquad \hat{L}\mapsto \hat{L}_{|X_S^\nu}.$$

For any $\delta : \hat{G}_S \to G$ as above we have $\delta_*\widehat{\underline{d}_S} = \underline{d}_S + \underline{c}^\delta$.

Proof. A divisor on $G-S$ is also a divisor on G, so the first part follows trivially by definition. Next, recall that $\underline{c}_v^\delta$ is the number of edges mapped to $v\in V(G)$ by δ. Hence $\underline{c}_v^\delta=0$ if $\delta^{-1}(v)=v$, and $\underline{c}_v^\delta=1$ otherwise. Since the value of $\widehat{\underline{d}_S}$ on exceptional vertices is 1 we have

$$(\delta_*\widehat{\underline{d}_S})_v = \begin{cases} (\widehat{\underline{d}_S})_v & \text{if } \delta^{-1}(v)=v \\ (\widehat{\underline{d}_S})_v+1 & \text{otherwise.} \end{cases}$$

Hence $(\delta_*\widehat{\underline{d}_S} - \underline{c}^\delta)_v = (\widehat{\underline{d}_S})_v = (\underline{d}_S)_v$. ♣

4.3. **Combinatorics of compactified Jacobians.** We shall now connect to the material of the earlier sections.

Lemma 4.3.1. *Let G be connected of genus g.*

(a) Let $\underline{d} \in \mathrm{Div}^{g-1}(G)$. There exists a 0-orientation, O, s.t. $\underline{d} = \underline{d}^O$ if and only if $|\underline{d}_Z| \geq g(Z) - 1$ for all $Z \subset V$.

(b) For any $\underline{d} \in \Sigma^1(G)$ there exists a 1-orientation, O, on G such that $\underline{d} = \underline{d}^O$.

Proof. Part (a) is well known, for example in graph theory as a version of Hakimi's Theorem (for a modern formulation see [3, Thm 4.8]).

For part (b), fix a vertex v of G. Let $\underline{d}' := \underline{d} - \underline{v}$ so that $\underline{d}' \in \mathrm{Div}^{g-1}(G)$. We have $|\underline{d}'_Z| \geq g(Z) - 1$ for all $Z \subset V$. Indeed, if $v \in Z$, we get $|\underline{d}'_Z| = |\underline{d}_Z| - 1 > g(Z) - 2$; thus $|\underline{d}'_Z| \geq g(Z) - 1$. If $v \notin Z$ we get $|\underline{d}'_Z| = |\underline{d}_Z| > g(Z) - 1$. Thus, by part (a), we can choose a 0-orientation O' on G such that $\underline{d}' = \underline{d}^{O'}$. Since $\underline{d} \in \Sigma^1(G)$, we have $|\underline{d}_{G-v}| > g(G-v) - 1$, hence

$$\underline{d}_v = g - |\underline{d}_{G-v}| < g - g(G-v) + 1 \leq g(v) + \deg v - 1 + 1 = g(v) + \deg v$$

(the "$\leq$" above is a "$=$" iff $G - v$ is connected). On the other hand

$$\underline{d}_v = \underline{d}'_v + 1 = g(v) + \underline{t}_v^{O'}.$$

Therefore $\underline{t}_v^{O'} < \deg v$, hence O' has an edge, e, whose source is v. Biorienting e gives a 1-orientation, O, with $\underline{d} = \underline{d}^O$. ♣

Recall that we denote by $\mathcal{O}^0(G)$ (resp. $\mathcal{O}^1(G)$) the set of totally cyclic (resp. rooted) orientations on G, and by $\mathcal{OP}^0_G$ (resp. $\mathcal{OP}^1_G$) the poset of totally cyclic (resp. rooted) orientations on spanning subgraphs of G. On such sets we defined an equivalence relation whose class-sets are marked by an overline. Finally, recall the notation introduced in 4.2.3.

Lemma 4.3.2. *Let $b = 0, 1$. Let G be a graph of genus g and $S, T \subset E$. Consider the following map*

$$\overline{\mathcal{OP}}^b_G \longrightarrow \mathrm{Div}(G); \qquad \overline{O}_S \mapsto \underline{d}^{O_S}.$$

(a) The map induces a bijection between $\overline{\mathcal{O}}^b(G - S)$ and $\Sigma^b(G - S)$.

(b) If O_S is a b-orientation with $\underline{d}^{O_S} \in \Sigma^b(G - S)$, then $O_S \in \mathcal{O}^b(G - S)$.

Proof. The map is well defined and injective by Definition 1.6.1. Its image lies in $\Sigma^b(G - S)$ by Lemma 1.5.2 in case $b = 0$ and by Lemma 1.5.3 in case $b = 1$. Moreover, its image is the whole of $\Sigma^b(G - S)$ by Lemma 4.3.1. This proves (a), and (b) follows from it. ♣

Remark 4.3.3. By 4.2.4 the points of $\overline{P}_X^{g-1+b}$ correspond to equivalence classes of stably balanced line bundles, and two such line bundles are equivalent if they are defined on the same $\hat{X}_S$ and if their restrictions to X_S^ν are isomorphic. Denote by $\underline{d}_S$ a stable divisor of $G - S$ and by $P_X^{\underline{d}_S} \subset \overline{P}_X^{g-1+b}$

the set of equivalence classes of stably balanced line bundles on $\hat{X}_S$ whose restriction to X_S^ν has multidegree $\underline{d}_S$. By Lemma 4.3.2, there exists a unique $\overline{O}_S \in \overline{\mathcal{O}}^b(G-S)$ such that $\underline{d}_S = \underline{d}^{O_S}$, and every stable divisor on $G-S$ is obtained in this way. Therefore we define, for any $\overline{O}_S \in \mathcal{O}^b(G-S)$

$$P_X^{O_S} := P_X^{\underline{d}_S}. \tag{22}$$

Theorem 4.3.4. *Let X be a stable curve of genus g and G its dual graph, let $b=0,1$. Then the following is a graded stratification of $\overline{P}_X^{g-1+b}$ by $\overline{\mathcal{OP}}_G^b$*

$$\overline{P}_X^{g-1+b} = \bigsqcup_{\overline{O}_S \in \overline{\mathcal{OP}}^b(G)} P_X^{O_S}, \tag{23}$$

and we have natural isomorphisms for every $\overline{O}_S \in \overline{\mathcal{OP}}_G^b$

$$P_X^{O_S} \cong \mathrm{Pic}^{\underline{d}^{O_S}}(X_S^\nu). \tag{24}$$

Proof. The case $b=0$ follows directly from results of [16]. Our proof in case $b=1$ also works for $b=0$, so we include it for completeness.

As in Remark 4.3.3, we denote by $P_X^{\underline{d}_S} \subset \overline{P}_X^{g-1+b}$ the set of equivalence classes of stably balanced line bundles on $\hat{X}_S$ whose restriction to X_S^ν has degree $\underline{d}_S$, for $\underline{d}_S$ a stable divisor of $G-S$. By Fact 4.2.4 we have

$$\overline{P}_X^{g-1+b} = \bigsqcup_{\substack{S \subset E \\ \underline{d}_S \in \Sigma^b(G-S)}} P_X^{\underline{d}_S}. \tag{25}$$

Now, as noted above, we have $P_X^{O_S} = P_X^{\underline{d}_S}$ for a unique class $\overline{O}_S \in \mathcal{O}^b(G-S)$ such that $\underline{d}_S = \underline{d}^{O_S}$. Moreover, by Lemma 4.3.2 every $\underline{d}_S \in \Sigma^b(G-S)$ is obtained in this way, for every $S \subset E$. Hence (25) yields (23).

Also, we obviously have $P_X^{\underline{d}_S} \cong \mathrm{Pic}^{\underline{d}_S}(X_S^\nu)$, from which (24) follows.

Next, recalling Definition 1.3.2, we prove the following

$$P_X^{O_S} \subset \overline{P_X^{O_T}} \quad \Leftrightarrow \quad \overline{O}_S \leq \overline{O}_T.$$

By [12, Prop. 5.1] (revised using graphs) we have $P_X^{\underline{d}_S} \cap \overline{P_X^{\underline{d}_T}} \neq \emptyset$ if and only if $P_X^{\underline{d}_S} \subset \overline{P_X^{\underline{d}_T}}$. Moreover, $P_X^{\underline{d}_S} \subset \overline{P_X^{\underline{d}_T}}$ if and only if $T \subset S$ and the edges in $S \smallsetminus T$ can be oriented so that, denoting by t_v the number of edges in $S \smallsetminus T$ with target a vertex v, we have

$$(\underline{d}_T)_v = (\underline{d}_S)_v + t_v.$$

Assume $P_X^{\underline{d}_S} \subset \overline{P_X^{\underline{d}_T}}$ and denote by O'_T the orientation on $G-T$ which extends O_S to $S \smallsetminus T$ by the orientation we just defined (where $\overline{O}_S \in \overline{\mathcal{O}}(G-S)$ is such that $\underline{d}^{O_S} = \underline{d}_S$, by the previous part). Of course $O_S \leq O'_T$ and, as $\underline{d}^{O_T} = \underline{d}_T$ for some $\overline{O}_T \in \overline{\mathcal{O}}(G-T)$, we have

$$\underline{d}^{O_T} = \underline{d}_T = \underline{d}^{O'_T},$$

hence $O'_T \sim O_T$. We conclude that $\overline{O}_S \leq \overline{O}_T$. The converse is obvious.

Finally, we need to show that the stratification (25) is graded. Recall that the generalized Jacobian of X_S^ν is an irreducible variety of dimension $g(G-S)$, hence so is $\mathrm{Pic}^{\underline{d}_S}(X_S^\nu)$, hence so is $P_X^{O_S}$. By Proposition 2.3.4, the map $\overline{O}_S \mapsto g(G-S)$ is a rank $\overline{\mathcal{OP}}_G^b$, hence we are done. ♣

4.4. **Specialization of polarized curves.** We shall be interested in (flat, projective) families of curves over a one-dimensional nonsingular base, specializing to a given curve X. Up to restricting the base we can assume that away from X the family is topologically trivial, i.e. that every fiber different from X has the same dual graph of some fixed curve Y. We shall refer to such a family as a *specialization from Y to X*. Since X has only nodes as singularities, the same holds for Y.

Suppose our curves X and Y are "polarized", i.e. endowed with a line bundle, $L \in \mathrm{Pic}(X)$ and $M \in \mathrm{Pic}(Y)$. We say that (Y, M) specializes to (X, L) if there is a specialization of Y to X under which M specializes to L.

Remark 4.4.1. Let us define all of the above more rigorously. The family under which Y specializes to X is a projective morphism $f : \mathcal{X} \to B$ where B is a smooth, connected, one-dimensional variety with a point b_0 such that $f^{-1}(b_0) \cong X$, and the restriction of f away from b_0 is locally trivial, moreover $f^{-1}(b) \cong Y$ for some $b \neq b_0$. As an étale base change of f determines again a specialization of Y to X we are free to replace f by an étale base change. For the polarized version, to say that M specializes to L means that $\mathcal{X}$ is endowed with a line bundle whose restriction to Y is M and whose restriction to X is L.

Proposition 4.4.2. *Let X and Y be two nodal curves and G and H their respective dual graphs. Let $L \in \mathrm{Pic}(X)$ and $M \in \mathrm{Pic}(Y)$ such that (Y, M) specializes to (X, L). Then there exists a contraction $\gamma : G \to H$ such that*

$$\gamma_* \underline{\deg}(L) = \underline{\deg}(M).$$

In the opposite direction, we have the following.

Proposition 4.4.3. *Let $\gamma : G \to H$ be a contraction between two graphs. Then for any curve X dual to G and for any $L \in \mathrm{Pic}(X)$ there exist a curve Y dual to H and a line bundle $M \in \mathrm{Pic}(Y)$ such that $\gamma_* \underline{\deg}(L) = \underline{\deg}(M)$ and such that (Y, M) specializes to (X, L).*

Proof. We prove Propositions 4.4.2 and 4.4.3 together as their proofs are closely related. They extend [14, Thm 4.7 (2)] to polarized nodal curves.

To prove Proposition 4.4.2, assume (Y, M) specializes to (X, L). Under such a specialization every node of Y specializes to a node of X and different nodes specialize to different nodes. Hence we partition $E(G) = S_0 \sqcup T$ so that S_0 is the set of nodes of X which are not specializations of nodes of Y. We let $\gamma : G \to G/S_0$, and, arguing as for [14, Thm 4.7], we have $G/S_0 = H$.

For any vertex $w \in V(H)$ we write $D_w \subset Y$ for the irreducible component corresponding to w. As shown in loc. cit., the specialization from Y to X

induces a specialization of D_w to $\cup_{\gamma(v)=w}C_v$ (as a subcurve of X). Now, M specializes to L and hence $M_{|D_w}$ specializes to the restriction of L to $\cup_{\gamma(v)=w}C_v$. Therefore

$$\underline{\deg}(M)_w = \deg_{D_w} M = \deg L_{|\cup_{\gamma(v)=w}C_v} = \sum_{\gamma(v)=w} \underline{\deg}(L)_v = \gamma_*\underline{\deg}(L)_w.$$

This proves Proposition 4.4.2.

For Proposition 4.4.3, let $\gamma : G \to G/S_0 = H$ be a contraction, for some $S_0 \in E(G)$; write $E(G) = S_0 \sqcup T$ so that T is identified with $E(H)$. Let X be a curve dual to G and let X^ν_T be its normalization at T, so that $G - T$ is the dual graph of X^ν_T. The curve X^ν_T is endowed with $|T|$ pairs of marked smooth points, namely the branches over the nodes in T. Observe that the connected components of X^ν_T are in bijection with the connected components of $H - T$, and hence with the vertices of H. We can therefore decompose X^ν_T as follows

$$X^\nu_T = \sqcup_{w\in V(H)} Z_w$$

with Z_w a connected nodal curve whose genus, $g(Z_w)$, is equal to the weight of w as a vertex in H. Therefore we can find a family of smooth curves of genus $g(Z_w)$ specializing to Z_w, i.e. we have a smooth curve, W_w, specializing to Z_w. Considering the union for $w \in V(H)$ we get a specialization of $\sqcup_{w\in V(H)} W_w$ to $\sqcup_{w\in V(H)} Z_w = X^\nu_T$

Now, up to étale cover, such a specialization can be endowed with $|T|$ pairs of sections specializing to the $|T|$ pairs of branch points of X^ν_T. By gluing together each such pair of sections we get a specialization to our X from a curve, Y, whose dual graph is H.

Clearly, the contraction $\gamma : G \to H$ corresponds to this specialization from Y to X.

Now, using the notation of Remark 4.4.1, let $f : \mathcal{X} \to B$ be a family under which Y specializes to X, and consider its relative Picard scheme, $\mathrm{Pic}_{\mathcal{X}/B} \to B$. Its fiber over b_0 is $\mathrm{Pic}(X)$ and its fiber over b is $\mathrm{Pic}(Y)$. Write $\underline{d} = \deg L$; we claim that, in the relative Picard scheme, $\mathrm{Pic}^{\underline{d}}(X)$ is the specialization of $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$. Indeed, $\mathrm{Pic}^{\underline{d}}(X)$ must be the specialization of some connected component of $\mathrm{Pic}(Y)$ (even if this Picard scheme were not separated, every connected component of its fiber over b_0 is the specialization of some connected component of the general fiber), and this component is necessarily $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$ by the same computation we used to prove Proposition 4.4.2.

Now, as $\mathrm{Pic}^{\underline{d}}(X)$ is the specialization of $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$, any $L \in \mathrm{Pic}^{\underline{d}}(X)$ is the specialization of some $M \in \mathrm{Pic}^{\gamma_*\underline{d}}(Y)$, and we are done. ♣

4.5. **Compactified universal Jacobians.** We fix $d \in \mathbb{Z}$. In this paper we are interested in $d = g$ and $d = g - 1$, so we shall restrict to these two cases even though some of the preliminary results quoted in this subsection hold more generally for every d. We also assume $b = 0, 1$ so that $d = g - 1 + b$.

We let $\overline{M}_g$ be the moduli space of stable curves of genus $g \geq 2$, an irreducible projective variety.

Fact 4.5.1. *The following is a graded stratification of $\overline{M}_g$ by $\mathcal{SG}_g$:*

$$\overline{M}_g = \bigsqcup_{G \in \mathcal{SG}_g} M_G$$

where M_G parametrises curves having G as dual graph.

Indeed, the function $G \mapsto \dim M_G$ equals the rank on $\mathcal{SG}_g$ defined in Proposition 2.1.3.

Now, from [12] we introduce, for every $d \in \mathbb{Z}$, the compactified universal degree-d Jacobian

$$\psi_{g,d} : \overline{P}^d_g \longrightarrow \overline{M}_g.$$

We sometimes write $\psi = \psi_{g,d}$ for simplicity. Recall that $\overline{P}^d_g$ is the GIT quotient of a Hilbert scheme, and that ψ is a projective morphism whose fiber over $X \in \overline{M}_g$ is isomorphic to $\overline{P}^d_X/\mathrm{Aut}(X)$. Set

$$P^d_G := \psi^{-1}_{g,d}(M_G).$$

Pick a stable curve $X \in M_G$. Then we have a canonical map

$$\mu_X : \overline{P}^d_X \to P^d_G. \tag{26}$$

Corollary 4.5.2. *Let $G, H \in \mathcal{SG}_g$. Then*

$$P^d_G \subset \overline{P^d_H} \qquad \textit{if and only if} \qquad H \geq G.$$

Proof. It suffices to use Fact 4.5.1 and that $\psi : \overline{P}^d_g \to \overline{M}_g$ is projective. ♣

In the next remark we recall the basic moduli properties of $\overline{P}^d_g$.

Remark 4.5.3. Let $f : \mathcal{X} \to B$ be a family of quasistable curves of genus g and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ whose restriction, L_b, to every fiber over $b \in B$ is stably balanced of degree d (in the sense of Definition 4.2.2). Then there is a *moduli morphism*, $\mu_{\mathcal{L}} : B \to \overline{P}^d_g$ such that the image of $b \in B$ is the equivalence class of L_b.

Consider the case of a fixed curve rather than a family. So $B = \{b\}$ and $\mathcal{X} = \hat{X}_S$ is a fixed quasistable curve. Let $L, L' \in \mathrm{Pic}(\hat{X}_S)$ be stably balanced. If the restriction of L and L' away from the exceptional components are isomorphic (i.e. if $L_{X^\nu_S} \cong L'_{X^\nu_S}$) then $\mu_L(b) = \mu_{L'}(b)$.

Fix G and $S \subset E(G)$. Let $f : \mathcal{X} \to B$ be a family of stable curves all having dual graph identified with G, hence S can be identified with $|S|$ (set-theoretic) sections of f corresponding to the nodes in S. Denote by $f_S : \mathcal{X}_S \to B$ the desingularization at these sections, so that every fiber of f_S has dual graph $G - S$. For every $e \in S$ we have a pair of sections $(\sigma_{h^+_e}, \sigma_{h^-_e})$ of f_S (where h^+_e, h^-_e are the half-edges of e). We glue to $\mathcal{X}_S$ a copy of $\mathbb{P}^1 \times B$ by identifying its 0 and ∞ section to $\sigma_{h^+_e}$ and $\sigma_{h^-_e}$. By repeating this for every $e \in S$ we obtain a family of quasistable curves $\widehat{f_S} : \hat{\mathcal{X}}_S \to B$ with dual graph $\hat{G}_S$.

Let now $\underline{d}_S$ be a divisor on $G - S$, denote by $\mathrm{Pic}^{\underline{d}_S}_{f_S}$ the corresponding connected component of the Picard scheme Pic_{f_S}. Similarly, denote by $\mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}}$ the connected component of $\mathrm{Pic}_{\widehat{f_S}}$ corresponding to $\widehat{\underline{d}_S} \in \mathrm{Div}(\hat{G}_S)$.

Now, using the notation in Lemma 4.2.5, we have

Lemma 4.5.4. *Let $f : \mathcal{X} \to B$ be as above. Let $b = 0, 1$ and $\underline{d}_S \in \Sigma^b(G-S)$. Then there exist a moduli morphism $\mu_{\widehat{\underline{d}_S}} : \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}} \to \overline{P}^d_g$ and a morphism $\mu_{\underline{d}_S} : \mathrm{Pic}^{\underline{d}_S}_{f_S} \to P^d_G$ such that*

$$\mu_{\widehat{\underline{d}_S}} : \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}} \xrightarrow{\varphi} \mathrm{Pic}^{\underline{d}_S}_{f_S} \xrightarrow{\mu_{\underline{d}_S}} P^d_G,$$

where φ is given by restriction away from the exceptional components.

Proof. We have a polarized family of quasistable curves

$$\mathcal{L} \longrightarrow \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}} \times_B \hat{\mathcal{X}}_S \longrightarrow \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}}$$

where $\mathcal{L}$ is the tautological (Poincaré) line bundle, which, by hypothesis, is relatively stably balanced. By Remark 4.5.3 we have a moduli morphism $\mu_{\mathcal{L}} : \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}} \to \overline{P}^d_g$. Set $\mu_{\widehat{\underline{d}_S}} = \mu_{\mathcal{L}}$, it is clear that its image lies in P^d_G.

We let $\varphi : \mathrm{Pic}^{\widehat{\underline{d}_S}}_{\widehat{f_S}} \to \mathrm{Pic}^{\underline{d}_S}_{f_S}$ be the map given by restricting a line bundle away from the exceptional components, so φ is the analog of the map used in Lemma 4.2.5. Now, as we said in Remark 4.5.3, if two line bundles have the same image under φ (i.e. their restriction away from the exceptional components are isomorphic) they also have the same image under $\mu_{\mathcal{L}}$. By applying a standard argument using that $\overline{P}^d_g$ is a GIT quotient, we conclude that there exists a map $\mu_{\underline{d}_S} : \mathrm{Pic}^{\underline{d}_S}_{f_S} \to P^d_G$ such that $\mu_{\widehat{\underline{d}_S}}$ factors a stated. ♣

4.6. The strata of $\overline{P}^d_g$.

Our goal is to find a stratification of $\overline{P}^d_g$ compatible with the one of $\overline{M}_g$. By [1, Prop. 3.4.1], the stratum M_G has the following expression

$$\widetilde{M}_G := \Pi_{v \in V} M_{w(v),\deg(v)} \xrightarrow{\pi} M_G = \widetilde{M}_G / \mathrm{Aut}(G) \tag{27}$$

where $M_{w(v),\deg(v)}$ is the moduli space of smooth curves of genus $w(v)$ with $\deg(v)$ marked points representing the branches/half-edges over the nodes/edges.

More generally, with the notation of [1, Subsection 2.1], for every $S \subset E$, consider the $2|S|$-marked graph **G-S**, whose underlying (unmarked) graph is $G - S$, and whose $2|S|$-marking is given by the half-edges corresponding to S. Then **G-S** is stable as marked graph and we have a moduli space, $M_{\mathbf{G\text{-}S}}$, of stable curves with $2|S|$ marked points and dual graph **G-S**. In particular, if $S = E$ then $\widetilde{M}_G = M_{\mathbf{G\text{-}E}}$ and the map π above factors:

$$\pi : \widetilde{M}_G \xrightarrow{\pi_S} M_{\mathbf{G\text{-}S}} = \widetilde{M}_G/\mathrm{Aut}(\mathbf{G\text{-}S}) \longrightarrow M_G.$$

For our goal we need a universal curve over $\widetilde{M}_G$, but it is well known that this may fail to exist over some $M_{w(v),\deg(v)}$. However (see [4] for example), a universal curve exists over some finite cover of it. We choose a finite cover $M'_{w(v),\deg(v)} \to M_{w(v),\deg(v)}$ of large enough order (the same for all pairs $w(v), \deg(v)$) so that we have a universal curve over each $M'_{w(v),\deg(v)}$. We let $\widetilde{M}'_G$ be the product of the $M'_{w(v),\deg(v)}$ for $v \in V$ so that, composing with (27), we have a finite map $\pi' : \widetilde{M}'_G \to M_G$. The action of $\mathrm{Aut}(G)$ on M_G lifts naturally to an action on $\widetilde{M}'_G$

We denote by $\mathcal{C}_{w(v),\deg(v)} \to M'_{w(v),\deg(v)}$ the universal curve, and we have the following family

$$\widetilde{\mathcal{X}}_G := \sqcup_{v\in V}\mathcal{C}_{w(v),\deg(v)} \longrightarrow \widetilde{M}'_G,$$

together with $2|E|$ sections, $\sigma_h : \widetilde{M}'_G \to \widetilde{\mathcal{X}}_G$, indexed by the half-edges of G.

Fix $S \subset E$. Let $\widetilde{\mathcal{X}}_G \to \mathcal{X}^S_G$ be the gluing along pairs $(\sigma_{h_e^+}, \sigma_{h_e^-})$ for every $e \notin S$. Then $\mathcal{X}^S_G$ is a family over the space $Z^S_G := \widetilde{M}'_G/\mathrm{Aut}(\mathbf{G\text{-}S})$. Let

$$f_S : \mathcal{X}^S_G \to Z^S_G$$

be this family of curves, all of whose fibers have dual graph $G - S$. Since Z^S_G is a finite cover of $M_{\mathbf{G\text{-}S}}$, the map π' factors through finite maps:

$$\pi' : \widetilde{M}'_G \longrightarrow Z^S_G \longrightarrow M_G.$$

Fixing a stable multidegree $\underline{d}^{O_S}$ on $G-S$, by Lemma 4.5.4 we get a morphism

$$\mu_{O_S} : \mathrm{Pic}^{\underline{d}^{O_S}}_{f_S} \longrightarrow P^d_G.$$

We define $P^{O_S}_G$ to be the image of this map.

Lemma 4.6.1. *Let $G \in \mathcal{SG}_g$ and $O_S \in \mathcal{OP}^b_G$ with $b = 0, 1$. Then $P^{O_S}_G$ is quasiprojective, irreducible of dimension $3g - 3 - |E(G)| + g(G - S)$.*

If $O_T \equiv O_S$ for some $O_T \in \mathcal{OP}^b_G$, then $P^{O_S}_G = P^{O_T}_G$.

Proof. The morphism μ_{O_S} is finite because so is the morphism $Z^S_G \to M_G$. Moreover μ_{O_S} exhibits $P^{O_S}_G$ as the image of an irreducible quasiprojective variety of dimension

$$\dim \mathrm{Pic}^{\underline{d}^{O_S}}_{f_S} = \dim Z^S_G + g(G - S) = 3g - 3 - |E(G)| + g(G - S)$$

(as $\dim Z^S_G = \dim M_G$). So the first part of the statement is proved.

Now suppose $O_T \equiv O_S$, then $O_T = \alpha_* O_S$ for some $\alpha \in \mathrm{Aut}(G)$. Hence $\alpha_*\underline{d}^{O_S} = \underline{d}^{O_T}$, and α induces an isomorphism between Z^S_G and Z^T_G, a corresponding isomorphism between $\mathcal{X}^S_G$ and $\mathcal{X}^T_G$, and an isomorphism $\mathrm{Pic}^{\underline{d}^{O_S}}_{f_S} \cong$

$\mathrm{Pic}^{d^{O_T}}_{f_T}$. The latter induces an isomorphism between the respective Poincaré line bundles. Therefore the images of μ_{O_S} and μ_{O_T} in P^d_G get identified; see the second part of Remark 4.5.3. ♣

We define for any $[O_S] \in [\overline{\mathcal{OP}}^b_G]$

$$P^{[O_S]}_G := P^{O_S}_G,$$

by Lemma 4.6.1, this definition does not depend on the choice of the representative in $[O_S]$.

4.7. Stratifications of universal Jacobians in degree $g-1$ and g.

Theorem 4.7.1. *The following is a graded stratification of* $\overline{P}^{g-1+b}_g$ *by* $[\overline{\mathcal{OP}}^b_g]$

$$\overline{P}^{g-1+b}_g = \bigsqcup_{(G,[O_S])\in[\overline{\mathcal{OP}}^b_g]} P^{[O_S]}_G.$$

Proof. We have

$$\overline{P}^{g-1+b}_g = \bigsqcup_{G\in\mathcal{SG}_g}\left(\bigsqcup_{[O_S]\in[\overline{\mathcal{OP}}^b_G]} P^{[O_S]}_G\right) = \bigsqcup_{(G,[O_S])\in[\overline{\mathcal{OP}}^b_g]} P^{[O_S]}_G.$$

Indeed, the only thing that might not be clear is that the union is disjoint. Suppose two different strata $P^{[O_S]}_G$ and $P^{[O_T]}_G$ intersect and let us show they coincide. Let $X \in \overline{M}_g$ be such that $P^{[O_S]}_G \cap P^{[O_T]}_G \cap \psi^{-1}(X)$ is not empty. Recall that the strata $P^{O_S}_X$ and $P^{O_T}_X$ are disjoint in $\overline{P}^{g-1+b}_X$. Since automorphisms of X obviously map strata to strata in $\overline{P}^{g-1+b}_X$, the images via μ_X (see (26)) of $P^{O_S}_X$ and $P^{O_T}_X$ are no longer disjoint if and only if there is an automorphism α_X of X identifying them. Then one easily checks that the induced automorphism α on G maps $\overline{O}_S$ to $\overline{O}_T$ in $\overline{\mathcal{OP}}^b_G$. Hence $[O_S] = [O_T]$.

Lemma 4.6.1 gives that $P^{[O_S]}_G$ is quasiprojective, irreducible, of dimension

$$\dim P^{[O_S]}_G = \dim P^{O_S}_G = 3g-3-|E(G)|+g(G-S),$$

and by Proposition 3.5.3 the right hand side is a rank on $[\overline{\mathcal{OP}}^b_g]$.

To complete the proof we must show that we have a stratification in the sense of Definition 1.3.2. We will do that in the next two propositions.

Proposition 4.7.2. *Let* $(G,\overline{O}_S),(H,\overline{O}_T) \in \overline{\mathcal{OP}}^b_g$. *If* $(G,\overline{O}_S) \leq (H,\overline{O}_T)$ *then* $P^{[O_S]}_G \subset \overline{P^{[O_T]}_H}$.

Proof. Consider $\psi : \overline{P}^{g-1+b}_g \to \overline{M}_g$. For a fixed $X \in M_G$ we have

$$P^{[O_S]}_G \cap \psi^{-1}([X]) = \bigcup_{\overline{O}_{S'}\equiv\overline{O}_S} \mu_X(P^{O_{S'}}_X),$$

with μ_X defined in (26). It suffices to show that for every such X and every $\overline{O}_{S'} \equiv \overline{O}_S$, every point in $P_X^{O_{S'}}$ is a specialization of line bundles parametrized by $P_H^{O_T}$, so that that $\mu_X(P_X^{O_{S'}}) \subset \overline{P_H^{O_T}}$.

By the proof of Proposition 3.5.3 we have that for any $\overline{O}_{S'} \equiv \overline{O}_S$ there is $\overline{O}_{T'} \equiv \overline{O}_T$ with $\overline{O}_{S'} \leq \overline{O}_{T'}$. Since $P_G^{[O_S]} = P_G^{[O_{S'}]}$ and $P_G^{[O_T]} = P_G^{[O_{T'}]}$ we can replace $\overline{O}_{S'}$ by $\overline{O}_S$ without loss of generality.

By hypothesis, there exists a curve Y dual to H which specializes to X; let $\gamma : G \to H$ be the associated contraction. Under the corresponding specialization of compactified Picard varieties, $\overline{P}_Y^{g-1+b}$ specializes to $\overline{P}_X^{g-1+b}$.

Now, $\overline{\gamma}_*\overline{O}_S \in \overline{\mathcal{OP}}_H^b$, hence $\underline{d}^{\gamma_* O_S}$ is stable, and hence, by 4.2.5, $P_Y^{\gamma_* O_S}$ parametrizes stably balanced line bundles on $\hat{Y}_R$ of degree $\widehat{\underline{d}^{\gamma_* O_S}}$, where $R = \gamma_* S$. We begin by showing that $P_Y^{\gamma_* O_S}$ specializes to $P_X^{O_S}$. To the contraction γ we associate the contraction

$$\hat{\gamma} : \hat{G}_S \longrightarrow \hat{H}_R = \hat{G}_S/\hat{S}_0$$

(where $\hat{S}_0 = \delta_E^{-1}(S_0)$ for $\delta : \hat{G}_S \to G$). Now, with the notation introduced before 4.2.5, consider $\widehat{\underline{d}^{O_S}}$ and $\widehat{\underline{d}^{\gamma_* O_S}}$. We claim

$$\widehat{\underline{d}^{\gamma_* O_S}} = \hat{\gamma}_* \widehat{\underline{d}^{O_S}}. \tag{28}$$

Let $v \in V(\hat{H}_R)$. If $v = v_e$ for $e \in R$ then v_e is also an exceptional vertex of $\hat{G}_S$ mapped to v_e by $\hat{\gamma}$. Hence both divisors appearing in (28) have value 1 on v_e. Now suppose $v \in V(H)$, then, by Proposition 3.2.1,

$$(\widehat{\underline{d}^{\gamma_* O_S}})_v = (\underline{d}^{\gamma_* O_S})_v = (\gamma_* \underline{d}^{O_S})_v + \underline{c}_v^{\gamma, S} = \sum_{z \in \gamma_V^{-1}(v)} \underline{d}_z^{O_S} + \underline{c}_v^{\gamma, S} = (\hat{\gamma}_* \widehat{\underline{d}^{O_S}})_v$$

where the last equality follows as $\underline{c}_v^{\gamma, S}$ is equal to the number of exceptional vertices of $\hat{G}_S$ that are mapped to v by $\hat{\gamma}$. (28) is proved.

We can now apply Proposition 4.4.3, to obtain that any line bundle $\hat{L} \in \operatorname{Pic}(\hat{X}_S)$ such that $\underline{\deg}\hat{L} = \widehat{\underline{d}^{O_S}}$ is obtained as specialization of a line bundle $\hat{M} \in \operatorname{Pic}(\hat{Y}_R)$ such that

$$\underline{\deg}\hat{M} = \hat{\gamma}_* \underline{\deg}\hat{L} = \hat{\gamma}_* \widehat{\underline{d}^{O_S}} = \widehat{\underline{d}^{\gamma_* O_S}}.$$

This proves that $P_Y^{\gamma_* O_S}$ specializes to $P_X^{O_S}$. Therefore $\mu_X(P_X^{O_S}) \subset \overline{P_H^{\gamma_* O_S}}$. Now, by Theorem 4.3.4 and the hypothesis $\overline{\gamma}_*\overline{O}_S \leq \overline{O}_T$, in $\overline{P}_Y^{g-1+b}$ we have $P_Y^{\gamma_* O_S} \subset \overline{P_Y^{O_T}}$. Hence $\mu_X(P_X^{O_S}) \subset \overline{P_H^{\gamma_* O_S}} \subset \overline{P_H^{O_T}}$. The Proposition is proved. ♣

Proposition 4.7.3. *Let $(G, [O_S])$ and $(H, [O_T])$ be in $[\mathcal{OP}_g^b]$. The following are equivalent*

(a) $P_G^{[O_S]} \cap \overline{P_H^{[O_T]}} \neq \emptyset$.

(b) $(G, [O_S]) \leq (H, [O_T])$.

(c) $P_G^{[O_S]} \subset \overline{P_H^{[O_T]}}$.

Proof. (a) $\Rightarrow$(b). By hypothesis, we have a specialization of polarized curves, $(\hat{Y}_T, \hat{M})$ to $(\hat{X}_S, \hat{L})$, where X and Y are curves dual to G and H respectively, and $\hat{L}$ and $\hat{M}$ are stably balanced line bundles on $\hat{X}_S$ and $\hat{Y}_T$ such that $\underline{\deg}_{X_S^\nu}\hat{L} = \underline{d}^{O_{S'}}$ and $\underline{\deg}_{Y_T^\nu}\hat{M} = \underline{d}^{O_{T'}}$ for some $\overline{O}_{S'} \in [O_S]$ and $\overline{O}_{T'} \in [O_T]$. It suffices to prove that $\overline{O}_{S'} \leq \overline{O}_{T'}$.

To simplify the notation, from now on we drop the indices and write $\overline{O}_{S'} = \overline{O}_S$ and $\overline{O}_{T'} = \overline{O}_T$. We denote by $\hat{G}_S$ and $\hat{H}_T$ the dual graphs of $\hat{X}_S$ and $\hat{Y}_T$. By Proposition 4.4.2, the above specialization is associated to a contraction

$$\hat{\gamma} : \hat{G}_S \longrightarrow \hat{H}_T,$$

such that $\hat{\gamma}_*\underline{\deg}\hat{L} = \underline{\deg}\hat{M}$. Now, every exceptional component of $\hat{Y}_T$ specializes to an exceptional component of $\hat{X}_S$, hence we have a specialization of Y to X and the associated contraction $\gamma : G \to H = G/S_0$. We have an inclusion $T \subset S$ induced by $E(H) \subset E(G)$.

Denote by $\hat{O}$ the orientation on $\hat{G}_S$ obtained from O_S by orienting all edges adjacent to exceptional vertices towards the exceptional vertex. Then the degree of $\underline{d}^{\hat{O}}$ on each exceptional component is 1 and $\underline{d}^{\hat{O}} = \widehat{(\underline{d}^{O_S})}$.

We first assume $T = \emptyset$, then $\hat{H}_T = H$ and we have a commutative diagram

$$\begin{tikzcd} \hat{G}_S \arrow[rr, "\hat{\gamma}"] \arrow[dr, "\delta"'] & & H \\ & G \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Here δ is given as follows: every exceptional vertex v_e in $\hat{G}_S$ has two adjacent edges h_e and j_e, both $\hat{O}$-oriented towards v_e. Defining δ amounts to choosing one of the two for every exceptional vertex. If $e \in S_0$, we can contract any of the two, as $\hat{\gamma}$ contracts both. If $e \notin S_0$ we choose the one contracted by $\hat{\gamma}$. This choice clearly makes the diagram commutative. Set $O' := \delta_*\hat{O}$ on G. Since $\hat{\gamma}_*\underline{\deg}\hat{L} = \underline{\deg}\hat{M}$, i.e. $\hat{\gamma}_*(\underline{d}^{\hat{O}}) = \underline{d}^{O_\emptyset}$ we get

$$\gamma_*O' = \gamma_*(\delta_*\hat{O}) = \hat{\gamma}_*(\hat{O}) \sim O_\emptyset$$

where the equivalence at the end follows from Proposition 3.2.1 (c), (with $\underline{c}^{\hat{\gamma},\emptyset} = 0$ because O' is defined on the whole graph). By construction we have $O'_{|G-S} = O_S$, i.e. $O_S \leq O'$, and thus by Proposition 3.2.1 (e):

$$\gamma_*O_S \leq \gamma_*O' \sim O_\emptyset,$$

which proves the claim in case $T = \emptyset$.

In general, we have $T \subset S$ and, of course, $T \cap S_0 = \emptyset$. Therefore the restriction of γ to $G - T$ is

$$\gamma_{|G-T} : G - T \longrightarrow \frac{G-T}{S_0} = H - T.$$

Write $G' = G - T$, $H' = H - T$ and $\gamma' = \gamma_{|G-T}$. Then write $O' = (O_T)_{|H'}$ and $O'_{S'} = (O_S)_{|G'}$ with $S' = S \smallsetminus T$. By the previous case $\overline{\gamma'_* O'_{S'}} \leq \overline{O'}$, i.e.

$$\gamma'_* O'_{S'} \sim O'_{|H' - \gamma'_* S'}. \tag{29}$$

Now, O_S is defined on $G - S \subset G - T$, hence

$$\gamma'_* O'_{S'} = (\gamma_{|G-T})_* (O_S)_{|G-T} = \gamma_* O_S.$$

Also, as O_T is defined on $H' = H - T$, we have

$$O'_{|H' - \gamma'_* S'} = ((O_T)_{|H-T})_{|H-T-\gamma'_* S'} = (O_T)_{H - \gamma_* S}$$

($\gamma'_* S' \cup T = S' \smallsetminus S'_0 \cup T = S \smallsetminus S_0 = \gamma_* S$ as $T \cap S_0 = \emptyset$). Combining with (29) gives $\gamma_* O_S \sim (O_T)_{H - \gamma_* S}$ and we are done with the implication (a) $\Rightarrow$(b).

(b) $\Rightarrow$(c). By hypothesis, $\overline{O}_{S'} \leq \overline{O}_{T'}$ for some $\overline{O}_{S'} \in [\overline{O}_S]$ and $\overline{O}_{T'} \in [\overline{O}_T]$. By Proposition 4.7.2, we have $P_G^{[O_{S'}]} \subset \overline{P_H^{[O_{T'}]}}$, hence we conclude as follows

$$P_G^{[O_S]} = P_G^{[O_{S'}]} \subset \overline{P_H^{[O_{T'}]}} = \overline{P_H^{[O_T]}}.$$

(c) $\Rightarrow$(a) is obvious. ♣

Theorem 4.7.1 is proved, and so is Theorem 1.1.1. ♣

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