

# TORELLI THEOREM FOR STABLE CURVES

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ABSTRACT. We study the Torelli morphism from the moduli space of stable curves to the moduli space of principally polarized stable semi-abelic pairs. We give two characterizations of its fibers, describe its injectivity locus, and give a sharp upper bound on the cardinality of the finite fibers. We also bound the dimension of the infinite fibers. In particular, we obtain that the compactified Torelli map is injective away from curve with separating nodes if and only if the genus is at most 4.

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## 1. INTRODUCTION

**1.1. The problem.** The classical Torelli theorem asserts the injectivity of the Torelli map  $t_g : M_g \rightarrow A_g$  from the moduli scheme  $M_g$  of smooth projective curves of genus  $g$  to the moduli scheme  $A_g$  of principally polarized abelian varieties of dimension  $g$ .

As  $M_g$  and  $A_g$  are not complete varieties if  $g \geq 1$ , the problem of finding good compactifications for them has been thoroughly investigated, and solved in various

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ways. For  $M_g$ , the most widely studied compactification is the moduli space of Deligne-Mumford stable curves,  $\overline{M}_g$ .

Observe now that the Torelli map  $t_g$  does not extend to a regular map from  $\overline{M}_g$  to  $A_g$ . More precisely, the largest subset of  $\overline{M}_g$  admitting a regular map to  $A_g$  extending  $t_g$  is the locus of curves of compact type (i.e. every node is a separating node). Therefore the following question naturally arises: does there exist a good compactification of  $A_g$  which is the image of an extended Torelli morphism from the whole of  $\overline{M}_g$ ? If so, what are the properties of such an extended map?

It was known to D. Mumford (see [Nam76b]) that  $t_g$  extends to a morphism

$$\overline{t}_g^{Vor} : \overline{M}_g \rightarrow \overline{A}_g^{Vor},$$

where  $\overline{A}_g^{Vor}$  is the second Voronoi toroidal compactification of  $A_g$ ; see [AMRT75], [Nam80], [FC90]. On the other hand, the Torelli theorem does not extend. Indeed if  $g \geq 3$  the map  $\overline{t}_g^{Vor}$  has positive dimensional fibers over the locus of curves having a separating node (see [Nam80, Thm 9.30(vi)]); furthermore although it has finite fibers away from this locus, it fails to be injective (see [V03]). The precise generalization of the Torelli theorem with respect to the above map  $\overline{t}_g^{Vor}$  remains an open problem, since the pioneering work of Y. Namikawa.

In recent years, the space  $\overline{A}_g^{Vor}$  and the map  $\overline{t}_g^{Vor}$  have been placed in a new modular framework by V. Alexeev ([Ale02], [Ale04]). As a consequence, there exists a different compactification of the Torelli morphism, whose geometric interpretation ties in well with the modular descriptions of  $\overline{M}_g$  and of the compactified Jacobian. More precisely, in [Ale02] a new moduli space is constructed, the coarse moduli space  $\overline{A}_g^{\text{mod}}$ , parametrizing principally polarized “semi-abelic stable pairs”. The Voronoi compactification  $\overline{A}_g^{Vor}$  is shown to be the normalization of the irreducible component of  $\overline{A}_g^{\text{mod}}$  containing  $A_g$ ; see Theorem 1.2.5 below. Next, in [Ale04], a new compactified Torelli morphism,  $\overline{t}_g$ , factoring through  $\overline{t}_g^{Vor}$ , is defined:

$$\overline{t}_g : \overline{M}_g \xrightarrow{\overline{t}_g^{Vor}} \overline{A}_g^{Vor} \rightarrow \overline{A}_g^{\text{mod}}.$$

$\overline{t}_g$  is the map sending a stable curve  $X$  to the principally polarized semi-abelic stable pair  $(J(X) \curvearrowright \overline{P}_X^{g-1}, \Theta(X))$ . Here  $J(X)$  is the generalized Jacobian of  $X$ ,  $\overline{P}_X^{g-1}$  is a stable semi-abelic variety, called the compactified Picard scheme (in degree  $g-1$ ), acted upon by  $J(X)$ ; finally  $\Theta(X) \subset \overline{P}_X^{g-1}$  is a Cartier, ample divisor, called the Theta divisor. As proved in [Ale04],  $\overline{P}_X^{g-1}$  coincides with the previously constructed compactified Picard schemes of [OS79], [Sim94], and [Cap94]; moreover the definition of the theta divisor extends the classical one very closely.

The goal of the present paper is to establish the precise analogue of the Torelli theorem for stable curves, using the compactified Torelli morphism  $\overline{t}_g$ . This is done in Theorem 2.1.4, our main result. We characterize curves having the same image via  $\overline{t}_g$  in two different ways. One characterization is based on the classifying morphism of the generalized Jacobian, and it is too technical to be further described in this introduction, we refer to Section 3 for details. On the other hand the other characterization, used in the statement of Theorem 2.1.4, is quite explicit as we are going to illustrate.

Let  $X$  and  $X'$  be two stable curves free from separating nodes (this is the key case, from which the general case follows); our main theorem states that  $\overline{t}_g(X) = \overline{t}_g(X')$  if and only if they are “C1-equivalent”, i.e. if the following holds. First,  $X$  and  $X'$  have the same normalization,  $Y$ , and the same set of gluing points,  $G \subset Y$ . Second, the set  $G$  is endowed with a so-called “C1-partition”, the same for  $X$

and  $X'$ , defined in terms of the codimension-one strata of the compactified Picard scheme  $\overline{P_X^{g-1}}$ .

The C1-partition of the gluing set  $G$  is important, let us describe it on the simplest examples. If  $X$  is irreducible,  $\overline{P_X^{g-1}}$  has one codimension-one stratum for every node of  $X$ ; hence the C1-partition completely determines  $X$ , as it is the partition into pairs of corresponding branches over the nodes; so we have that the Torelli map is injective on the locus of irreducible curves, a fact that was already known to Y. Namikawa. The next example is more interesting; let  $X$  be a cycle of  $h$  smooth components,  $C_1, \dots, C_h$ , with  $h$  nodes; then  $G = \{p_1, q_1, \dots, p_h, q_h\}$  with  $p_i, q_i \in C_i$ . The scheme  $\overline{P_X^{g-1}}$  is irreducible, and has a unique codimension-one stratum. This yields that the C1-partition of  $G$  is trivial, i.e. it contains no proper subset. Therefore all curves of genus  $g$  whose normalization is  $\bigcup_1^h C_i$  and whose gluing points are  $\{p_1, q_1, \dots, p_h, q_h\}$  have the same image via  $\bar{t}_g$ .

Let us outline the proof of our main theorem, which occupies Section 4.

The difficult part is the necessary condition: assume that two curves, stable and free from separating nodes, have the same image, denoted  $(J \curvearrowright \overline{P}, \Theta)$ , under the Torelli map; as we said, we must prove that they are C1-equivalent (see 2.1.3). First, the structure of  $J$ -scheme of  $\overline{P}$  induces a stratification, whose (unique) smallest stratum determines the normalization of the curves, apart from rational components. Second, the intricate combinatorics of the stratification of  $\overline{P}$  (the  $J$ -strata form a set partially ordered by inclusion of closures) carries information about the combinatorics of the curves themselves, so that it determines the “cyclic equivalence class” (see 1.2.2) of the dual graphs of the curves. This second part requires a combinatorial analysis, carried out in Section 2. From these first steps one easily obtains that the two curves have the same normalization. It remains to prove that the gluing sets of the normalization maps are the same, together with their C1-partition. Here is where we use the Theta divisor,  $\Theta$ , its geometry and its connection with the Abel maps of the curves. See Subsection 4.2 for details.

The proof of the converse is based on a second characterization of C1-equivalence, which we temporarily call “T-equivalence” (the “T” stands for Torelli). The crux of the matter is to prove that C1-equivalence and T-equivalence coincide; we do that in Section 3. Then the proof of the sufficiency follows directly from the general theory of compactifications of principally polarized semiabelian varieties, on which our definition of T-equivalence is inspired.

Our paper ends with a fifth section where we compute the upper bounds on the cardinality (Theorem 5.1.4), and on the dimension (Proposition 5.2.1), of the fibers of  $\bar{t}_g$ . We prove that the finite fibers have cardinality at most  $\left\lceil \frac{(g-2)!}{2} \right\rceil$ ; in particular, since our bound is sharp, we have that, away from curves with a separating node,  $\bar{t}_g$  is injective if and only if  $g \leq 4$ . We also give, in Theorem 5.1.4, a geometric description of the injectivity locus of  $\bar{t}_g$ .

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**1.2. Preliminaries.** We work over an algebraically closed field  $k$ . A variety over  $k$  is a reduced scheme of finite type over  $k$ . A curve is a projective variety of pure dimension 1.

Throughout the paper  $X$  is a connected nodal curve of arithmetic genus  $g$ , and  $Y$  is a nodal curve, non necessarily connected. We denote by  $g_Y$  the arithmetic genus of  $Y$ .

A node  $n$  of  $Y$  is called a *separating node* if the number of connected components of  $Y \setminus n$  is greater than the number of connected components of  $Y$ . We denote by  $Y_{\text{sep}}$  the set of separating nodes of  $Y$ .

For any subset  $S \subset X_{\text{sing}} := \{\text{nodes of } X\}$ , we denote by  $\nu_S : Y_S \rightarrow X$  the partial normalization of  $X$  at  $S$ . We denote by  $\gamma_S$  the number of connected components of  $Y_S$ . The (total) normalization of  $X$  will be denoted

$$\nu : X^\nu \longrightarrow X = \coprod_{i=1}^{\gamma} C_i$$

where the  $C_i$  are the connected components of  $X^\nu$ . The points  $\nu^{-1}(X_{\text{sing}}) \subset X^\nu$  will be usually called “gluing points” of  $\nu$ .

The dual graph of  $Y$  will be denoted  $\Gamma_Y$ . The irreducible components of  $Y$  correspond to the vertices of  $\Gamma_Y$ , and we shall often identify these two sets. Likewise we shall identify the set of nodes of  $Y$  to the set,  $E(\Gamma_Y)$ , of edges of  $\Gamma_Y$ .

A graph  $\Gamma$  is a *cycle* if it is connected and has  $h$  edges and  $h$  vertices (each of valency 2) for some  $h \geq 1$ . A curve whose dual graph is a cycle will be called a cycle curve.

**1.2.1.** *The graph  $\Gamma_X(S)$ .* Let  $S \subset X_{\text{sing}}$  be a set of nodes of  $X$ ; we associate to  $S$  a graph,  $\Gamma_X(S)$ , defined as follows.  $\Gamma_X(S)$  is obtained from  $\Gamma_X$  by contracting to a point every edge not in  $S$ . Consider  $\nu_S : Y_S \rightarrow X$  (the normalization of  $X$  at  $S$ ). Then the vertices of  $\Gamma_X(S)$  correspond to the connected components of  $Y_S$ . For example,  $\Gamma_X(X_{\text{sing}}) = \Gamma_X$ , and  $\Gamma_X(\emptyset)$  is a point.

**1.2.2.** Recall that, in graph theory, two graphs  $\Gamma$  and  $\Gamma'$  are called *cyclically equivalent* (or “two-isomorphic”), in symbols  $\Gamma \equiv_{\text{cyc}} \Gamma'$ , if there exists a bijection  $\epsilon : E(\Gamma) \rightarrow E(\Gamma')$  inducing a bijection between the cycles of  $\Gamma$  and the cycles of  $\Gamma'$  (such an  $\epsilon$  will be called a cyclic bijection). In other words, if for any orientation on  $\Gamma$  there exists an orientation on  $\Gamma'$  such that the following diagram is commutative:

$$\begin{array}{ccc} C_1(\Gamma, \mathbb{Z}) & \xrightarrow[\cong]{\epsilon_C} & C_1(\Gamma', \mathbb{Z}) \\ \uparrow & & \uparrow \\ H_1(\Gamma, \mathbb{Z}) & \xrightarrow[\cong]{\epsilon_H} & H_1(\Gamma', \mathbb{Z}) \end{array}$$

where the vertical arrows are the inclusions,  $\epsilon_C$  is the (linear) isomorphism induced by  $\epsilon$  and  $\epsilon_H$  the restriction of  $\epsilon_C$  to  $H_1(\Gamma, \mathbb{Z})$ .

**1.2.3.** *The moduli space  $\overline{A}_g^{\text{mod}}$ .*

**Definition 1.2.4.** [Ale02] A *principally polarized stable semi-abelic pair* (ppSSAP for short) over  $k$  is the datum of  $(G \curvearrowright P, \Theta)$  where

- (i)  $G$  is a semiabelian variety over  $k$ , that is an algebraic group which is an extension of an abelian variety  $A$  by a torus  $T$ :

$$1 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0.$$

- (ii)  $P$  is a seminormal, connected, projective variety of pure dimension equal to  $\dim G$ .
- (iii)  $G$  acts on  $P$  with finitely many orbits, and with connected and reduced stabilizers contained in the toric part  $T$  of  $G$ .
- (iv)  $\Theta$  is an effective ample Cartier divisor on  $P$  which does not contain any  $G$ -orbit, and such that  $h^0(P, \mathcal{O}_P(\Theta)) = 1$ .

The datum of  $(G \curvearrowright P)$  satisfying the first three properties above is called a *stable semi-abelic variety*. The datum of  $(G \curvearrowright P, \Theta)$  satisfying all the above properties, except  $h^0(P, \mathcal{O}_P(\Theta)) = 1$ , is called a *principally polarized stable semi-abelic pair*.

When  $G$  is an abelian variety, the word “semi-abelic” is replaced by “abelic”.

Recall that a homomorphism  $\Phi = (\phi_0, \phi_1) : (G \curvearrowright P, \Theta) \rightarrow (G' \curvearrowright P', \Theta')$  between two ppSSAP is given by a homomorphism of algebraic groups  $\phi_0 : G \rightarrow G'$ , and a morphism  $\phi_1 : P \rightarrow P'$ , satisfying the following two conditions:

- (1)  $\phi_0$  and  $\phi_1$  are compatible with the actions of  $G$  on  $P$  and of  $G'$  on  $P'$ .
- (2)  $\phi_1^{-1}(\Theta') = \Theta$

$\Phi = (\phi_0, \phi_1)$  is an isomorphism if  $\phi_0$  and  $\phi_1$  are isomorphisms.

One of the main results of [Ale02] is the following

**Theorem 1.2.5.** *There exists a projective scheme  $\overline{A}_g^{\text{mod}}$  which is a coarse moduli space for principally polarized stable semi-abelic pairs. Moreover the open subset parametrizing principally polarized stable abelian pairs is naturally isomorphic to  $A_g$ . The normalization of the irreducible component of  $\overline{A}_g^{\text{mod}}$  containing  $A_g$  (the “main component”) is isomorphic to the second toroidal Voronoi compactification  $\overline{A}_g^{\text{Vor}}$ .*

To the best of our knowledge, it is not known whether the main component of  $\overline{A}_g^{\text{mod}}$  is normal; see [Bri07] for an expository account.

**1.2.6.** *The compactified Torelli map  $\overline{t}_g : \overline{M}_g \rightarrow \overline{A}_g^{\text{mod}}$ .*

We shall now recall the modular description of the compactified Torelli map  $\overline{t}_g : \overline{M}_g \rightarrow \overline{A}_g^{\text{mod}}$ .

**Definition 1.2.7.** Let  $Y$  be a nodal curve of arithmetic genus  $g_Y$ . Let  $M$  be a line bundle on  $Y$  of multidegree  $\underline{d}$  and degree  $g_Y - 1$ . We say that  $M$ , or its multidegree  $\underline{d}$ , is *semistable* if for every subcurve  $Z \subset Y$  of arithmetic genus  $g_Z$ , we have

$$(1.1) \quad g_Z - 1 \leq d_Z,$$

where  $d_Z := \deg_Z M$ . We say that  $M$  (or its multidegree  $\underline{d}$ ) is *stable* if the equality holds in (1.1) exactly for every subcurve  $Z$  which is a union of connected components of  $Y$ . We denote by  $\Sigma(Y)$  the set of stable multidegrees on  $Y$ .

We denote by  $\text{Pic}^{\underline{d}}Y$  the variety of line bundles of multidegree  $\underline{d}$  on  $Y$ ; the variety of line bundles having degree 0 on every component of  $Y$ ,  $\text{Pic}^0Y = J(Y)$ , is identified with the generalized Jacobian. Using the notation of 1.2 and 1.2.1, we now recall some properties of the compactified Jacobian  $\overline{P}_X^{g-1}$  (see [Ale04], [Cap07]).

**Fact 1.2.8.** *Let  $X$  be a connected nodal curve of genus  $g$ , and  $J(X)$  its.*

- (i)  $\overline{P}_X^{g-1}$  is a SSAV with respect to the natural action  $J(X)$ .
- (ii) The orbits of the action of  $J(X)$  give a stratification of  $\overline{P}_X^{g-1}$ :

$$(1.2) \quad \overline{P}_X^{g-1} = \coprod_{\substack{\emptyset \subseteq S \subseteq X_{\text{sing}} \\ \underline{d} \in \Sigma(Y_S)}} P_S^{\underline{d}},$$

where each stratum  $P_S^{\underline{d}}$  is canonically isomorphic to  $\text{Pic}^{\underline{d}}Y_S$ .

- (iii)  $\Sigma(Y_S)$  is not empty if and only if  $Y_S$  has no separating node. In particular, if  $\Sigma(Y_S)$  is not empty then  $X_{\text{sep}} \subseteq S$ .

- (iv) Each stratum  $P_S^{\underline{d}}$  is a torsor under the generalized Jacobian  $J(Y_S)$  of  $Y_S$ , and the action of  $J(X)$  on  $P_S^{\underline{d}}$  factorizes through the pull-back map  $J(X) \rightarrow J(Y_S)$ . Hence every nonempty stratum  $P_S^{\underline{d}}$  has dimension

$$(1.3) \quad \dim P_S^{\underline{d}} = \dim J(Y_S) = g - \#S + \gamma_S - 1 = g - b_1(\Gamma_X(S)).$$

(v) If  $P_{S'}^{d'} \subset \overline{P_S^d}$  then  $S \subset S'$  and  $\underline{d} \geq \underline{d}'$  (i.e.  $d_i \geq d'_i, \forall i = 1, \dots, \gamma$ ).

(vi) The smooth locus  $P_X^{g-1}$  of  $\overline{P_X^{g-1}}$  consists of the strata of maximal dimension:

$$P_X^{g-1} = \coprod_{\underline{d} \in \Sigma(Y_{X_{\text{sep}}})} P_S^{\underline{d}}.$$

The irreducible components of  $\overline{P_X^{g-1}}$  are the closure of the maximal dimension strata.

Before recalling the definition of the theta divisor we introduce some notation.

For any multidegree  $\underline{d}$  on a curve  $Y$  and for any  $r \geq 0$  we set

$$(1.4) \quad W_{\underline{d}}^r(Y) := \{L \in \text{Pic}^{\underline{d}} Y : h^0(Y, L) > r\};$$

when  $r = 0$  the superscript is usually omitted:  $W_{\underline{d}}^0(Y) = W_{\underline{d}}(Y)$ .

The normalization of  $X$  at its set of separating nodes,  $X_{\text{sep}}$ , will be denoted by

$$(1.5) \quad \tilde{X} = \prod_{i=1}^{\tilde{\gamma}} \tilde{X}_i$$

where the  $\tilde{X}_i$  are connected (and all free from separating nodes). Note that  $\tilde{\gamma} = \#X_{\text{sep}} + 1$ . We denote by  $\tilde{g}_i$  the arithmetic genus of  $\tilde{X}_i$ .

The subsequent facts summarize results of [E97], [Ale04] and [Cap07].

**Definition 1.2.9.** The theta divisor  $\Theta(X)$  of  $\overline{P_X^{g-1}}$  is

$$\Theta(X) := \overline{\bigcup_{\underline{d} \in \Sigma(\tilde{X})} W_{\underline{d}}(\tilde{X})} \subset \overline{P_X^{g-1}}.$$

**Fact 1.2.10.** (i) The pair  $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$  is a ppSSAP. In particular  $\Theta(X)$  is Cartier, ample and  $h^0(\overline{P_X^{g-1}}, \Theta(X)) = 1$ .

(ii) The stratification of  $\overline{P_X^{g-1}}$  given by 1.2.8(ii) induces the stratification

$$(1.6) \quad \Theta(X) = \coprod_{\substack{\emptyset \subseteq S \subseteq X_{\text{sing}} \\ \underline{d} \in \Sigma(Y_S)}} \Theta_S^{\underline{d}},$$

where  $\Theta_S^{\underline{d}} := \{M \in P_S^{\underline{d}} : h^0(Y_S, M) > 0\} \cong W_{\underline{d}}^0(Y_S)$  is a divisor in  $P_S^{\underline{d}}$ .

(iii) Let  $Y_S = \cup_{i=1}^{\gamma_S} Y_i$  be the decomposition of  $Y_S$  in connected components, and let  $\underline{d} \in \Sigma(Y_S)$ . The irreducible components of  $\Theta_S^{\underline{d}}$  are given by

$$(\Theta_S^{\underline{d}})_i = \{L \in P_S^{\underline{d}} : h^0(Y_i, L|_{Y_i}) > 0\},$$

for every  $1 \leq i \leq \gamma_S$  such that the arithmetic genus of  $Y_i$  is positive.

**Remark 1.2.11.** From the description 1.2.8, we derive that there exists a unique  $J(X)$ -stratum in  $\overline{P_X^{g-1}}$  contained in the closure of every other stratum, namely

$$P_{X_{\text{sing}}}^{(g_1-1, \dots, g_\gamma-1)} = \prod_{i=1}^{\gamma} \text{Pic}^{g_i-1} C_i.$$

We refer to this stratum as the *smallest stratum* of  $\overline{P_X^{g-1}}$ . Moreover, according to stratification (1.6), the restriction of  $\Theta(X)$  to the smallest stratum is given by

$$(1.7) \quad \Theta(X)|_{P_{X_{\text{sing}}}^{(g_1-1, \dots, g_\gamma-1)}} = \prod_{i=1}^{\gamma} \text{Pic}^{g_i-1} C_i \times \cdots \times \Theta(C_i) \times \cdots \times \text{Pic}^{g_\gamma-1} C_\gamma.$$

We can now state the following result of Alexeev ([Ale04]):

**Theorem 1.2.12.** *The classical Torelli morphism is compactified by the morphism  $\bar{t}_g : \bar{M}_g \rightarrow \bar{A}_g^{mod}$  which sends a stable curve  $X$  to the ppSSAP  $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$ .*

**1.3. First reductions.** We shall now show that the ppSSAP  $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$  depends only on the stabilization of every connected component of the partial normalization of  $X$  at its separating nodes. Most of what is in this subsection is well known to the experts.

We often have deal with disconnected curves. Let  $Y = \coprod_{i=1}^h Y_i$  be such a curve and  $g_Y$  its arithmetic genus, so that  $g_Y = \sum g_{Y_i} - h$ . We have

$$(1.8) \quad \overline{P_Y^{g_Y-1}} = \prod_{i=1}^h \overline{P_{Y_i}^{g_{Y_i}-1}} \quad \text{and} \quad \Theta(Y) = \bigcup_{i=1}^h \pi_i^*(\Theta(Y_i))$$

where  $\pi_i : \overline{P_Y^{g_Y-1}} \rightarrow \overline{P_{Y_i}^{g_{Y_i}-1}}$  is the  $i$ -th projection.

The next Lemma illustrates the recursive structure of  $(\overline{P_X^{g-1}}, \Theta(X))$ . For  $S \subset X_{\text{sing}}$  such that  $\Sigma(Y_S)$  is non empty (i.e.  $Y_S$  has no separating nodes), denote

$$(1.9) \quad \overline{P_S} := \overline{\bigcup_{d \in \Sigma(Y_S)} P_S^d} \subset \overline{P_X^{g-1}} \quad \text{and} \quad \overline{\Theta_S} := \Theta(X) \cap \overline{P_S}.$$

**Lemma 1.3.1.** *Assumptions as above. There is a natural isomorphism  $\overline{P_S} \cong \overline{P_{Y_S}^{g_{Y_S}-1}}$ , inducing an isomorphism between  $\overline{\Theta_S}$  and  $\Theta(Y_S)$ .*

*Proof.* Recall that  $\overline{P_X^{g-1}}$  is a GIT-quotient,  $V_X \xrightarrow{q} \overline{P_X^{g-1}} = V_X/G$  where  $V_X$  is contained in a certain Hilbert scheme of curves in projective space (there are other descriptions of  $\overline{P_X^{g-1}}$  as a GIT-quotient, to which the subsequent proof can be easily adjusted). Denote  $V_Y := q^{-1}(\overline{P_S})$  so that  $V_Y$  is a  $G$ -invariant, reduced, closed subscheme of  $V_X$  and  $\overline{P_S}$  is the GIT-quotient

$$(1.10) \quad V_Y \longrightarrow V_Y/G = \overline{P_S}.$$

The restriction to  $V_Y$  of the universal family over the Hilbert scheme is a family of nodal curves  $\mathcal{Z} \rightarrow V_Y$  endowed with a semistable line bundle  $\mathcal{L} \rightarrow \mathcal{Z}$ . Let  $Z$  be any fiber of  $\mathcal{Z} \rightarrow V_Y$ ; then  $Z$  has  $X$  as stabilization, and the stabilization map  $Z \rightarrow X$  blows-up some set  $S'$  of nodes of  $X$ ; note that  $S'$  certainly contains  $S$ . Therefore the exceptional divisors corresponding to  $s \in S$  form a family over  $V_Y$

$$\mathcal{Z} \supset \mathcal{E}_S \rightarrow V_Y,$$

such that  $\mathcal{E}_S = \coprod_{s \in S} \mathcal{E}_s$  and every  $\mathcal{E}_s$  is a  $\mathbb{P}^1$ -bundle over  $V_Y$ . Consider the family of curves obtained by removing  $\mathcal{E}_S$ :

$$\mathcal{Y} := \overline{\mathcal{Z} \setminus \mathcal{E}_S} \rightarrow V_Y.$$

By construction the above is a family of nodal curves, all admitting a surjective map to  $Y_S$  which blows down some exceptional component (over a dense open subset of  $V_Y$  the fiber of  $\mathcal{Y} \rightarrow V_Y$  is isomorphic to  $Y_S$ ). The restriction  $\mathcal{L}_{\mathcal{Y}}$  of  $\mathcal{L}$  to  $\mathcal{Y}$  is a relatively semistable line bundle. Therefore  $\mathcal{L}_{\mathcal{Y}}$  determines a unique moduli map  $\mu$  from  $V_Y$  to the compactified Picard variety of  $Y_S$ , i.e.  $\mu : V_Y \rightarrow \overline{P_{Y_S}^{g_{Y_S}-1}}$ . The map  $\mu$  is of course  $G$ -invariant, and therefore it descends to a unique map

$\bar{\mu} : V_Y/G \longrightarrow \overline{P_{Y_S}^{g_{Y_S}-1}}$ . Summarizing, we have a commutative diagram

$$\begin{array}{ccc} V_Y & \xrightarrow{\mu} & \overline{P_{Y_S}^{g_{Y_S}-1}} \\ & \searrow & \nearrow \bar{\mu} \\ & V_Y/G = \overline{P_S} & \end{array}$$

By Fact 1.2.8 the morphism  $\bar{\mu}$  is a bijection. Since  $\overline{P_{Y_S}^{g_{Y_S}-1}}$  is seminormal,  $\bar{\mu}$  is an isomorphism. Finally, by Fact 1.2.10 we conclude that  $\bar{\mu}$  maps  $\overline{\Theta_S}$  isomorphically to  $\Theta(Y_S)$ . ■

We say that a ppSSAP  $(G \curvearrowright P, \Theta)$  is *irreducible* if every irreducible component of  $P$  contains a unique irreducible component of  $\Theta$ .

**Corollary 1.3.2.** (i) If  $X_{\text{sep}} = \emptyset$  then  $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$  is irreducible.  
(ii) In general, we have the decomposition into irreducible non-trivial ppSSAP:

$$(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) = \prod_{\tilde{g}_i > 0} (J(\tilde{X}_i) \curvearrowright \overline{P_{\tilde{X}_i}^{\tilde{g}_i-1}}, \Theta(\tilde{X}_i)).$$

*Proof.* The first assertion follows from [Cap07, Thm 3.1.2]. Concerning the second assertion, by 1.2.8 we have  $J(X) = \prod_{i=1}^{\tilde{r}} J(\tilde{X}_i)$ .

We apply Lemma 1.3.1 to  $S = X_{\text{sep}}$ . Note that in this case  $\overline{P_S} = \overline{P_X^{g-1}}$ , and hence  $\overline{\Theta_S} = \Theta(X)$ . Therefore we get

$$(\overline{P_X^{g-1}}, \Theta(X)) \cong (\overline{P_{\tilde{X}}^{\tilde{g}-1}}, \Theta(\tilde{X})) \cong \prod_{\tilde{g}_i > 0} (\overline{P_{\tilde{X}_i}^{\tilde{g}_i-1}}, \Theta(\tilde{X}_i)).$$

■

**1.3.3. Invariance under stabilization.** A connected nodal curve  $X$  of arithmetic genus  $g \geq 0$  is called *stable* if each smooth rational component  $E \subsetneq X$  meets the complementary subcurve  $E^c = \overline{X} \setminus E$  in at least three points. So, when  $g = 0$  the only stable curve is  $\mathbb{P}^1$ . If  $g = 1$  a stable curve is either smooth or irreducible with one node. If  $g \geq 2$  stable curves are Deligne-Mumford stable curves.

Given any nodal connected curve  $X$ , the *stabilization* of  $X$  is defined as the curve  $\overline{X}$  obtained by contracting to a point every exceptional component  $E$ , i.e. every  $E \subsetneq X$  such that  $\#E \cap E^c \leq 2$  and  $E \cong \mathbb{P}^1$ . It is easy to check that  $\overline{X}$  is unique up to isomorphism. The stabilization of a non connected curve will be defined as the union of the stabilizations of its connected components.

From the moduli properties of  $\overline{A}_g^{\text{mod}}$ , and the fact that it is a projective scheme, one derives the following useful

*Remark 1.3.4.* Let  $X$  be a connected nodal curve of arithmetic genus  $g \geq 0$ , and let  $\overline{X}$  be its stabilization. Then

$$(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) \cong (J(\overline{X}) \curvearrowright \overline{P_{\overline{X}}^{g-1}}, \Theta(\overline{X})).$$

## 2. STATEMENT OF THE MAIN THEOREM

**2.1. C1-equivalence.** Assume that  $X_{\text{sep}} = \emptyset$ . We begin by introducing two posets (i.e. partially ordered sets) naturally associated to the stratification of  $\overline{P_X^{g-1}}$  into  $J(X)$ -orbits, described in (1.2).

- The *poset of strata*, denoted  $\mathcal{ST}_X$ , is the set  $\{P_S^d\}$  of all strata of  $\overline{P_X^{g-1}}$ , endowed with the following partial order:

$$P_S^d \geq P_T^e \iff \overline{P_S^d} \supset P_T^e.$$

- The *poset of (strata) supports*, denoted  $\mathcal{SP}_X$ , is the set of all subsets  $S \subset X_{\text{sing}}$  such that the partial normalization of  $X$  at  $S$ ,  $Y_S$ , is free from separating nodes:

$$\mathcal{SP}_X := \{S \subset X_{\text{sing}} \mid (Y_S)_{\text{sep}} = \emptyset\}.$$

Its partial order is defined as follows:

$$S \geq T \iff S \subseteq T.$$

There is a natural map

$$\text{Supp}_X : \mathcal{ST}_X \longrightarrow \mathcal{SP}_X; \quad P_S^d \mapsto S.$$

$\text{Supp}_X$  is order preserving (by Fact 1.2.8(v)), and surjective (by Fact 1.2.8(iii)).

We have the integer valued function,  $\text{codim}$ , on  $\mathcal{SP}_X$  (cf. 1.2.1 and (1.3)):

$$(2.1) \quad \text{codim}(S) := \dim J(X) - \dim J(Y_S) = b_1(\Gamma_X(S)).$$

Notice that  $\text{codim}(S)$  is the codimension in  $\overline{P_X^{g-1}}$  of every stratum  $P_S^d \in \text{Supp}_X^{-1}(S)$ . Moreover  $\text{codim}$  is strictly order reversing.

**Lemma - Definition 2.1.1.** *Assume  $X_{\text{sep}} = \emptyset$ ; let  $S \in \mathcal{SP}_X$ . We say that  $S$  is a C1-set if the two equivalent conditions below hold.*

- (1)  $\text{codim}(S) = 1$ .
- (2) The graph  $\Gamma_X(S)$  (defined in 1.2.1) is a cycle.

We denote by  $\text{Set}^1 X$  the set of all C1-sets of  $X$ .

*Proof.* The equivalence of the two conditions follows from (2.1), together with the fact that for any  $S \subset X_{\text{sing}}$  the graph  $\Gamma_X(S)$  is connected and free from separating edges (because the same holds for  $\Gamma_X$ ).  $\blacksquare$

*Remark 2.1.2.* Let  $S \in \text{Set}^1 X$  and consider  $Y_S$ , the normalization of  $X$  at  $S$ . By definition  $Y_S$  has  $\#S$  connected components, and  $\Gamma_X(S)$  can be viewed as the graph whose vertices are the connected components of  $Y_S$ , and whose edges correspond to  $S$ . Since  $\Gamma_X(S)$  is a cycle, if  $X$  is stable every connected component of  $Y_S$  has positive arithmetic genus.

The main result of this paper, Theorem 2.1.4 below, is based on the following

**Definition 2.1.3** (C1-equivalence.). Let  $X$  and  $X'$  be connected nodal curves free from separating nodes; denote by  $\nu : X^\nu \rightarrow X$  and  $\nu' : X'^\nu \rightarrow X'$  their normalizations.  $X$  and  $X'$  are *C1-equivalent* if the following conditions hold

- (A) There exists an isomorphism  $\phi : X^\nu \xrightarrow{\cong} X'^\nu$ .
- (B) There exists a bijection between their C1-sets, denoted by

$$\text{Set}^1 X \rightarrow \text{Set}^1 X'; \quad S \mapsto S'$$

$$\text{such that } \phi(\nu^{-1}(S)) = \nu'^{-1}(S').$$

We say that two non connected nodal curves  $Y$  and  $Y'$  are C1-equivalent if there exists a bijection between their connected components such that every two corresponding components are C1-equivalent.

We give a non trivial, yet simple, example in 5.1.2.

In the next subsection, we shall see that the C1-sets form a partition of  $X_{\text{sing}}$ . Hence they determine a partition of the set of gluing points  $\nu^{-1}(X_{\text{sing}}) \subset X^\nu$  of  $X$ , to which we refer as the *C1-partition*. With this terminology, two curves free

from separating nodes are C1-equivalent if they have the same normalization,  $Y$ , the same set of gluing points  $G \subset Y$ , and the same C1-partition of  $G$ .

**Theorem 2.1.4.** *Let  $X$  and  $X'$  be two stable curves of genus  $g$ .*

*Assume that  $X$  and  $X'$  are free from separating nodes. Then  $\bar{t}_g(X) = \bar{t}_g(X')$  if and only if  $X$  and  $X'$  are C1-equivalent.*

*In general, let  $\tilde{X}$  and  $\tilde{X}'$  be the normalizations of  $X$  and  $X'$  at their separating nodes. Then  $\bar{t}_g(X) = \bar{t}_g(X')$  if and only if the stabilization of  $\tilde{X}$  is C1-equivalent to the stabilization of  $\tilde{X}'$ .*

**2.2. The C1-set partition of  $X_{\text{sing}}$ .** In this subsection we collect a few useful results concerning C1-sets. The set  $\text{Set}^1 X$  is a combinatorial invariant, in other words it depends only on the dual graph of  $X$ . So it makes sense to denote  $\text{Set}^1 \Gamma_X := \text{Set}^1 X$ . C1-sets of graphs have been independently studied, and turn out to be a useful tool, in graph theory. In particular, we have the following simple lemma, which is just a rephrasing of its graph theoretic analogue, Lemma 2.3.2 in [CV09].

**Lemma 2.2.1.** *Let  $X$  be a connected curve free from separating nodes.*

- (1) *Every node of  $X$  is contained in a unique C1-set.*
- (2) *Two nodes of  $X$  belong to the same C1-set if and only if the corresponding edges of the dual graph  $\Gamma_X$  belong to the same cycles of  $\Gamma_X$ .*
- (3) *Two nodes  $n_1$  and  $n_2$  of  $X$  belong to the same C1-set if and only if the normalization of  $X$  at  $n_1$  and  $n_2$  is disconnected.*

As a consequence we have the following useful

*Remark 2.2.2.* Let  $S$  and  $T$  be two distinct C1-sets of  $X$ . Then  $T$  is entirely contained in a unique connected component of  $Y_S$ .

Indeed, recall that  $Y_S$  has  $\#S$  connected components, all free from separating nodes. By Lemma 2.2.1 the set  $T$  is contained in the singular locus of  $Y_S$ . Let  $n_1, n_2 \in T$ , and let  $X^*$  and  $Y_S^*$  be the normalizations at  $n_1$  of, respectively,  $X$  and  $Y_S$ . Then by Lemma 2.2.1(3)  $n_2$  is a separating node of  $X^*$  and hence of  $Y_S^*$ . Since  $Y_S$  has no separating node we infer that  $n_1$  belongs to the same connected component as  $n_2$ .

We conclude with a technical lemma which will be used later on.

**Lemma 2.2.3.** *Let  $\Gamma$  be an oriented connected graph free from separating edges. Then the inclusion  $H_1(\Gamma, \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$  factors naturally as follows*

$$H_1(\Gamma, \mathbb{Z}) \hookrightarrow \bigoplus_{S \in \text{Set}^1 \Gamma} H_1(\Gamma(S), \mathbb{Z}) \hookrightarrow C_1(\Gamma, \mathbb{Z})$$

where the graphs  $\Gamma(S)$  have the orientation induced by that of  $\Gamma$ .

*Proof.* Let  $S \in \text{Set}^1 \Gamma$  and consider the natural map  $\sigma_S : \Gamma \rightarrow \Gamma(S)$  contracting all edges not in  $S$ . Recall that  $\Gamma(S)$  is a cycle whose set of edges is  $S$ . By Lemma 2.3.2 in [CV09] (or Lemma 2.2.1) we have the following commutative diagram with exact rows

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_1(\Gamma \setminus S, \mathbb{Z}) & \longrightarrow & C_1(\Gamma, \mathbb{Z}) & \longrightarrow & C_1(\Gamma(S), \mathbb{Z}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H_1(\Gamma \setminus S, \mathbb{Z}) & \longrightarrow & H_1(\Gamma, \mathbb{Z}) & \xrightarrow{\sigma_{S*}} & H_1(\Gamma(S), \mathbb{Z}) \longrightarrow 0 \end{array}$$

where  $\Gamma \setminus S \subset \Gamma$  is the subgraph obtained by removing  $S$  from  $E(\Gamma)$ . We claim that we have the following commutative diagram

$$(2.3) \quad \begin{array}{ccc} H_1(\Gamma, \mathbb{Z}) & \hookrightarrow & \bigoplus_{S \in \text{Set}^1 \Gamma} H_1(\Gamma(S), \mathbb{Z}) \\ \downarrow & & \downarrow \\ C_1(\Gamma, \mathbb{Z}) & \longleftarrow & \bigoplus_{S \in \text{Set}^1 \Gamma} C_1(\Gamma(S), \mathbb{Z}) \end{array}$$

where the vertical arrows are the usual inclusions. The bottom horizontal arrow is the obvious map mapping an edge  $e \in E(\Gamma(S)) = S \subset E(\Gamma)$  to itself. It is injective because two different C1-sets of  $\Gamma$  are disjoint (by 2.2.1), it is surjective as  $\Gamma$  has no separating edges. Finally, the top horizontal arrow is the sum of the maps  $\sigma_{S^*}$  defined in the previous diagram; it is injective because the other maps in the diagram are injective.  $\blacksquare$

**2.3. Gluing points and gluing data.** Let  $X$  be such that  $X_{\text{sep}} = \emptyset$ , and  $S \in \text{Set}^1 X$  a C1-set of cardinality  $h$ . The partial normalization  $Y_S$  of  $X$  at  $S$  has a decomposition  $Y_S = \coprod_{i=1}^h Y_{S,i}$ , with  $Y_{S,i}$  connected and free from separating nodes. We denote by  $Y_{S,i}^\nu$  the normalization of  $Y_{S,i}$ . We set

$$(2.4) \quad G_S := \nu^{-1}(S) \subset X^\nu.$$

Each of the connected components  $Y_{S,i}$  of  $Y_S$  contains exactly two of the points in  $G_S$ , let us call them  $p_i$  and  $q_i$ . This enables us to define a unique fixed-point free involution on  $G_S$ , denoted  $\iota_S$ , such that  $\iota_S$  exchanges  $p_i$  and  $q_i$  for every  $1 \leq i \leq h$ .

The involutions  $\iota_S$  and the curves  $Y_{S,i}^\nu$  are the same for C1-equivalent curves, by the next

**Lemma 2.3.1.** *Let  $X$  be free from separating nodes. The data of  $X^\nu$  and of the sets  $G_S \subset X^\nu$  for every  $S \in \text{Set}^1 X$  uniquely determine the curves  $Y_{S,i}^\nu \subset X^\nu$  and the involution  $\iota_S$ , for every  $S \in \text{Set}^1 X$ .*

*Proof.* Pick a C1-set  $S$  and let  $h = \#S$ . Denote  $G_S := \{r_1, \dots, r_{2h}\}$  and  $Y_S = \coprod_{i=1}^h Y_i$ . We have

$$(2.5) \quad \#G_S \cap Y_i^\nu = 2$$

for every  $i$ . Consider the point  $r_1$  and let us show that the component, call it  $Y_1^\nu$ , containing  $r_1$  is uniquely determined. Let  $C_1 \subset X^\nu$  be the irreducible component containing  $r_1$ ; of course  $C_1 \subset Y_1^\nu$ .

Now, by Remark 2.2.2, for every  $T \in \text{Set}^1 X$  such that  $S \neq T$ , we have that if  $G_T \cap C_1 \neq \emptyset$  then  $T$  is entirely contained in the singular locus of  $Y_1$ . In particular every irreducible component of  $X^\nu$  intersecting  $G_T$  is contained in  $Y_1^\nu$ . Define the following subcurve of  $X^\nu = \cup C_i$

$$Z_1 := C_1 \cup \bigcup_{\substack{\exists T \in \text{Set}^1 X \setminus \{S\}: \\ C_i \cap G_T \neq \emptyset, C_1 \cap G_T \neq \emptyset}} C_i.$$

We can apply Remark 2.2.2 to  $Z_1$  as above, obtaining that if  $X$  has a C1-set  $T \neq S$  such that  $G_T$  intersects  $Z_1$ , then again  $T \subset (Y_1)_{\text{sing}}$ ; therefore every component of  $X^\nu$  intersecting  $G_T$  is contained in  $Y_1^\nu$ . We can hence inductively define the following subcurve of  $Y_1^\nu$ : for  $n \geq 2$  set

$$Z_n := Z_{n-1} \cup \bigcup_{\substack{\exists T \in \text{Set}^1 X \setminus \{S\}: \\ C_i \cap G_T \neq \emptyset, Z_{n-1} \cap G_T \neq \emptyset}} C_i.$$

Since  $Y_1$  is connected and all of its nodes belong to some C1-set of  $X$ , for  $n$  large enough we have

$$Z_n = Z_{n+1} = \dots = Y_1^\nu.$$

Hence  $Y_1^\nu$  is uniquely determined. Now, by (2.5) we have that  $Y_1^\nu \cap G_S = \{r_1, r_j\}$  for a unique  $j \neq 1$ ; therefore we must have  $\iota_S(r_1) = r_j$ . This proves that the curves  $Y_{S,i}^\nu \subset X^\nu$  are all determined, and so are the involutions  $\iota_S$ .  $\blacksquare$

**2.3.2. Gluing data of  $X$ .** What extra data should one specify to reconstruct  $X$  from its C1-equivalence class? We are going to give an answer to this question. Fix  $S \in \text{Set}^1 X$ , let  $h = \#S$  and  $Y_S = \coprod_1^h Y_i$ . By Lemma 2.3.1 the C1-equivalence class of  $X$  determines the involution  $\iota_S$  of  $G_S$ . This enables us to write  $G_S = \{p_1, q_1, \dots, p_h, q_h\}$  with  $p_i, q_i \in Y_i^\nu$ . Of course this is not enough to determine how  $G_S$  is glued on  $X$ . To describe what is further needed, we introduce an abstract set of cardinality  $2h$ , denoted  $G_h = \{s_1, t_1, \dots, s_h, t_h\}$ , endowed with the involution  $\iota_h$  defined by  $\iota_h(s_i) = t_i$  for every  $1 \leq i \leq h$ .

Pick either one of the two cyclic orientations of  $\Gamma_X(S)$ . We claim that the gluing data of  $G_S$  determine, and are uniquely determined by, the following two items.

- (1) A marking  $\psi_S : (G_h, \iota_h) \xrightarrow{\cong} (G_S, \iota_S)$ , where  $\psi_S$  is a bijection between  $G_h$  and  $G_S$  mapping the pair  $(s_i, t_i)$  to the pair  $(p_i, q_i)$ .
- (2) A cyclic permutation  $\sigma_S \in \mathcal{S}_h$ , free from fixed points.

Indeed the points  $\psi_S(s_i)$  and  $\psi_S(t_i)$  correspond, respectively, to the sources and targets of the orientation of  $\Gamma_X(S)$ ; the permutation  $\sigma_S$  is uniquely determined by the fact that the point  $\psi_S(s_i)$  is glued to the point  $\psi_S(t_{\sigma_S(i)})$ . The opposite cyclic orientation of  $\Gamma_X(S)$  corresponds to changing

$$(2.6) \quad (\sigma_S, \psi_S) \mapsto (\sigma_S^{-1}, \psi_S \circ \iota_h);$$

the above transformation defines an involution on the set of pairs  $(\sigma_S, \psi_S)$  as above. We call the equivalence class  $[(\sigma_S, \psi_S)]$ , with respect to the above involution, the *gluing data* of  $S$  on  $X$ .

Conversely, it is clear that a nodal curve  $X$  is uniquely determined, within its C1-equivalence class, by an equivalence class  $[(\sigma_S, \psi_S)]$  as above for each C1-set  $S \in \text{Set}^1 X$ . In fact,  $X$  is given as follows

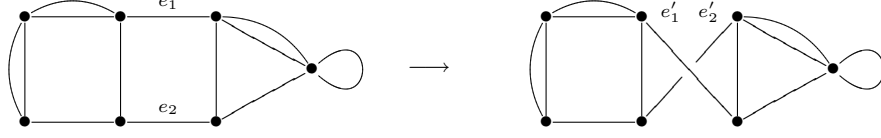
$$X = \overline{\coprod_{S \in \text{Set}^1 X} \{ \psi_S(s_i) = \psi_S(t_{\sigma_S(i)}) : 1 \leq i \leq \#S \}}.$$

The previous analysis would enable us to explicitly, and easily, bound the cardinality of any C1-equivalence class. We postpone this to the final section of the paper; see Lemma 5.1.5.

**2.4. Dual graphs of C1-equivalent curves.** In this subsection, we shall prove that two C1-equivalent curves have cyclically equivalent dual graphs. As a matter of fact, we will prove a slightly stronger result. We need the following

**Definition 2.4.1.** Let  $\Gamma$  and  $\Gamma'$  be two graphs free from separating edges. We say that  $\Gamma$  and  $\Gamma'$  are *strongly cyclically equivalent* if they can be obtained from one another via iterated applications of the following move, called *twisting at a separating pair of edges*:

The above picture means the following. Since  $e_1, e_2$  is a separating pair of edges, we have that  $\Gamma \setminus \{e_1, e_2\}$  has two connected components, call them  $\Gamma_a$  and  $\Gamma_b$ . For  $i = 1, 2$  call  $v_i^a$  (resp.  $v_i^b$ ) the vertex of  $\Gamma_a$  (resp. of  $\Gamma_b$ ) adjacent to  $e_i$ . Then  $\Gamma'$  is obtained by joining the two graphs  $\Gamma_a$  and  $\Gamma_b$  by an edge  $e'_1$  from  $v_1^a$  to  $v_2^b$  and by another edge  $e'_2$  from  $v_2^a$  to  $v_1^b$ . Notice that if  $v_1^a = v_2^a$  and  $v_1^b = v_2^b$ , our twisting operation does not change the isomorphism class of the graph.

FIGURE 1. A twisting at the separating pair of edges  $\{e_1, e_2\}$ .

*Remark 2.4.2.* If  $\Gamma$  and  $\Gamma'$  are strongly cyclically equivalent then they are cyclically equivalent.

This fact is intuitively clear. A cyclic bijection  $E(\Gamma) \rightarrow E(\Gamma')$  can be obtained by mapping every separating pair of edges at which a twisting is performed to its image (which is a separating pair of edges of  $\Gamma'$ ). To check that this bijection preserves the cycles it suffices to observe that if two edges form a separating pair then they belong to the same cycles.

Alternatively, the twisting at a separating pair of edges is a particular instance of the so-called second move of Whitney, and it is well-known that this move does not change the cyclic equivalence class of a graph (see [Whi33]).

**Proposition 2.4.3.** *Let  $X$  and  $X'$  be free from separating nodes and  $C1$ -equivalent. Then  $\Gamma_X$  and  $\Gamma_{X'}$  are strongly cyclically equivalent (and hence cyclically equivalent).*

*Proof.* It will be enough to show that for every  $C1$ -set  $S \in \text{Set}^1 X$ , any two gluing data  $(\sigma_S, \psi_S)$  associated to  $S$  can be transformed into one another by a sequence of edge twistings of the type described in 2.4.1. Moreover, it is enough to consider one  $C1$ -set at the time, in fact by 2.2.2, the twisting at a separating pair of edges  $\{e_1, e_2\}$  belonging to  $S \in \text{Set}^1 X$  does not affect the gluing data of the other  $C1$ -sets.

So let us fix  $S \in \text{Set}^1 X$  of cardinality  $h$  and let  $[(\sigma_S, \psi_S)]$  be the gluing data of  $S$  on  $X$ . We consider two types of edge-twisting, as in 2.4.1:

- (a) Fix a component  $Y_j$  of  $Y_S$ , exchange the two gluing points lying on  $Y_j$ ,  $\psi_S(s_j)$  and  $\psi_S(t_j)$ , and leave everything else unchanged. On  $\Gamma_X$  this operation corresponds to a twisting at the separating pair of edges of  $S$  that join  $\Gamma_{Y_j}$  with  $\Gamma_{Y_S \setminus Y_j}$  (both viewed as subgraphs of  $\Gamma_X$ ). The gluing data are changed according to the rule

$$[(\sigma_S, \psi_S)] \mapsto [(\sigma_S, \psi_S \circ \text{inv}_j)],$$

where  $\text{inv}_j$  is the involution of  $\{s_1, t_1, \dots, s_h, t_h\}$  exchanging  $s_j$  with  $t_j$  and fixing everything else.

- (b) Fix a component  $Y_j$  of  $Y_S$  and an integer  $1 \leq a \leq h-1$ . Consider the curve

$$Z = Y_j \amalg Y_{\sigma_S(j)} \amalg \dots \amalg Y_{\sigma_S^a(j)} \subset Y_S.$$

Now change the gluing data between  $Z$  and  $Y_S \setminus Z$  by exchanging the two points of  $Z$  that are glued to  $Y_S \setminus Z$ , and leaving everything else unchanged. On  $\Gamma_X$  this operation corresponds to a twisting at the separating pair of edges of  $S$  that join  $\Gamma_Z$  to  $\Gamma_{Y_S \setminus Z}$ . The gluing data are changed according to the rule

$$[(\sigma_S, \psi_S)] \mapsto [(\tau_{j,a} \circ \sigma_S \circ \tau_{j,a}^{-1}, \psi_S \circ \text{inv}_{j,a})],$$

where  $\tau_{j,a}$  is the element of  $\mathcal{S}_h$  defined by

$$\tau_{j,a} := \prod_{0 \leq b \leq \lfloor a-1/2 \rfloor} (\sigma_S^b(j) \sigma_S^{a-b}(j))$$

and  $\text{inv}_{j,a}$  is the involution of  $\{s_1, t_1, \dots, s_h, t_h\}$  that exchanges  $s_k$  with  $t_k$ , for all  $k = j, \sigma_S(j), \dots, \sigma_S^a(j)$ , and fixes all the other elements.

The proof will be complete if we show that all the possible gluing data of  $S$  can be obtained starting from  $[(\sigma_S, \psi_S)]$  and performing operations of type (a) and (b).

First of all observe that, by iterating operations of type (a), it is possible to arbitrarily modify the marking  $\psi_S$ , while keeping the cyclic permutation  $\sigma_S$  fixed.

On the other hand, using the fact that any two cyclic permutations of  $\mathcal{S}_h$  are conjugate, and that  $\mathcal{S}_h$  is generated by transpositions, it will be enough to show that for any transposition  $(jk) \in \mathcal{S}_h$ , by iterating operations of type (b), we can pass from the gluing data  $[(\sigma_S, \psi_S)]$  to gluing data of the form  $[(jk) \circ \sigma_S \circ (jk)^{-1}, \psi'_S]$  for some marking  $\psi'_S$ . If the transposition  $(jk)$  is such that  $k = \sigma_S(j)$  (resp.  $k = \sigma_S^2(j)$ ), then it is enough to apply the operation (b) with respect to the component  $Y_j$  and the integer  $a = 1$  (resp.  $a = 2$ ). In the other cases, we can write  $k = \sigma_S^a(j)$  with  $3 \leq a \leq h - 1$  and then we apply the operation (b) two times: first with respect to the component  $Y_{\sigma_S(j)}$  and the integer  $a - 2$ ; secondly with respect to the component  $Y_j$  and the integer  $a$ . After these two operations the cyclic permutation  $\sigma_S$  gets changed to  $(jk) \circ \sigma_S \circ (jk)^{-1}$  since

$$(jk) = (j\sigma_S^a(j)) = \tau_{j,a} \circ \tau_{\sigma_S(j),a-2}.$$

■

### 3. T-EQUIVALENCE: A SECOND VERSION OF THE TORELLI THEOREM

The statement of Theorem 2.1.4 characterizes curves having isomorphic ppSSAV in terms of their normalization, and of the C1-partition of their gluing points, determined by the codimension-one strata of the compactified Picard scheme.

In this section we shall give a different characterization, based on the classifying morphism of the generalized Jacobian. From the general theory of semiabelian varieties, recall that the generalized Jacobian  $J(X)$  of a nodal curve  $X$  is an extension

$$1 \longrightarrow H^1(\Gamma_X, k^*) = \mathbb{G}_m^{b_1(\Gamma_X)} \longrightarrow J(X) \longrightarrow J(X^\nu) = \prod_{i=1}^{\gamma} J(C_i) \longrightarrow 0$$

where recall that we denote by  $\cup_{i=1}^{\gamma} C_i = X^\nu$  the normalization of  $X$ . The above extension is determined by the so-called classifying morphism, from the character group of the torus  $H^1(\Gamma_X, k^*)$ , i.e. from  $H_1(\Gamma_X, \mathbb{Z})$ , to the dual abelian variety of  $J(X^\nu)$ . Since  $J(X^\nu)$  is polarized by the Theta divisor, its dual variety can be canonically identified with  $J(X^\nu)$  itself. So the classifying morphism in our case takes the form

$$c_X : H_1(\Gamma_X, \mathbb{Z}) \longrightarrow J(X^\nu).$$

This morphism  $c_X$  has a very explicit description, as we are going to illustrate. We shall use the groups of divisors and line bundles having degree 0 on every component:

$$\prod_{i=1}^{\gamma} \text{Div}^0 C_i = \text{Div}^0 X^\nu \longrightarrow \text{Pic}^0 X^\nu = \prod_{i=1}^{\gamma} \text{Pic}^0 C_i = J(X^\nu).$$

#### 3.1. Definition of T-equivalence.

**3.1.1.** Fix an orientation of  $\Gamma_X$  and consider the source and target maps

$$s, t : E(\Gamma_X) \rightarrow V(\Gamma_X).$$

Observe that  $s(e)$  and  $t(e)$  correspond naturally to the two points of  $X^\nu$  lying over the node corresponding to  $e$ . We call  $s_e, t_e \in X^\nu$  such points.

The usual boundary map is defined as follows

$$\partial : C_1(\Gamma_X, \mathbb{Z}) \rightarrow C_0(\Gamma_X, \mathbb{Z}); \quad e \mapsto t(e) - s(e)$$

and  $H_1(\Gamma_X, \mathbb{Z}) = \ker \partial$ . We now introduce the map

$$\widetilde{\eta}_X : C_1(\Gamma_X, \mathbb{Z}) \rightarrow \text{Div}^0 X^\nu; \quad e \mapsto t_e - s_e.$$

We will denote by  $\eta_X$  the restriction of  $\widetilde{\eta}_X$  to  $H_1(\Gamma_X, \mathbb{Z})$ , which is easily seen to take values in the subgroup,  $\text{Div}^0 X^\nu$ , of divisors having degree 0 on every component.

Summarizing, we have a commutative diagram

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \text{Div}^0 X^\nu \\ \downarrow & & \downarrow \\ C_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\widetilde{\eta}_X} & \text{Div}^0 X^\nu. \end{array}$$

The classifying morphism  $c_X : H_1(\Gamma_X, \mathbb{Z}) \rightarrow J(X^\nu)$  of  $J(X)$  is obtained by composing the homomorphism  $\eta_X : H_1(\Gamma_X, \mathbb{Z}) \rightarrow \text{Div}^0 X^\nu$  with the quotient map  $\text{Div}^0 X^\nu \rightarrow \text{Pic}^0 X^\nu = J(X^\nu)$  sending a divisor to its linear equivalence class. See [Ale04, Sec. 2.4] or [Bri07, Sec. 1.3].

**3.1.2.** There are automorphisms of  $\text{Pic}^0 X^\nu$  and  $\text{Div}^0 X^\nu$  that do not change the isomorphism class of  $J(X)$ . We need to take those into account. In order to do that, consider the group  $K_\gamma := (\mathbb{Z}/2\mathbb{Z})^\gamma$ ; note that it acts diagonally as subgroup of automorphisms,  $K_\gamma \hookrightarrow \text{Aut}(\text{Div}^0 X^\nu)$ ,  $K_\gamma \hookrightarrow \text{Aut}(\text{Div} X^\nu)$ , and  $K_\gamma \hookrightarrow \text{Aut}(\text{Pic}^0 X^\nu)$ , via multiplication by  $-1$  on each factor. We shall usually identify  $K_\gamma$  with the image of the above monomorphisms.

For example, if  $X^\nu = C_1 \cup C_2$  then  $K_2 \subset \text{Aut}(\text{Div}^0 X^\nu)$  is generated by the involutions  $(D_1, D_2) \mapsto (-D_1, D_2)$  and  $(D_1, D_2) \mapsto (D_1, -D_2)$ .

**Definition 3.1.3** (T-equivalence). We say that two nodal connected curves  $X$  and  $X'$  are *T-equivalent* if the following conditions hold.

- (a) There exists an isomorphism  $\phi : X^\nu \xrightarrow{\cong} X'^\nu$  between their normalizations.
- (b)  $\Gamma_X \cong_{\text{cyc}} \Gamma_{X'}$ .
- (c) For every orientation on  $\Gamma_X$  there exists an orientation on  $\Gamma_{X'}$  and an automorphism  $\alpha \in K_\gamma \subset \text{Div}^0 X^\nu$  such that the following diagram commutes

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \text{Div}^0 X^\nu \\ \cong \downarrow \epsilon_H & & \cong \downarrow \phi_D \circ \alpha \\ H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \text{Div}^0 X'^\nu \end{array}$$

where  $\epsilon_H$  is defined in 1.2.2 and  $\phi_D : \text{Div}^0 X^\nu \rightarrow \text{Div}^0 X'^\nu$  is the isomorphism induced by  $\phi$ .

We say that two non connected nodal curves  $Y$  and  $Y'$  are T-equivalent if there exists a bijection between their connected components such that every two corresponding components are T-equivalent.

We shall prove in 3.2.1 that two curves free from separating nodes are T-equivalent if and only if they are C1-equivalent, thereby getting a new statement of Theorem 2.1.4. We first need some observations.

*Remark 3.1.4.* Let  $X$  and  $X'$  be T-equivalent and free from separating nodes. Then part (c) of the definition implies that

$$\phi(\nu^{-1}(X_{\text{sing}})) = \nu'^{-1}(X'_{\text{sing}}),$$

where  $X^\nu \xrightarrow{\nu} X$  and  $X'^\nu \xrightarrow{\nu'} X'$  are the normalization maps.

*Remark 3.1.5.* Suppose that  $\Gamma_X$  and  $\Gamma_{X'}$  are cyclically equivalent and fix a cyclic bijection  $\epsilon : E(\Gamma_X) \rightarrow E(\Gamma_{X'})$ . Then by [CV09, Cor. 2.3.5]) there is a natural bijection from the C1-sets of  $X$  to the C1-sets of  $X'$ , mapping  $S$  to  $\epsilon(S)$ . For this bijection we shall always use the following notation

$$\text{Set}^1 X \longrightarrow \text{Set}^1 X'; \quad S \mapsto S'.$$

Notice also that  $\#S = \#S'$ .

**Lemma 3.1.6.** *Let  $X$  and  $X'$  be T-equivalent connected curves, free from separating nodes; pick a pair of corresponding C1-sets,  $S \in \text{Set}^1 X$  and  $S' \in \text{Set}^1 X'$ . Then the normalization of  $X$  at  $S$  is T-equivalent to the normalization of  $X'$  at  $S'$ .*

*Proof.* Let  $Y$  be the normalization of  $X$  at  $S$  and  $Y'$  the normalization of  $X'$  at  $S'$ . It is obvious that  $Y$  and  $Y'$  have isomorphic normalizations, since this holds for  $X$  and  $X'$ , by definition.

Observe that  $\Gamma_Y = \Gamma_X \setminus S$  and  $\Gamma_{Y'} = \Gamma_{X'} \setminus S'$ . The bijection  $\epsilon : E(\Gamma_X) \rightarrow E(\Gamma_{X'})$  maps the edges of  $S$  to the edges of  $S'$ ; hence it induces a bijection  $\epsilon_Y : E(\Gamma_Y) \rightarrow E(\Gamma_{Y'})$ . To see that  $\epsilon_Y$  induces a bijection on the cycles it suffices to observe that the cycles of  $\Gamma_Y = \Gamma_X \setminus S$  are precisely the cycles of  $\Gamma_X$  which do not contain  $S$  (by Lemma 2.2.1), and the same holds for  $Y'$ . Therefore  $\Gamma_Y$  and  $\Gamma_{Y'}$  are cyclically equivalent.

Finally, let us pick an orientation on  $\Gamma_Y$  and extend it to an orientation on  $\Gamma_X$ . The map  $\eta_Y$  naturally factors

$$\eta_Y : H_1(\Gamma_X \setminus S, \mathbb{Z}) \hookrightarrow H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{\eta_X} \text{Div}^0 X^\nu.$$

Choose an orientation on  $\Gamma_{X'}$  so that condition (c) holds. Then we have a commutative diagram

$$\begin{array}{ccccc} \eta_Y : H_1(\Gamma_Y, \mathbb{Z}) & \hookrightarrow & H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \text{Div}^0 X^\nu \\ \cong \downarrow & & \cong \downarrow \epsilon_H & & \cong \downarrow \alpha \\ \eta_{Y'} : H_1(\Gamma_{Y'}, \mathbb{Z}) & \hookrightarrow & H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \text{Div}^0 X'^\nu \end{array}$$

This proves that condition (c) holds for  $Y$  and  $Y'$ , so we are done.  $\blacksquare$

### 3.2. C1-equivalence equals T-equivalence.

**Proposition 3.2.1.** *Let  $X$  and  $X'$  be connected curves free from separating nodes. Then  $X$  and  $X'$  are T-equivalent if and only if they are C1-equivalent.*

*Proof.* Suppose that  $X$  and  $X'$  are T-equivalent. Then property (A) of Definition 2.1.3 obviously holds. Let us simplify the notation by identifying  $X^\nu = X'^\nu$ . Since the dual graphs of  $X$  and  $X'$  are cyclically equivalent, we have a cardinality preserving bijection between the C1-sets of  $X$  and  $X'$ , by Remark 3.1.5. To prove part (B) of Definition 2.1.3 let  $S, S'$  be any pair as in 3.1.5, and denote, as usual,

$$G_S := \nu^{-1}(S) \subset X^\nu \quad \text{and} \quad G_{S'} := \nu'^{-1}(S) \subset X'^\nu.$$

We must prove that  $G_S = G_{S'}$ . Since  $X$  and  $X'$  are T-equivalent, by Remark 3.1.4 the gluing sets are the same:

$$(3.1) \quad G_{X_{\text{sing}}} = G_{X'_{\text{sing}}}.$$

Let  $Y$  be the normalization of  $X$  at  $S$  and  $Y'$  the normalization of  $X'$  at  $S'$ . By Lemma 3.1.6  $Y$  and  $Y'$  are T-equivalent. Now, the normalization of  $Y$  and  $Y'$  is  $X^\nu$ , and by Remark 3.1.4 applied to  $Y$  and  $Y'$  we obtain

$$(3.2) \quad G_{Y_{\text{sing}}} = G_{Y'_{\text{sing}}} \subset X^\nu.$$

Now, it is clear that  $G_S = G_{X_{\text{sing}}} \setminus G_{Y_{\text{sing}}}$  and  $G_{S'} = G_{X'_{\text{sing}}} \setminus G_{Y'_{\text{sing}}}$ . Therefore by (3.1) and (3.2) we get  $G_S = G_{S'}$  as wanted.

Conversely, assume that  $X$  and  $X'$  are C1-equivalent. By 2.4.3 their graphs are cyclically equivalent. Let us identify  $X^\nu = X'^\nu$ , so that by hypothesis  $G_S = G_{S'}$  for every pair of corresponding C1-sets. It remains to prove that property (c) of Definition 3.1.3 holds.

We begin with a preliminary definition. From 3.1.2, recall that the group  $K_\gamma = (\mathbb{Z}/2\mathbb{Z})^\gamma$  acts as subgroup of automorphisms of  $\text{Div } X^\nu = \prod_{i=1}^\gamma \text{Div } C_i$ , by the natural diagonal action defined in 3.1.2 (so that any  $\alpha \in K_\gamma$  acts on each  $\text{Div } C_i$  either as the identity or as multiplication by  $-1$ ). For every  $S \in \text{Set}^1 X$  denote as usual  $Y_1, \dots, Y_h$  the connected components of  $Y_S$  and let  $Y_i^\nu$  be the normalization of  $Y_i$ . We have  $\text{Aut}(\text{Div } X^\nu) = \prod_{i=1}^h \text{Aut}(\text{Div } Y_i^\nu)$ ; we define a subgroup of  $K_\gamma$

$$K_\gamma(S) := \{\alpha \in K_\gamma \subset \text{Aut}(\text{Div } X^\nu) : \alpha|_{\text{Div } Y_i^\nu} = \pm 1\}.$$

For example, if  $Y_S$  is connected, then  $K_\gamma(S) \cong \{+1, -1\}$ .

Let  $S$  and  $S'$  be corresponding C1-sets, as above. Let  $\Gamma = \Gamma_X$  and  $\Gamma' = \Gamma_{X'}$ . The graphs  $\Gamma(S)$  and  $\Gamma'(S')$  are cycles of length  $h = \#S = \#S'$ , whose sets of edges are naturally identified with  $S$  and  $S'$  respectively. Hence there is a natural inclusion  $C_1(\Gamma(S), \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$ ; ditto for  $S'$ . Set (notation in 3.1.1)

$$\widetilde{\eta}(S) := \widetilde{\eta}_{X|C_1(\Gamma(S))} : C_1(\Gamma(S)) \longrightarrow \text{Div}^0 X^\nu \subset \text{Div } X^\nu; \quad e \mapsto t_e - s_e$$

(where above and throughout the rest of the proof we omit  $\mathbb{Z}$ ). For any orientation on  $\Gamma(S)$  we let  $\eta(S)$  be the restriction of  $\widetilde{\eta}(S)$  to  $H_1(\Gamma(S))$  we have a map

$$(3.3) \quad \widetilde{\eta}_{X|H_1(\Gamma(S))} = \eta(S) : H_1(\Gamma(S)) \longrightarrow \text{Div } X^\nu.$$

We define  $\eta(S') : H_1(\Gamma'(S')) \rightarrow \text{Div } X^\nu$  analogously. Let us describe  $\eta(S)$  and  $\eta(S')$ . Since  $\Gamma(S)$  is a cycle for any choice of orientation we have a choice of two generators of  $H_1(\Gamma(S)) \cong \mathbb{Z}$ . We pick one of them and call it  $c_S$ . Write  $G_S = \{p_1, q_1; \dots; p_h, q_h\}$  as in (2.3.2). Up to reordering the components  $Y_1, \dots, Y_h$  and switching  $p_i$  with  $q_i$  we may assume that

$$(3.4) \quad \eta(S)(c_S) = \sum_{i=1}^h (q_i - p_i).$$

Notice that the choice of orientation is irrelevant; more precisely for any orientation and any generator  $\widetilde{c}_S$  of  $H_1(\Gamma(S))$  we have that  $\eta(S)(\widetilde{c}_S) = \pm \sum_{i=1}^h (q_i - p_i)$ .

Similarly, make a choice of orientation for  $\Gamma'(S')$  and pick a generator  $c_{S'}$  of  $H_1(\Gamma'(S'))$ . Then one easily checks that there exists a partition  $\{1, \dots, h\} = F \cup G$  in two disjoint sets,  $F$  and  $G$ , such that we have

$$(3.5) \quad \eta(S')(c_{S'}) = \sum_{i \in F} (q_i - p_i) + \sum_{i \in G} (p_i - q_i).$$

Let  $\alpha(S) \in K_\gamma(S) \subset \text{Aut}(\text{Div } X^\nu)$  be the automorphism whose restriction to  $\text{Div } Y_i^\nu$  is the identity for  $i \in F$ , and it is multiplication by  $-1$  for  $i \in G$ . Now let

$$\epsilon(S) : H_1(\Gamma(S)) \xrightarrow{\cong} H_1(\Gamma'(S'))$$

be the isomorphism mapping  $c_S$  to  $c_{S'}$ . By construction  $\eta(S) = \alpha(S) \circ \eta(S') \circ \epsilon(S)$ , i.e. the map  $\eta(S)$  factors as follows

$$(3.6) \quad \eta(S) : H_1(\Gamma(S)) \xrightarrow{\epsilon(S)} H_1(\Gamma'(S')) \xrightarrow{\eta(S')} \text{Div } X^\nu \xrightarrow{\alpha(S)} \text{Div } X^\nu.$$

We repeat the above construction for every pair of corresponding C1-sets  $(S, S')$ .

Using Lemma 2.2.3 and (3.3) we have

$$\eta_X = \left( \bigoplus_{S \in \text{Set}^1 X} \eta(S) \right)_{|H_1(\Gamma)} \quad \text{and} \quad \eta_{X'} = \left( \bigoplus_{S' \in \text{Set}^1 X'} \eta(S') \right)_{|H_1(\Gamma')}.$$

Now let

$$\alpha := \prod_{S \in \text{Set}^1 X} \alpha(S) \in K_\gamma$$

where the product above means composition of the  $\alpha(S)$  is any chosen order. We claim that for every fixed  $S \in \text{Set}^1 X$  we have

$$\alpha \circ \eta(S') = \pm \alpha(S) \circ \eta(S').$$

Indeed, by 2.2.2, for any  $T \in \text{Set}^1 X$ , with  $T \neq S$ ,  $S$  is entirely contained in the singular locus of a unique connected component of  $Y_T$ , call it  $Y_{T,1}$ . Therefore the gluing set  $G_{S'} = G_S$  is entirely contained in  $Y_{T,1}'$ . By definition,  $\alpha(T)$  acts either as the identity or as multiplication by  $-1$  on every divisor of  $X'$  supported on  $Y_{T,1}'$ ; in particular  $\alpha(T)$  acts by multiplication by  $\pm 1$  on  $\eta(S')(c_{S'})$ . The claim is proved.

As a consequence of this claim and of 3.6 we have

$$\alpha \circ \eta(S') \epsilon(S) = \pm \eta(S).$$

Now, if for a certain  $S$  we have that the above identity holds with a minus sign on the right, we change the isomorphism  $\epsilon(S)$  into  $-\epsilon(S)$ , but we continue to denote it  $\epsilon(S)$  for simplicity.

Using again Lemma 2.2.3 we let  $\epsilon_X : H_1(\Gamma) \xrightarrow{\cong} H_1(\Gamma')$  be the restriction to  $H_1(\Gamma)$  of the isomorphism

$$\bigoplus_{S \in \text{Set}^1 X} : \bigoplus_{S \in \text{Set}^1 \Gamma} H_1(\Gamma(S)) \xrightarrow{\cong} \bigoplus_{S' \in \text{Set}^1 \Gamma'} H_1(\Gamma'(S')).$$

It is trivial to check that  $\epsilon_X$  is an isomorphism. In fact by the proof of Proposition 2.4.3 it is clear that  $\epsilon_X$  induces the given bijection between the C1-sets of  $X$  and  $X'$ . Combining and concluding, we have a commutative diagram

$$(3.7) \quad \begin{array}{ccc} \eta_X : H_1(\Gamma) \hookrightarrow \bigoplus_{S \in \text{Set}^1 \Gamma} H_1(\Gamma(S)) \xrightarrow{\bigoplus \eta(S)} \text{Div } X^\nu & & \\ \epsilon_X \downarrow \cong & & \alpha \downarrow \cong \\ \eta_{X'} : H_1(\Gamma') \hookrightarrow \bigoplus_{S' \in \text{Set}^1 \Gamma'} H_1(\Gamma'(S')) \xrightarrow{\bigoplus \eta(S')} \text{Div } X'^\nu & & \end{array}$$

so we are done. ■

#### 4. PROOF OF THE MAIN THEOREM

The hard part of the proof of Theorem 2.1.4 is the necessary condition, i.e. the fact that if two stable curves,  $X$  and  $X'$ , with no separating nodes, have the same image under the Torelli map (i.e. have isomorphic ppSSAV) they are C1-equivalent. Let us outline the proof, to which the next subsections, till 4.3, are devoted. Using the  $J(X)$ -stratification of  $\overline{P_X^{g-1}}$  we shall prove that the graphs of  $X$  and  $X'$  are cyclically equivalent, and hence that there is a bijection between their C1-sets. The main technical tool here is Lemma 4.1.5(2), which is the goal of subsection 4.1. Next, the fact that the normalizations of  $X$  and  $X'$  are isomorphic follows easily. Finally, to recover the gluing points, together with their partitions determined by the C1-sets, we use the theta divisors. This is done in subsection 4.2.

**4.1. Recovering the combinatorics of  $X$  from that of  $\overline{P_X^{g-1}}$ .** We now study the precise relation between the posets  $\mathcal{ST}_X$  and  $\mathcal{SP}_X$ , defined in 2.1. The goal of this subsection is to prove, in 4.1.5, that the support map  $\text{Supp}_X : \mathcal{ST}_X \rightarrow \mathcal{SP}_X$  is a quotient of posets, i.e. given  $S, T \in \mathcal{SP}_X$  we have  $S \geq T$  if and only if there exists  $P_S^{\underline{d}}$  and  $P_T^{\underline{e}}$  in  $\mathcal{ST}_X$  such that  $P_S^{\underline{d}} \geq P_T^{\underline{e}}$ .

The poset  $\mathcal{SP}_X$  can be defined purely in terms of the dual graph of  $X$ . Namely as the poset  $\mathcal{SP}_{\Gamma_X}$  of all  $S \subset E(\Gamma_X)$  such that  $\Gamma_X \setminus S$  is free from separating edges, ordered by reverse inclusion. In [CV09] we studied  $\mathcal{SP}_\Gamma$  for any graph and used it to characterize the cyclic equivalence class of  $\Gamma$  in a way that we shall apply in the present paper.

To unravel the combinatorial nature of  $\mathcal{ST}_X$ , observe that its elements correspond to pairs,  $(S, \underline{d})$  where  $S \in \mathcal{SP}_X$  and  $\underline{d}$  is a stable multidegree on the curve  $Y_S$ . Now, stable multidegrees can be defined in terms of certain orientations, called totally cyclic, on  $\Gamma_X$ . Let us make this precise

**Definition 4.1.1.** If  $\Gamma$  is a connected graph, an orientation of  $\Gamma$  is *totally cyclic* if there exists no proper non-empty subset  $W \subset V(\Gamma)$  such that the edges between  $W$  and its complement  $V(\Gamma) \setminus W$  go all in the same direction.

If  $\Gamma$  is not connected, an orientation is totally cyclic if its restriction to each connected component of  $\Gamma$  is totally cyclic.

The poset  $\mathcal{OP}_\Gamma$  is defined as the set

$$\mathcal{OP}_\Gamma = \{ \phi_S : \phi_S \text{ is a totally cyclic orientation on } \Gamma \setminus S, \forall S \in \mathcal{SP}_\Gamma \}$$

together with the following partial order:

$$\phi_S \geq \phi_T \Leftrightarrow S \subset T \text{ and } \phi_T = (\phi_S)|_{\Gamma \setminus T}.$$

**4.1.2.** The connection with stable multidegrees is the following (see [B77, Lemma 2.1] and [Cap07, sec.1.3.2]). Pick  $Y_S$  and any totally cyclic orientation  $\phi_S$  on  $\Gamma_{Y_S} = \Gamma \setminus S$ ; for every vertex  $v_i$  call  $d^+(\phi_S)_{v_i}$  the number of edges of  $\Gamma \setminus S$  that start from  $v_i$  according to  $\phi_S$ . Now we define a multidegree  $\underline{d}(\phi_S)$  on  $Y_S$  as follows

$$(4.1) \quad \underline{d}(\phi_S)_{v_i} := g_i - 1 + d^+(\phi_S)_{v_i}, \quad i = 1, \dots, \gamma,$$

where  $g_i$  is the geometric genus of the component corresponding to  $v_i$ . With this notation, a multidegree  $\underline{d}$  is stable on  $Y_S$  if and only if there exists a totally cyclic orientation  $\phi_S$  such that  $\underline{d} = \underline{d}(\phi_S)$ .

Obviously, two totally cyclic orientations define the same multidegree if and only if the number of edges departing from every vertex is the same. We shall regard two such orientations as equivalent:

**Definition 4.1.3.** Two orientations  $\phi_S$  and  $\phi_T$  of  $\mathcal{OP}_\Gamma$  are equivalent if  $S = T$  and if  $d^+(\phi_S)_v = d^+(\phi_T)_v$  for every vertex  $v$ . The set of equivalence classes of orientations will be denoted by  $\overline{\mathcal{OP}_\Gamma}$ .

The quotient map  $\mathcal{OP}_\Gamma \rightarrow \overline{\mathcal{OP}_\Gamma}$  induces a unique poset structure on  $\overline{\mathcal{OP}_\Gamma}$ , such that two classes  $[\phi_S], [\phi_T] \in \overline{\mathcal{OP}_\Gamma}$  satisfy  $[\phi_S] \geq [\phi_T]$  if there exist respective representatives  $\phi_S$  and  $\phi_T$  such that  $\phi_S \geq \phi_T$  in  $\mathcal{OP}_\Gamma$ .

*Remark 4.1.4.* We have natural maps

$$\begin{array}{ccc} \text{Supp}_\Gamma : \mathcal{OP}_\Gamma & \longrightarrow & \overline{\mathcal{OP}_\Gamma} \xrightarrow{\text{Supp}_\Gamma} \mathcal{SP}_\Gamma \\ \phi_S \mapsto & & [\phi_S] \mapsto S \end{array}$$

which are order-preserving and surjective (the first by definition, the second by [CV09, Lemma 2.4.3(1)]). Moreover all of the above maps are quotients of posets, by [CV09, Lemma 5.3.1]. In particular, the poset  $\mathcal{SP}_\Gamma$  is completely determined by either  $\mathcal{OP}_\Gamma$  or  $\overline{\mathcal{OP}_\Gamma}$ .

Now, by what we said, to every  $\phi_S \in \mathcal{OP}_\Gamma$  we can associate a stable multidegree  $\underline{d}(\phi_S)$  of  $Y_S$  (see (4.1)); moreover two equivalent orientations define the same multidegree. This enables us to associate a unique stratum of  $\overline{P_X^{g-1}}$  to every orientation in  $\mathcal{OP}_\Gamma$ , and to every orientation class; i.e. we can define two maps  $st_X$  and  $\overline{st_X}$

$$(4.2) \quad \begin{array}{ccc} st_X : \mathcal{OP}_{\Gamma_X} & \longrightarrow & \overline{\mathcal{OP}_\Gamma} \xrightarrow{\overline{st_X}} ST_X \\ \phi_S \mapsto & & [\phi_S] \mapsto P_S^{\underline{d}(\phi_S)}. \end{array}$$

**Lemma 4.1.5.** *Let  $X$  be connected and free from separating nodes. Then*

- (1) *The map  $st_X : \overline{\mathcal{OP}_{\Gamma_X}} \rightarrow ST_X$  is an isomorphism of posets.*
- (2) *The poset  $\mathcal{SP}_X$  is completely determined by the poset  $ST_X$ .*

*Proof.* The analysis preceding the statement can be summarized in a commutative diagram of posets:

$$(4.3) \quad \begin{array}{ccc} \mathcal{OP}_{\Gamma_X} & \xrightarrow{st_X} & ST_X \\ \downarrow \text{Supp}_{\Gamma_X} & \searrow & \nearrow \cong \\ & \overline{\mathcal{OP}_{\Gamma_X}} & \nearrow \overline{st_X} \\ \downarrow \text{Supp}_{\Gamma_X} & \swarrow \overline{\text{Supp}_{\Gamma_X}} & \downarrow \text{Supp}_X \\ \mathcal{SP}_{\Gamma_X} & \xlongequal{\quad} & \mathcal{SP}_X \end{array}$$

The maps  $st_X$  and  $\overline{st_X}$  are surjective by what we said in 4.1.2. Moreover, by [Cap94, Prop. 5.1], they are morphisms of posets. From the definitions (4.1) and 4.1.3 it is clear that  $\overline{st_X}$  is bijective, and hence an isomorphism of posets.

Now, the map  $\overline{\text{Supp}_{\Gamma_X}} : \overline{\mathcal{OP}_{\Gamma_X}} \rightarrow \mathcal{SP}_X$  completely determines  $\mathcal{SP}_X$  as poset (see Remark 4.1.4), hence part (2) follows from part (1).  $\blacksquare$

## 4.2. Recovering gluing points from the Theta divisor.

**Lemma 4.2.1.** *Let  $P_S^{\underline{d}}$  be a codimension-1 stratum of  $\overline{P_X^{g-1}}$  and let  $h$  be the number of irreducible components of  $\Theta(X) \cap P_S^{\underline{d}}$ . Then  $S$  is a  $C1$ -set of cardinality  $h$ .*

*Proof.* We have already proved most of the statement in 2.1. The only part that needs to be justified is the one concerning  $\Theta(X)$ . By 2.1.2 every connected component of  $Y_S$  has positive genus. Now, according to Fact 1.2.10(iii), the number  $h$  of irreducible components of  $\Theta(X) \cap P_S^{\underline{d}}$  is equal to  $\gamma_S = \#S$ .  $\blacksquare$

**4.2.2.** The following set-up will be fixed throughout the rest of this subsection.  $X$  is a stable curve of genus  $g$ ,  $X_{\text{sep}} = \emptyset$ , and  $S \in \text{Set}^1 X$  is a  $C1$ -set of cardinality  $h$ . As usual  $\nu_S : Y_S \rightarrow X$  denotes the normalization at  $S$ . We have  $Y_S = \coprod_{i=1}^h Y_i$ , with  $Y_i$  connected, of arithmetic genus  $g_i > 0$ , free from separating nodes. The gluing set of  $\nu_S$  is denoted  $\{p_1, q_1, \dots, p_h, q_h\}$  with  $\nu_S(p_j) = \nu_S(q_{j+1})$  and  $p_j, q_j \in Y_j$ .

The pull-back via the partial normalization  $\nu_S$  induces an exact sequence

$$0 \rightarrow k^* \rightarrow \text{Pic } X \xrightarrow{\nu_S^*} \text{Pic } Y_S = \prod_{i=1}^h \text{Pic } Y_i \rightarrow 0.$$

In the following statement we use notation (1.4).

**Lemma 4.2.3.** *Fix  $\underline{d} \in \Sigma(X)$ . Let  $M$  a general line bundle in  $\text{Pic}^{\underline{d}} Y_S$ ,  $M_i := M|_{Y_i}$  and  $\underline{d}_i := \underline{\deg} M_i$ . Let  $y_i$  be a fixed smooth point of  $Y_i$ . Then for  $i = 1, \dots, h$  the following properties hold.*

- (i)  $h^0(Y_i, M_i) = 1$  (hence  $h^0(Y_S, M) = h$ ).

- (ii) Set  $\underline{d}_i(-y_i) := \underline{\deg} M_i(-y_i)$ . Then  $\underline{d}_i(-y_i) \in \Sigma^{ss}(Y_i)$ .
- (iii)  $M_i$  does not have a base point in  $y_i$  (i.e.  $h^0(Y_i, M_i(-y_i)) = 0$ ).
- (iv) The restriction of the pull-back map,  $\nu_S^* : W_{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}} Y_S$ , is birational.
- (v)  $\dim W_{\underline{d}_i}^1(Y_i) \leq g_i - 2$  for every  $1 \leq i \leq h$ .
- (vi) For any point  $p_k$ , define

$$(4.4) \quad T_{p_k} = \left\{ M \in \text{Pic}^{\underline{d}} Y_S \setminus W_{\underline{d}}^h(Y_S) : \begin{array}{l} h^0(Y_S, M(-p_k)) = h \text{ and} \\ h^0(Y_S, M(-q_j)) < h, \forall 1 \leq j \leq h. \end{array} \right\}.$$

Define  $T_{q_k}$  by replacing  $p_k$  with  $q_k$  and  $q_j$  with  $p_j$  in (4.4). Then

$$\text{Pic}^{\underline{d}} Y_S \setminus \nu_S^*(W_{\underline{d}}(X)) = \bigcup_{k=1}^h (T_{p_k} \cup T_{q_k}).$$

*Proof.* Since  $\underline{d}$  is stable, Theorem 3.1.2 of [Cap07] yields that  $W_{\underline{d}}(X)$  is irreducible of dimension  $g - 1$ .

For any  $M \in \text{Pic} Y_S$  we set  $F_M(X) := \{L \in \text{Pic} X : \nu^* L = M\} \cong k^*$ .

To prove (i), observe that the stability of  $\underline{d}$  yields  $\underline{\deg}(M_i) = g_i$ . Therefore the theorem of Riemann-Roch gives  $h^0(Y_i, M_i) \geq 1$ . Suppose, by contradiction, that  $h^0(Y_i, M_i) > 1$  for some  $i$ . Then  $h^0(Y_S, M) = \sum_{i=1}^h h^0(Y_i, M_i) \geq h + 1$  for every  $M \in \text{Pic}^{\underline{d}} Y_S$ .

This implies that  $F_M(X) \subset W_{\underline{d}}(X)$  (indeed there are at most  $h$  conditions on the global sections of  $M$  to descend to a global section of a fixed  $L \in F_M(X)$ ). Therefore

$$\dim W_{\underline{d}}(X) = \dim \text{Pic}^{\underline{d}} Y_S + 1 = \sum_{i=1}^h g_i + 1 = g$$

a contradiction. This proves (i).

For (ii) and (iii), set  $\underline{d}'_i = \underline{d}_i(-y_i)$ ; observe that  $|\underline{d}'_i| = g_i - 1$ . Let  $Z \subset Y_i$  be a subcurve of  $Y_i$  and let  $\tilde{Z} := \nu(Z) \subset X$ . Then, of course,  $g_Z \leq g_{\tilde{Z}}$ . Denoting  $d'_{i,Z} = |(\underline{d}'_i)|_Z|$  the total degree of  $\underline{d}'_i$  restricted to  $Z$ , and by  $d_{\tilde{Z}} = |\underline{d}_Z|$ , we have

$$d'_{i,Z} = \begin{cases} d_{\tilde{Z}} \geq g_{\tilde{Z}} \geq g_Z & \text{if } y_i \notin Z, \\ d_{\tilde{Z}} - 1 \geq g_{\tilde{Z}} - 1 \geq g_Z - 1 & \text{if } y_i \in Z, \end{cases}$$

where we used that  $d_{\tilde{Z}} \geq g_{\tilde{Z}}$  ( $\underline{d}$  is stable). So (ii) is proved. We can therefore use a result due to Beauville (see Proposition 1.3.7 in [Cap07]), stating that every irreducible component of  $W_{\underline{d}'_i}(Y_i)$  has dimension equal to  $g_i - 1$ , and in particular that  $W_{\underline{d}'_i}(Y_i) \neq \text{Pic}^{\underline{d}'_i}(Y_i)$ . Therefore for the general  $M \in \text{Pic}^{\underline{d}}(Y_S)$ , we have that  $h^0(Y_i, M_i(-y_i)) = 0$  and this proves part (iii).

In order to prove (iv), we need to make the isomorphism  $F_M(X) \cong k^*$  explicit. Any  $c \in k^*$  determines a unique  $L^c \in F_M(X)$ , defined as follows. For every  $j = 1, \dots, h$  consider the two fibers of  $M$  over  $p_j$  and  $q_{j+1}$  (with  $q_{h+1} = q_1$  as usual; recall that  $\nu_S$  glues  $p_j$  with  $q_{j+1}$ ) and fix an isomorphism between them. Then  $L^c \in F_M(X)$  is obtained by gluing  $M_{p_j}$  to  $M_{q_{j+1}}$  via the isomorphism

$$\begin{cases} M_{p_j} \xrightarrow{\cong} M_{q_{j+1}} & \text{for } j = 1, \dots, h-1, \\ M_{p_h} \xrightarrow{\cdot c} M_{q_1} \end{cases}$$

where the last isomorphism is given by multiplication by  $c$ . Conversely, every  $L \in F_M(X)$  is of type  $L^c$ , for a unique  $c \in k^*$ .

Now, by (i) we know that a general  $M \in \text{Pic}^{\underline{d}}(Y_S)$  does not belong to  $W_{\underline{d}}^h(Y_S)$ , i.e. we have  $h^0(Y_i, M_i) = 1$  for all  $i = 1, \dots, h$ . Take a generator, call it  $\alpha_i$ , of  $H^0(Y_i, M_i)$  and set  $\alpha_i^p := \alpha_i(p_i)$  and  $\alpha_i^q := \alpha_i(q_i)$ . A section  $\alpha = \sum_{i=1}^h x_i \alpha_i \in$

$H^0(Y_S, M)$  descends to a section of  $L^c \in F_M(X)$  on  $X$  if and only if it verifies the following system of equations:

$$(4.5) \quad \begin{cases} x_i a_i^p = \alpha(p_i) = \alpha(q_{i+1}) = x_{i+1} a_{i+1}^q & \text{for } 1 \leq i \leq h-1, \\ c x_h a_h^p = c \alpha(p_h) = r(q_1) = x_1 a_1^q. \end{cases}$$

The above system of  $h$  equations in the  $h$  unknown  $x_1, \dots, x_h$  admits a non-zero solution if and only if the determinant of the associated matrix is zero, that is if and only if

$$(4.6) \quad c \cdot \prod_{i=1}^h a_i^p = \prod_{i=1}^h a_i^q.$$

Since a general  $M \in \text{Pic}^{\underline{d}}(Y_S)$  verifies  $a_i^p \neq 0$  and  $a_i^q \neq 0$  for every  $i$  (by (iii)), the above equation has a unique solution  $c$  and therefore  $F_M(X)$  has a unique point in  $W_{\underline{d}}(X)$ . This proves (iv) since, clearly,  $\dim W_{\underline{d}}(X) = \dim \text{Pic}^{\underline{d}}(Y_S) = g-1$ .

Now we prove (v). The fiber of the birational map  $\nu^* : W_{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}} Y_S$  over  $W_{\underline{d}}^h(Y_S)$  has dimension 1; hence, as  $W_{\underline{d}}(X)$  is irreducible of dimension  $\sum_{i=1}^h g_i$  we have  $\dim W_{\underline{d}}^h(Y_S) \leq \sum_{i=1}^h g_i - 2$ . Since  $W_{\underline{d}}^h(Y) = \bigcup_{i=1}^h (\pi_i)^{-1}(W_{\underline{d}_i}^1(Y_i))$ , where  $\pi_i : \text{Pic}^{\underline{d}}(Y_S) \rightarrow \text{Pic}^{\underline{d}_i}(Y_i)$  is the projection, we deduce that  $\dim W_{\underline{d}_i}^1(Y_i) \leq g_i - 2$ .

Finally (vi). As observed before, we have

$$\text{Pic}^{\underline{d}} Y_S \setminus \nu_S^*(W_{\underline{d}}(X)) \subset \text{Pic}^{\underline{d}} Y_S \setminus W_{\underline{d}}^h(Y_S).$$

With the above notation, a line bundle  $M \in \text{Pic}^{\underline{d}} Y_S \setminus W_{\underline{d}}^h(Y_S)$  does not belong to  $\nu^*(W_{\underline{d}}(X))$  if and only if the equation (4.6) does not admit a solution  $c \in k^*$ . This happens precisely when either  $a_k^p = 0$  for at least one  $k$  and  $a_i^q \neq 0$  for any  $i$ , or if  $a_k^q = 0$  for at least one  $k$  and  $a_i^p \neq 0$  for any  $i$ . These conditions are easily seen to be equivalent to the fact that  $M \in \cup_k (T_{p_k} \cup T_{q_k})$ .  $\blacksquare$

**Proposition 4.2.4.** *Let  $X$  be such that  $X_{\text{sep}} = \emptyset$ ; pick  $S \in \text{Set}^1 X$  and  $\underline{d} \in \Sigma(X)$ . The image of the map  $\nu_S^* : W_{\underline{d}}(X) \rightarrow \text{Pic}^{\underline{d}} Y_S$  uniquely determines  $\nu_S^{-1}(S)$ , the gluing set of  $\nu_S$ .*

*Proof.* Denote  $\text{Pic}^{\underline{d}} Y_S \setminus \nu_S^*(W_{\underline{d}}(X)) = T_1 \amalg T_2$  where, using 4.2.3 (vi) we have

$$(4.7) \quad T_1 := \cup_{k=1}^h T_{p_k}, \quad T_2 := \cup_{k=1}^h T_{q_k},$$

for some set  $\{p_1, \dots, p_h, q_1, \dots, q_h\}$  which we must prove is uniquely determined, up to reordering the  $p_i$  (or the  $q_i$ ) among themselves. Notice that, for any such set, two different points  $p_k, p_j$  lie in two different connected components of  $Y_S$ , and the same holds for any two  $q_k, q_j$ . Therefore  $T_1$  and  $T_2$  are connected; on the other hand they obviously do not intersect, therefore they are determined. It thus suffices to prove that  $T_1$  (and similarly  $T_2$ ) determines a unique set of  $h$  smooth points of  $Y_S$  such that  $T_1$  is expressed as in (4.7).

We begin with a preliminary analysis. Pick any smooth point of  $Y_S$ , let  $Y_k$  be the connected component on which it lies, name the point  $y_k$ , for notational purposes. By 4.2.3(ii) the multidegree  $\underline{d}'_k := \underline{d}_k(-y_k)$  is semistable on  $Y_k$ . Therefore we can apply Proposition 3.2.1 in [Cap07]. This yields that  $W_{\underline{d}'_k}(Y_k)$  contains an irreducible component (of dimension  $g_k - 1$ ) equal to the image of the  $\underline{d}'_k$ -th Abel map; we call  $A_k$  this component. We also have (by loc. cit.) that  $A_k$  does not have a fixed base point, and that  $h^0(Y_k, L) = 1$  for the general  $L \in A_k$ .

We can thus define an irreducible effective divisor, as follows

$$D_{y_k} := \{M \in \text{Pic}^{\underline{d}} Y_S \mid M_k(-y_k) \in A_k\}.$$

---

<sup>1</sup> $V \subset \text{Pic} Y$  has a fixed base point if there exists a  $y \in Y$  which is a base point for every  $L \in V$

Observe that  $D_{y_k}$  has no fixed base point other than  $y_k$ . Indeed, let  $M \in D_{y_k}$  be a general point. If  $j \neq k$  then  $M_j$  is general in  $\text{Pic}^{d_j} Y_j$ , hence by 4.2.3  $M_j$  has no fixed base point and  $h^0(Y_j, M_j) = 1$ . On the other hand  $M_k$  varies in a set of dimension  $g_k - 1$ , therefore  $h^0(Y_k, M_k) = 1$  by 4.2.3(v). Therefore, if every  $M_k$  had a base point in  $r \neq y_k$ , we would obtain

$$(4.8) \quad 1 = h^0(M_k) = h^0(M_k(-y_k)) = h^0(M_k(-r)) = h^0(M_k(-y_k - r)).$$

But  $M_k(-y_k) \in A_k$ , so every element of  $A_k$  would have a base point in  $r$ , which is not possible (see above).

Summarizing, the general  $M \in D_{y_k}$  satisfies the following properties

$$(4.9) \quad \begin{cases} h^0(Y_j, M_j) = 1 & \text{for any } j = 1, \dots, h. \\ h^0(Y_S, M) = h^0(Y_S, M(-y_k)) = h \\ h^0(Y_S, M(-r)) < h^0(Y_S, M) & \forall r \neq y_k \text{ smooth point of } Y_S. \end{cases}$$

Now, back to the proof of the proposition; it suffices to concentrate on  $T_1$ . By contradiction, suppose there are two different descriptions for  $T_1$  as follows

$$T_1 = \bigcup_{j=1}^h T_{p_j} = \bigcup_{j=1}^h T_{\tilde{p}_j};$$

we may assume  $\tilde{p}_1 \notin \{p_1, \dots, p_h\}$ . By (4.9) applied to  $y_k = \tilde{p}_1$ , together with 4.2.3(vi), we have

$$D_{\tilde{p}_1} \subset \overline{T_1}.$$

But then, since  $T_1 = \cup_j T_{p_j}$ , we conclude that  $D_{\tilde{p}_1}$  has a fixed base point in some  $p_j$ , which is impossible by the last property in (4.9).  $\blacksquare$

**4.3. Torelli theorem: proof of the necessary condition.** To prove the necessary condition of Theorem 2.1.4 it suffices to prove the following. Let  $X$  and  $X'$  be stable curves of genus  $g$  free from separating nodes; suppose  $\bar{t}_g(X) = \bar{t}_g(X')$ . Then  $X$  and  $X'$  are C1-equivalent. Fix an isomorphism

$$\Phi = (\phi_0, \phi_1) : (J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) \xrightarrow{\cong} (J(X') \curvearrowright \overline{P_{X'}^{g-1}}, \Theta(X')).$$

We divide the proof into Steps.

**Step 1.** *There exists a natural bijection*

$$\text{Set}^1 X \longrightarrow \text{Set}^1 X'; \quad S \mapsto S'$$

*such that  $\#S = \#S'$  for every  $S \in \text{Set}^1 X$ .*

The isomorphism  $\phi_1 : \overline{P_X^{g-1}} \xrightarrow{\cong} \overline{P_{X'}^{g-1}}$  induces an isomorphism between the posets of strata  $\mathcal{ST}_X \cong \mathcal{ST}_{X'}$ ; hence, by Lemma 4.1.5(2), it induces an isomorphism  $\mathcal{SP}_X \cong \mathcal{SP}_{X'}$  of the posets of supports, compatible with the support maps. In particular, we get a bijection  $\text{Set}^1 X \leftrightarrow \text{Set}^1 X'$ .

Fix a C1-set  $S$  of  $X$  and call  $S'$  the corresponding C1-set of  $X'$ . By what we just said, every stratum of type  $P_S^d$  is mapped isomorphically to a stratum of type  $P_{S'}^d$ . Moreover, as the theta divisor of  $X$  is mapped isomorphically to the theta divisor of  $X'$ , the intersection  $\Theta(X) \cap P_S^d$  is mapped isomorphically to  $\Theta(X') \cap P_{S'}^d$ ; in particular the number of irreducible components is the same. Hence, by Lemma 4.2.1,  $S$  and  $S'$  have the same cardinality.

**Step 2.**  $\Gamma_X \equiv_{\text{cyc}} \Gamma_{X'}$ .

By Prop. 2.3.9 and Thm. 5.3.2 in [CV09] two graphs  $\Gamma$  and  $\Gamma'$ , free from separating edges, are cyclically equivalent if there exists an isomorphism of posets  $\mathcal{SP}_\Gamma \cong \mathcal{SP}_{\Gamma'}$  such that its restriction to C1-sets,  $\text{Set}^1 \Gamma \leftrightarrow \text{Set}^1 \Gamma'$ , preserves the cardinality. Now, as we have shown in the previous step, there is an isomorphism

$\mathcal{SP}_{\Gamma_X} = \mathcal{SP}_X \cong \mathcal{SP}_{X'} = \mathcal{SP}_{\Gamma_{X'}}$ , inducing a cardinality preserving bijection between C1-sets (recall that  $\text{Set}^1 \Gamma_X = \text{Set}^1 X$ ). Hence the statement is proved.

**Step 3.**  $X^\nu \cong X''^\nu$ .

By the previous step, the number of irreducible components of  $X^\nu$  and  $X''^\nu$  is the same; indeed, the number of edges and the first Betti number of  $\Gamma_X$  and  $\Gamma_{X'}$  are the same, hence the number of vertices is the same. Denote by  $X_+^\nu \subset X^\nu$  and  $X''_+^\nu \subset X''^\nu$  the union of all components of positive genus. It is enough to show that

$$(4.10) \quad X_+^\nu \cong X''_+^\nu.$$

In Remark 1.2.11 we saw that  $\overline{P_X^{g-1}}$  has a unique stratum of smallest dimension, namely the unique stratum supported on  $X_{\text{sing}}$ . This smallest stratum is isomorphic to the product of the Jacobians of the components of  $X^\nu$ , and hence to the product of the Jacobians of the components of  $X^\nu$  having positive genus. It is clear that the smallest stratum of  $\overline{P_X^{g-1}}$  is mapped by  $\phi_1$  to the smallest stratum of  $\overline{P_{X'}^{g-1}}$ . Recall now (1.7), expressing the restriction of the Theta divisor to this smallest stratum in terms of the Theta divisors of the components of  $X^\nu$ . As a consequence the projection of the smallest stratum onto each of its factors determines the polarized Jacobian of all the positive genus components of the normalization. Hence, by the Torelli theorem for smooth curves, we obtain that the positive genus components of the normalizations of  $X$  and  $X'$  are isomorphic, so (4.10) is proved.

**Step 4.** *Condition (B) of Definition 2.1.3 holds.*

We use induction on the number of nodes. The base is the nonsingular case, i.e. the classical Torelli theorem. From now on assume  $X$  and  $X'$  singular.

As usual, we denote the normalizations of  $X$  and  $X'$  both by  $X^\nu$ .

Let  $S \in \text{Set}^1 X$  and  $S' \in \text{Set}^1 X'$  be a pair of corresponding C1-set, under the bijection described in the first step; set  $h := \#S = \#S'$ . Let  $\nu_S : Y_S \rightarrow X$  and  $\nu_{S'} : Y'_{S'} \rightarrow X'$  be the partial normalizations at  $S$  and  $S'$ , and call  $g_S = g - h$  their arithmetic genus. Recall that  $Y_S$  and  $Y'_{S'}$  have  $h$  connected components, each of which is free from separating nodes and has positive arithmetic genus. We claim that  $Y_S$  and  $Y'_{S'}$  are C1-equivalent.

Recall (see (1.9)) that we denote by  $\overline{P_S} \subset \overline{P_X^{g-1}}$  and  $\overline{P_{S'}} \subset \overline{P_{X'}^{g-1}}$  the closures of all strata supported, respectively, on  $S$  and  $S'$ . By what we said, the isomorphism  $\phi_1$  induces an isomorphism

$$(4.11) \quad \overline{P_S} \cong \overline{P_{S'}}.$$

By Lemma 1.3.1 we obtain that  $\overline{P_S}$  together with the restriction of the theta divisor and the action of  $J(Y_S)$  is naturally isomorphic to  $(J(Y_S) \curvearrowright \overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S))$ ; similarly for  $\overline{P_{S'}}$ . Therefore by (4.11) we have

$$(J(Y_S) \curvearrowright \overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S)) \cong (J(Y'_{S'}) \curvearrowright \overline{P_{Y'_{S'}}^{g_{S'}-1}}, \Theta(Y'_{S'})).$$

By Proposition 1.3.4, the same holds if  $Y_S$  and  $Y'_{S'}$  are replaced by their stabilizations,  $\overline{Y_S}$  and  $\overline{Y'_{S'}}$ . Therefore we can apply the induction hypothesis to  $\overline{Y_S}$  and  $\overline{Y'_{S'}}$  (which are stable, free from separating nodes, and have fewer nodes than  $X$  and  $X'$ ). We thus obtain that  $\overline{Y_S}$  is C1-equivalent, or T-equivalent, to  $\overline{Y'_{S'}}$ .

On the other hand the normalizations of  $Y_S$  and  $Y'_{S'}$  are isomorphic, as they are equal to the normalizations of  $X$  and  $X'$ . Furthermore, as  $\Gamma_X \equiv_{\text{cyc}} \Gamma_{X'}$  (by Step 2) the dual graphs of  $Y_S$  and  $Y'_{S'}$  are cyclically equivalent (by the same argument used for Lemma 3.1.6). Therefore, by Lemma 4.3.1,  $Y_S$  is T-equivalent, hence C1-equivalent, to  $Y'_{S'}$ . The claim is proved.

Next, consider the normalization maps

$$\nu : X^\nu \xrightarrow{\mu} Y_S \xrightarrow{\nu_S} X, \quad \nu' : X^\nu \xrightarrow{\mu'} Y'_{S'} \xrightarrow{\nu_{S'}} X'$$

where  $\mu$  and  $\mu'$  are the normalizations of  $Y_S$  and  $Y_{S'}$ . As  $Y_S$  and  $Y_{S'}$  are C1-equivalent, the gluing sets  $\mu^{-1}((Y_S)_{\text{sing}})$  and  $\mu'^{-1}((Y_{S'})_{\text{sing}})$  are the same (cf. 3.1.4). The gluing sets of  $\nu$  and  $\nu'$  are obtained by adding to the above set the gluing sets of  $S$  and  $S'$ .

By Proposition 4.2.4, we have that  $\nu_S^{-1}(S)$  and  $\nu_{S'}^{-1}(S')$  are uniquely determined by the ppSSAV of  $Y_S$  or of  $Y_{S'}$ , which are isomorphic. Therefore  $(\overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S))$  uniquely determines  $\mu^{-1}(\nu_S^{-1}(S)) = \nu^{-1}(S)$  and  $\mu'^{-1}(\nu_{S'}^{-1}(S')) = \nu'^{-1}(S')$  on  $X^\nu$ . This is to say that, up to automorphisms of  $X^\nu$ , the sets  $\nu^{-1}(S)$  and  $\nu'^{-1}(S')$  coincide. Denote  $G_S := \nu^{-1}(S) = \nu'^{-1}(S')$ . We also obtain that the gluing set of  $\nu$  is equal to the gluing set of  $\nu'$ ; we call it  $G_{X_{\text{sing}}} = \nu^{-1}(X_{\text{sing}}) = \nu'^{-1}(X'_{\text{sing}})$ . Of course  $G_{X_{\text{sing}}}$  is the disjoint union of all the gluing sets associated to all the C1-sets of  $X$ .

Now we apply the previous argument to every remaining pair of corresponding C1-sets, as follows. Pick a pair of corresponding C1-sets,  $U$  and  $U'$ , with  $U \neq S$ . Then, as before,  $Y_U$  and  $Y_{U'}$  are C1-equivalent, and their (same) ppSSAV uniquely determines

$$G_U := \nu^{-1}(U) = \nu'^{-1}(U') \subset G_{X_{\text{sing}}} \setminus G_S \subset X^\nu.$$

Therefore condition (B) of Definition 2.1.3 holds, i.e.  $X$  and  $X'$  are C1-equivalent. The proof is complete.  $\blacksquare$

We used the following basic

**Lemma 4.3.1.** *Let  $X$  and  $X'$  be free from separating nodes; suppose that their stabilizations are  $T$ -equivalent and that  $\Gamma_X \equiv_{\text{cyc}} \Gamma_{X'}$ . Then  $X$  and  $X'$  are  $T$ -equivalent.*

*Proof.* Let  $\overline{X}$  and  $\overline{X}'$  be the stabilizations of  $X$  and  $X'$ . Observe that the dual graph of  $\overline{X}$  is obtained from  $\Gamma_X$  by removing some vertices of valence 2 (corresponding to the exceptional components of  $X$ ) so that the two edges adjacent to every such vertex become a unique edge. Therefore there is a natural isomorphism  $H_1(\Gamma_X) \cong H_1(\Gamma_{\overline{X}})$ . Moreover, this isomorphism fits in a commutative diagram

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \text{Div}^0 X^\nu \\ \cong \downarrow & & \uparrow \\ H_1(\Gamma_{\overline{X}}, \mathbb{Z}) & \xrightarrow{\eta_{\overline{X}}} & \text{Div}^0 \overline{X}^\nu \end{array}$$

where the right vertical arrow is induced by the obvious injection  $\overline{X}^\nu \hookrightarrow X^\nu$ . The diagram immediately yields that the map  $\eta_{\overline{X}}$  is determined by  $\eta_X$ . The converse is also true, in fact if  $E \subset X$  is an exceptional component, and  $\pi_E : \text{Div}^0 X^\nu \rightarrow \text{Div}^0 E = \text{Div}^0 \mathbb{P}^1$  the projection ( $\text{Div}^0 E$  is a factor of  $\text{Div}^0 X^\nu$ ), then the map  $\pi_E \circ \eta_X$  is uniquely determined up to an automorphism of  $X$ . The same observation applies to  $X'$ , of course.

Now to prove the lemma, notice that  $X$  and  $X'$  have the same number of irreducible components, because their dual graphs are cyclically equivalent. Denote by  $X_+^\nu \subset X^\nu$ , respectively by  $X_+^{\nu'} \subset X^{\nu'}$ , the union of all components of  $X^\nu$ , respectively of  $X^{\nu'}$ , having positive genus. To show that  $X^\nu \cong X^{\nu'}$  it suffices to show that  $X_+^\nu \cong X_+^{\nu'}$ . This follows immediately from the fact that the normalizations of  $\overline{X}$  and  $\overline{X}'$  are isomorphic.

Finally, by the initial observation, the maps  $\eta_X$  and  $\eta_{X'}$  are determined by those of  $\overline{X}$  and  $\overline{X}'$ , and hence Property (c) of Definition 3.1.3 holds for  $X$  and  $X'$ , because it holds for their stabilizations.  $\blacksquare$

**4.4. Torelli theorem: proof of the sufficient condition.** By Corollary 1.3.2 and Remark 1.3.4 it suffices to prove the first part of Theorem 2.1.4, i.e. we can assume that  $X$  and  $X'$  are C1-equivalent curves free from separating nodes. By Proposition 3.2.1 C1-equivalence and T-equivalence coincide; so we can use the second concept, which is now more convenient. Indeed the proof consists in applying some well known (some quite deep) facts about ppSSAV, on which our definition of T-equivalence is based.

By [AN99], and by [Ale04, Sec. 5.5] (where a short description, ad-hoc for the present case, is given)  $\bar{t}_g(X)$  is determined by a set of “combinatorial data” (partly known also to Mumford and Namikawa, see [Nam79, Chap. 18] and [Nam80, Chap. 9.D]). Let us recall them. Denote by  $J(X^\nu)^t$  be the dual abelian variety of  $J(X^\nu)$ . Now let

$$\lambda_X : J(X^\nu) \xrightarrow{\cong} J(X^\nu)^t$$

be the isomorphism associated to the class of the Theta divisor of  $X^\nu$ .

Let  $\mathcal{P}$  be the universal, or Poincaré, line bundle on  $J(X^\nu) \times J(X^\nu)^t$ . Recall that its set of  $k$ -rational points,  $\mathcal{P}(k)$ , defines a biextension, the so-called Poincaré biextension, of  $J(X^\nu) \times J(X^\nu)^t$  by  $k^*$  ([Mum68, Sect. 2 p. 311]).

Then  $\bar{t}_g(X)$  is uniquely determined by the following data.

- (1) The free abelian group  $H_1(\Gamma_X, \mathbb{Z})$ .
- (2) the Delaunay decomposition of the real vector space  $H_1(\Gamma_X, \mathbb{R})$  associated to the lattice  $H_1(\Gamma_X, \mathbb{Z})$ , with respect to the Euclidean scalar product.
- (3) The classifying morphism of the semiabelian variety  $J(X)$ , together with its dual. In our present situation, this is the datum of the group homomorphism  $c_X : H_1(\Gamma_X, \mathbb{Z}) \rightarrow J(X^\nu)$  already described in 3.1.1, together with its dual

$$c_X^t : H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{c_X} J(X^\nu) \xrightarrow{\lambda_X} J(X^\nu)^t.$$

- (4) The equivalence class of a trivialization of the pull back to  $H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z})$  of the inverse of the Poincaré bi-extension; i.e. the class of a map

$$\tau_X : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \rightarrow (c_X^t \times c_X)^* \mathcal{P}^{-1}(k).$$

This is determined by composing

$$\eta_X \times \eta_X : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \rightarrow \text{Div}^0 X^\nu \times \text{Div}^0 X^\nu$$

with the Deligne symbol (see [SGA4III, XVII] and [Ale04, Sec. 5.5]).

Let us show that such data are the same for our T-equivalent curves  $X$  and  $X'$ .

As the graphs  $\Gamma_X$  and  $\Gamma_{X'}$  are cyclically equivalent, there is an isomorphism  $\epsilon_H : H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{\cong} H_1(\Gamma_{X'}, \mathbb{Z})$ . Such an isomorphism induces an isomorphism  $\text{Del}(\Gamma_X) \cong \text{Del}(\Gamma_{X'})$  between the Delaunay decompositions of  $X$  and of  $X'$  (see [CV09, Prop. 3.2.3(i)]). Therefore the data (1) and (2) are the same for  $X$  and  $X'$ .

Since  $X^\nu = X'^\nu$ , we have  $J(X^\nu) = J(X'^\nu)$  and the principal polarizations, of course, coincide:

$$\lambda_X = \lambda_{X'} : J(X^\nu) \rightarrow J(X'^\nu)^t.$$

The classifying morphism has been described in 3.1.1. From 3.1.3(c), we get the commutativity of the following diagram

$$\begin{array}{ccccc} c_X : H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \text{Div}^0 X^\nu & \twoheadrightarrow & J(X^\nu) \\ \cong \downarrow \epsilon_H & & \cong \downarrow \alpha & & \cong \downarrow \bar{\alpha} \\ c_{X'} : H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \text{Div}^0 X^\nu & \twoheadrightarrow & J(X'^\nu) \end{array}$$

where  $\bar{\alpha} \in \text{Aut}(J(X^\nu))$  is the automorphism induced by  $\alpha$  (recall that  $J(X^\nu) = \text{Pic}^0 X^\nu$ ). It is clear that the automorphisms of  $J(X^\nu)$  have no effect on the isomorphism class of the semiabelian variety corresponding to the classifying morphisms. This shows that data (3) are also the same for  $X$  and  $X'$ .

Let now  $\mathcal{P}'(k)$  be the Poincaré bi-extension of  $X'$ ; see (4). By what we said so far, it is clear that

$$(\epsilon_H \times \epsilon_H)^*(c_{X'}^t \times c_{X'})^* \mathcal{P}'^{-1}(k) \cong (c_X^t \times c_X)^* \mathcal{P}^{-1}(k).$$

Now, the class of the map  $\tau_X$  (respectively  $\tau_{X'}$ ) is constructed using the Deligne symbol which is canonically defined on the pull back of  $\mathcal{P}^{-1}(k)$  (respectively of  $\mathcal{P}'^{-1}(k)$ ) to  $\text{Div}^0 X^\nu \times \text{Div}^0 X^\nu$ . Therefore, using the above isomorphism and the commutative diagram of 3.1.3(c), we get

$$\tau_X = (\epsilon_H \times \epsilon_H)^* \tau_{X'} : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \rightarrow (c_X^t \times c_X)^* \mathcal{P}^{-1}(k).$$

Therefore the data of part (4) are also the same for  $X$  and  $X'$ . We thus proved that the data defining  $\bar{t}_g(X)$  and  $\bar{t}_g(X')$  are the same, hence we are done. ■

## 5. THE FIBERS OF THE TORELLI MORPHISM

### 5.1. Injectivity locus and fiber cardinality of the Torelli morphism.

Where, in  $\bar{M}_g$ , is the compactified Torelli morphism  $\bar{t}_g$  injective? At this point it is clear (as was already known to Namikawa, see [Nam80, Thm. 9.30(iv)]) that this is the case for irreducible curves; the question is thus really interesting for reducible curves. To give it a precise answer we introduce some terminology.

**5.1.1.** A connected graph is *3-edge connected* if it remains connected after removing any two of its edges. We need the following characterization (see corollary 2.3.4 of [CV09]). *A connected graph free from separating edges is 3-edge connected if and only if every C1-set has cardinality one.*

Note also that given two cyclically equivalent connected graphs, one is 3-edge connected if and only if the other one is.

In graph theory, the definition of a 3-edge connected graph is usually given for graphs having at least two vertices. Here we do not make this assumption, so for us a graph with one vertex is always 3-edge connected.

We shall call “Torelli-curves” those stable curves for which the Torelli map is actually injective. Such curves will be defined in 5.1.3. First we give an example to motivate their definition.

**Example 5.1.2.** The following is the simplest example of C1-equivalent stable curves. Let  $X^\nu = X'^\nu = C_1 \amalg C_2$ , where the  $C_i$  are smooth of genus  $g_i \geq 1$ . Let  $p_i, q_i \in C_i$  be distinct points; now define

$$X = \frac{C_1 \amalg C_2}{(p_1 = p_2, q_1 = q_2)} \quad \text{and} \quad X' = \frac{C_1 \amalg C_2}{(p_1 = q_2, q_1 = p_2)}.$$

It is clear that  $X$  and  $X'$  are C1-equivalent.

Observe now that they are not isomorphic, unless one of them,  $C_1$  say, has an automorphism switching  $p_1$  with  $q_1$ .

Indeed, suppose that there exists  $\alpha_1 \in \text{Aut} C_1$  such that  $\alpha_1(p_1) = q_1$  and  $\alpha_1(q_1) = p_1$ . Then the automorphism  $\phi \in \text{Aut} X^\nu$  which restricts to  $\alpha_1$  on  $C_1$  and to the identity on  $C_2$ , descends to an isomorphism between  $X$  and  $X'$ , since  $\nu' \circ \phi(p_1) = \nu' \circ \phi(p_2)$  and  $\nu' \circ \phi(q_1) = \nu' \circ \phi(q_2)$ . This example, when  $\alpha_1$  as above exists, is a special case of Torelli curve, defined as follows.

**Definition 5.1.3.** A stable curve  $X$  such that  $X_{\text{sep}} = \emptyset$  will be called a *Torelli curve* if for every C1-set  $S$  such that  $\#S = h \geq 2$ , conditions (1) and (2) hold.

- (1) For every  $i = 1, \dots, h-1$  there exists an automorphism  $\alpha_i \in \text{Aut}(Y_i)$  such that  $\alpha_i(p_i) = q_i$  and  $\alpha_i(q_i) = p_i$ , (where  $Y_1, \dots, Y_h$  are the connected components of  $Y_S$  and  $p_i, q_i \in Y_i$  are the two gluing points).
- (2) There is an isomorphism as marked curves  $(Y_i; p_i, q_i) \cong (Y_j; p_j, q_j)$  for every  $i, j \leq h-1$ ; or else  $h = 3$  and there exists  $\alpha_h \in \text{Aut}(Y_h)$  such that  $\alpha_h(p_h) = q_h$  and  $\alpha_h(q_h) = p_h$ .

In particular, by 5.1.1 if  $\Gamma_X$  is 3-edge connected,  $X$  is a Torelli curve.

**Theorem 5.1.4.** *Let  $X$  be a stable curve free from separating nodes. Then*

(1)

$$\#\bar{t}_g^{-1}(\bar{t}_g(X)) \leq \left\lceil \frac{(g-2)!}{2} \right\rceil.$$

*Furthermore the bound is sharp, and can be obtained with  $X$  a cycle curve equal to the union of  $g-1$  elliptic curves, no two of them isomorphic.*

(2)  $\bar{t}_g^{-1}(\bar{t}_g(X)) = \{X\}$  if and only if  $X$  is a Torelli-curve.

*Proof.* By Theorem 2.1.4 the set  $\bar{t}_g^{-1}(\bar{t}_g(X))$  is the C1-equivalence class of  $X$ . The bound on the cardinality of the C1-equivalence class follows from Lemma 5.1.7.

Now, let  $X$  be the union of  $g-1$  smooth curves  $C_1, \dots, C_{g-1}$  of genus 1, so that the dual graph of  $X$  is a cycle of length  $g-1$ . Suppose that  $C_i \not\cong C_j$  for all  $i \neq j$ . The curve  $X$  has a unique C1-set, namely  $S = X_{\text{sing}}$ , and each curve  $C_i$  contains exactly two points of  $G_S$ , which we call  $p_i$  and  $q_i$ . With the notation of 2.3.2, let  $[(\sigma_S, \psi_S)]$  be the gluing data of  $X$ . Since each  $C_i$  has an automorphism exchanging  $p_i$  with  $q_i$ , varying the marking  $\psi_S$  does not change the isomorphism class of the curve  $X$ . On the other hand, any change in  $\sigma_S$  (with the exception of  $\sigma_S^{-1}$  of course) changes the isomorphism class of the curve, because no two  $C_i$  are isomorphic. Therefore, we conclude that the number of non-isomorphic curves that are C1-equivalent to  $X$  is equal to 1 if  $g \leq 3$ , and  $(g-2)!/2$  if  $g \geq 4$ . Part (1) is proved.

For part (2) it suffices to prove the following. Let  $X$  be connected with  $X_{\text{sep}} = \emptyset$ ;  $X$  is a Torelli curve if and only if the only curve C1-equivalent to  $X$  is  $X$  itself.

Assume first that  $X$  is a Torelli curve. If  $\Gamma_X$  is 3-edge connected, then every C1-set has cardinality 1 by 5.1.1, therefore we conclude by Lemma 5.1.5. We can henceforth assume that  $\Gamma_X$  is not 3-edge connected.

Let  $S \in \text{Set}^1 X$  have cardinality  $h \geq 2$  (it exists by 5.1.1). We claim that  $\text{Aut } X$  acts transitively on the gluing data of  $S$ , described in 2.3.2. We use the notation of Definition 5.1.3. If  $h = 3$  and  $Y_i$  has an automorphism exchanging  $p_i$  with  $q_i$  for  $i = 1, 2, 3$ , then the claim trivially holds.

Next, assume that the first  $h-1$  marked components  $(Y_i; p_i, q_i)$  are isomorphic and have an automorphism switching the gluing points  $p_i, q_i$ . Using the set-up of 2.3.2, the gluing data are given by an ordering of the components, which we can assume has  $Y_h$  as last element, and by a marking of each pair  $(p_i, q_i)$  for all  $i = 1, \dots, \gamma-1$ . Now  $\text{Aut } X$  acts transitively on the orderings of the components, by permuting  $Y_1, \dots, Y_{h-1}$ , which are all isomorphic by isomorphisms preserving the gluing points. Moreover for  $i = 1, \dots, \gamma-1$  each pair of points  $(p_i, q_i)$  is permuted by the automorphism  $\alpha_i$ . The claim is proved. Of course, the claim implies that  $X$  is unique in its C1-equivalence class..

Conversely, let  $X$  be the unique curve in its C1-equivalence class. If every C1-set of  $X$  has cardinality 1 then  $\Gamma_X$  is 3-edge connected (by 5.1.1) and we are done.

So, let  $S \in \text{Set}^1 X$  be such that  $\#S \geq 2$  and let us check that the conditions of Definition 5.1.3 hold. With no loss of generality, and using the same notation as before, we may order the connected components of  $Y_S$  so that  $q_i$  is glued to  $p_{i+1}$  and

$p_i$  is glued to  $q_{i-1}$  (with the cyclic convention, so that  $p_1$  is glued to  $q_h$ ). Assume that  $Y_h$  has no automorphism exchanging  $p_h$  with  $q_h$ ; let us change the gluing data of  $X$  by switching  $p_h$  with  $q_h$ , and by leaving everything else unchanged. Then the corresponding curve is C1-equivalent to  $X$ , and hence it is isomorphic to  $X$ , by hypothesis. Therefore, the curve  $W = \overline{X \setminus Y_h}$  must admit an automorphism switching  $p_1$  with  $q_{h-1}$  (the two points glued to  $q_h$  and  $p_h$ ). Now it is easy to see, by induction on the number of components of  $W$ , that such an automorphism exists if and only if  $W$  is a union  $h-1$  of marked components,  $(Y_i; p_i, q_i)$ , all isomorphic to  $(Y_1; p_1, q_1)$ , and if  $Y_1$  has an involution switching  $p_1, q_1$ . Therefore  $X$  is a Torelli curve.

If instead  $Y_i$  has an automorphism exchanging the two gluing points for every  $i = 1, \dots, h$ , and no  $h-1$  among the  $Y_i$  are isomorphic, it is clear that for  $h \geq 4$  there exist different orderings of the  $Y_i$  giving different C1-equivalent curves. Therefore we must have  $h = 3$ , hence  $X$  is a Torelli curve. ■

The proof of the Theorem used the following lemmas.

**Lemma 5.1.5.** *Let  $X$  be a connected nodal curve free from separating nodes. Then the cardinality of the C1-equivalence class of  $X$  is at most*

$$\prod_{S \in \text{Set}^1 X} 2^{\#S-1} (\#S - 1)! .$$

*Proof.* By the discussion in 2.3.2, the number of curves that are C1-equivalent to  $X$  is bounded above by the product of the number of all gluing data for each C1-set  $X$ . The C1-sets with  $\#S = 1$  admit only one gluing data, so they do not contribute.

Let  $S$  be a C1-set of cardinality at least 2. Clearly there are  $2^{\#S}$  possible markings  $\psi_S$ , and  $(\#S-1)!$  possible choices for the cyclic permutation  $\sigma_S$ . Furthermore, recall that each gluing data can be given by two such pairs  $(\psi_S, \sigma_S)$ , namely the two conjugate pairs under the involution (2.6). This gives us a total of  $2^{\#S-1} (\#S-1)!$  gluing data. ■

We shall make use of the following elementary

*Remark 5.1.6.* Let  $E$  be a connected nodal curve of genus at most 1, free from separating nodes. For any two smooth points  $p, q$  of  $E$ , there exists an automorphism of  $E$  exchanging  $p$  and  $q$ .

**Lemma 5.1.7.** *Let  $X$  be a connected curve of genus  $g \geq 2$  free from separating nodes; let  $e$  be the number of its exceptional components. Then the C1-equivalence class of  $X$  has cardinality at most*

$$\left\lceil \frac{(g-2+e)!}{2} \right\rceil .$$

*Proof.* Throughout this proof, we denote by  $\{Y\}_{C1}$  the C1-equivalence class of a nodal curve  $Y$ . We will use induction on  $g$ .

We begin with the following claim. Let  $\overline{X}$  be the stabilization of  $X$ . If  $\Gamma_{\overline{X}}$  is 3-edge connected, then  $\#\{X\}_{C1} = 1$ .

Indeed there is a natural bijection between the C1-sets of  $X$  and those of  $\overline{X}$ ; which we denote by  $S \mapsto \overline{S}$ . By assumption, for every C1-set  $\overline{S}$  of  $\overline{X}$  the partial normalization  $\overline{Y}_{\overline{S}}$  of  $\overline{X}$  at  $\overline{S}$  is connected (since  $\#\overline{S} = 1$ ). Now, for any  $S \in \text{Set}^1 X$ , the partial normalization  $Y_S$  of  $X$  at  $S$  is equal to the disjoint union of  $\overline{Y}_{\overline{S}}$  together with some copies of  $\mathbb{P}^1$ . Using this explicit description and 5.1.6 we find that all the possible gluing data  $[(\sigma_S, \psi_S)]$  of  $S$  (see 2.3.2) give isomorphic curves, i.e.  $X$  is unique inside its C1-equivalence class. The claim is proved.

Now we start the induction argument. Let us treat the cases  $g = 2, 3$ .

Using the above claim, it is easy to see that to prove the Lemma for  $g = 2, 3$  we need only worry about curves  $X$  of genus 3, whose stabilization  $\overline{X}$  is the union of two components  $C_1$  and  $C_2$  of genus 1, meeting at two points. If  $e = 0$  it is easy to see that  $X$  is unique in its C1-equivalence class by using 5.1.6. If  $e > 0$  then the curves C1-equivalent to  $X$  are obtained by inserting two chains of exceptional components between  $C_1$  and  $C_2$ , one of length  $0 \leq e_1 \leq \lfloor e/2 \rfloor$  and the other of length  $e - e_1$ . This show that

$$\#\{X\}_{C1} = 1 + \left\lfloor \frac{e}{2} \right\rfloor \leq \left\lceil \frac{(e+1)!}{2} \right\rceil.$$

Assume now  $g \geq 4$  and let  $S \in \text{Set}^1 X$  such that  $\#S = h$ . As usual, we write  $Y_S = \coprod_{i=1}^h Y_i$ , with  $Y_i$  free from separating nodes and of genus  $g_i := g_{Y_i}$ . We order the connected components  $Y_i$  of  $Y_S$  in such a way that:

- $Y_1, \dots, Y_f$  have genus at least 4;
- $Y_{f+1}, \dots, Y_{f+k_3}$  have genus 3;
- $Y_{f+k_3+1}, \dots, Y_{f+k_3+k_2}$  have genus 2;
- $Y_{f+k_3+k_2+1}, \dots, Y_{f+k_3+k_2+k_1}$  have genus 1;
- $Y_{f+k_3+k_2+k_1+1}, \dots, Y_h$  have genus 0 and therefore are isomorphic to  $\mathbb{P}^1$ .

Let  $e_i$  be the number of exceptional components of  $X$  contained in  $Y_i$ ; then  $Y_i$  has at most  $e_i + 2$  exceptional components. We have the obvious relations

$$(*) \quad e = \sum_{i=1}^h e_i = \sum_{g_i \geq 2} e_i + \sum_{g_i=1} e_i + h - f - k_3 - k_2 - k_1$$

$$(**) \quad g - 1 = \sum_i g_i = \sum_{g_i \geq 2} g_i + k_1.$$

Consider now the gluing data  $[(\sigma_S, \psi_S)]$  associated to  $S$  (notation as in 2.3.2). Call, as usual,  $\{p_i, q_i\}$  the two points of  $G_S$  contained in the component  $Y_i$ . Since all the components  $Y_i$  with  $g_i \leq 1$  have an automorphism that exchanges  $p_i$  and  $q_i$  (by 5.1.6), if we compose the marking  $\psi_S$  with the involution of  $G_h$  that exchanges  $s_i$  with  $t_i$  (for all indices  $i$  such that  $g_i \leq 1$ ) the resulting curve will be isomorphic to the starting one. Therefore, the number of possible non-isomorphic gluing data associated to  $S$  is bounded above by  $(h-1)!2^{f+k_2+k_3-1}$  (here we use the hypothesis  $g \geq 4$  to ensure that this number be an integer). We conclude that

$$\#\{X\}_{C1} \leq (h-1)!2^{f+k_3+k_2-1} \prod_{i=1}^h \#\{Y_i\}_{C1}.$$

The components  $Y_i$  of genus at most 1 are unique inside their C1-equivalence class. For the components  $Y_i$  of genus  $g_i \geq 2$  we can apply the induction hypothesis (note that  $2 \leq g_i < g$ ) and we get that

$$\#\{Y_i\}_{C1} \leq \left\lceil \frac{(g_i - 2 + e_i + 2)!}{2} \right\rceil = \frac{(g_i + e_i)!}{2}$$

By substituting into the previous formula, we get

$$\#\{X\}_{C1} \leq (h-1)!2^{f+k_3+k_2-1} \prod_{i=1}^{f+k_3+k_2} \frac{(g_i + e_i)!}{2} = \frac{(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!}{2}.$$

The number of non-trivial factors of the product  $(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!$  is equal to  $h-2 + \sum_{g_i \geq 2} (g_i + e_i - 1)$ . Using the formulas (\*) and (\*\*), we get that

$$h-2 + \sum_{g_i \geq 2} (g_i + e_i - 1) = g-3 + e - \sum_{g_i=1} e_i \leq g-3 + e.$$

Since the factorial  $(g-2+e)!$  has a number of non-trivial factors equal to  $g-3+e$ , we conclude from the above inequalities that

$$\#\{X\}_{C1} \leq \frac{(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!}{2} \leq \frac{(g-2+e)!}{2},$$

which concludes the proof.  $\blacksquare$

**Corollary 5.1.8.**  $\bar{t}_g^{-1}(\bar{t}_g(X)) = \{X\}$  for every  $X \in \bar{M}_g$  with  $X_{\text{sep}} = \emptyset$  if and only if  $g \leq 4$ .

*Remark 5.1.9.* Consider a Torelli curve  $X$  of genus at least 5 with dual graph non 3-edge connected. It is not hard to see that  $X$  is the specialization of curves for which the Torelli morphism is not injective. On the other hand we just proved that  $\bar{t}_g^{-1}(\bar{t}_g(X)) = \{X\}$ . Therefore the Torelli morphism, albeit injective at  $X$ , necessarily ramifies at  $X$ .

**5.2. Dimension of the fibers.** Let  $X$  be a stable curve of genus  $g$ ; now we shall assume that  $X_{\text{sep}}$  is not empty and bound the dimension of the fiber of the Torelli map over  $X$ .

Recall the notation of (1.5); the normalization of  $X$  at  $X_{\text{sep}}$  is denoted  $\tilde{X}$ . We denote by  $\tilde{\gamma}_0$  the number of connected components of  $\tilde{X}$  of arithmetic genus 0, by  $\tilde{\gamma}_1$  the number of those of arithmetic genus 1, and by  $\tilde{\gamma}_+$  the number of those having positive arithmetic genus, that is:

$$\tilde{\gamma}_j := \#\{i : \tilde{g}_i = j\}, \quad j = 0, 1, \quad \text{and} \quad \tilde{\gamma}_+ := \#\{i : \tilde{g}_i \geq 1\}.$$

**Proposition 5.2.1.** *Let  $X$  be a stable curve of genus  $g \geq 2$ . Then*

$$\dim \bar{t}_g^{-1}(\bar{t}_g(X)) = 2\tilde{\gamma}_+ - \tilde{\gamma}_1 - 2$$

(i.e. the maximal dimension of an irreducible component of  $\bar{t}_g^{-1}(\bar{t}_g(X))$  is equal to  $2\tilde{\gamma}_+ - \tilde{\gamma}_1 - 2$ ).

*Proof.* According to Theorem 2.1.4,  $\bar{t}_g(X)$  depends on (and determines) the C1-equivalence class of the stabilizations  $\tilde{X}_i$  of the components of  $\tilde{X}$  such that  $\tilde{g}_i > 0$ . The C1-equivalence class of  $\tilde{X}_i$  determines  $\tilde{X}_i$  up to a finite choice. In particular, note that  $\tilde{\gamma}_0$  and the number, call it  $e$ , of exceptional components of  $\prod_{\tilde{g}_i > 0} \tilde{X}_i$  is not determined by  $\bar{t}_g(X)$ .

The dimension of the locus of curves in the fiber  $\bar{t}_g^{-1}(\bar{t}_g(X))$  having the same topological type of  $X$  is equal to

$$(5.1) \quad 2\#X_{\text{sep}} - 3\tilde{\gamma}_0 - \tilde{\gamma}_1 - e.$$

Indeed, each separating node gives two parameters of freedom, because we can arbitrarily choose the two branches of the node. The components  $\tilde{X}_i$  of arithmetic genus 0 reduce the parameters by 3 because they have a 3-dimensional automorphism group, similarly the components of arithmetic genus 1 reduce the parameters by 1. Finally, each exceptional component of  $\prod_{\tilde{g}_i > 0} \tilde{X}_i$  reduces the parameters by 1, because it contains at least one branch of one of the separating nodes and exactly two branches of non-separating nodes.

Formula (5.1) shows that the curves  $X'$  in the fiber  $\bar{t}_g^{-1}(\bar{t}_g(X))$  whose topological type attains the maximal dimension are the ones for which  $e' = 0$  (i.e. each positive genus component of  $\tilde{X}'$  is stable) and  $\tilde{\gamma}'_0 = 0$  (i.e.  $\tilde{X}'$  has no genus 0 component).

In particular, since  $\tilde{\gamma}'_+ = \tilde{\gamma}'$ , such a curve  $X'$  has  $\#X'_{\text{sep}} = \tilde{\gamma}'_+ - 1$  separating nodes. Applying formula (5.1) to the curve  $X'$  we obtain  $\dim \bar{t}_g^{-1}(\bar{t}_g(X)) \leq 2\tilde{\gamma}'_+ - \tilde{\gamma}'_1 - 2$ . To conclude that equality holds we must check that the locus of curves

$X'$  is not empty. This is easy: given  $\coprod_{\tilde{g}_i > 0} \widetilde{X}_i$  we can glue (in several ways) the stabilizations of the  $\widetilde{X}_i$  so that they form a tree. This, by our results, yields curves in  $\bar{t}_g^{-1}(\bar{t}_g(X))$ . ■

**Corollary 5.2.2.** *Let  $X$  be a stable curve of genus  $g$ . Then*

$$\begin{cases} \dim \bar{t}_g^{-1}(\bar{t}_g(X)) \leq g - 2 & \text{with equality iff } \tilde{g}_i \leq 2 \text{ for all } i, \\ \dim \bar{t}_g^{-1}(\bar{t}_g(X)) \geq \tilde{\gamma}_+ - 2 & \text{with equality iff } \tilde{g}_i \leq 1 \text{ for all } i. \end{cases}$$

*Proof.* The first inequality follows from the Proposition and

$$g = \tilde{\gamma}_1 + \sum_{\tilde{g}_i \geq 2} \tilde{g}_i \geq \tilde{\gamma}_1 + 2(\tilde{\gamma}_+ - \tilde{\gamma}_1) = 2\tilde{\gamma}_+ - \tilde{\gamma}_1,$$

with equality if and only if all  $\tilde{g}_i \leq 2$  for all  $i$ .

The second inequality follows from  $\tilde{\gamma}_1 \leq \tilde{\gamma}_+$ , with equality if and only if  $\tilde{g}_i \leq 1$ . ■

Using Theorem 5.1.4 and Corollary 5.2.2, one obtains that for  $g \geq 3$  the locus in  $\overline{M}_g$  where  $\bar{t}_g$  has finite fibers is exactly the open subset of stable curves free from separating nodes; see [Nam80, Thm. 9.30(vi)] and [V03, Thm. 1.1] for the analogous results for the map  $\bar{t}_g^{Vor}$ . On the other hand  $\bar{t}_g$  is an isomorphism for  $g = 2$ ; again see [Nam80, Thm. 9.30(v)].

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