

# GEOMETRY OF TROPICAL MODULI SPACES AND LINKAGE OF GRAPHS

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ABSTRACT. We prove a “linkage” theorem about graphs, study the topological properties of moduli spaces of tropical curves, and use the linkage theorem to prove that these moduli spaces are connected through codimension one. The linkage theorem says: two  $p$ -regular graphs with the same first Betti number can be obtained from one another by a finite alternating sequence of one-edge-contractions; moreover this preserves 3-edge-connectivity. We prove that the moduli space of tropical curves is Hausdorff.

## CONTENTS

1. Introduction	1
2. $p$ -regular hamiltonian graphs	4
3. The linkage theorem	10
4. Tropical curves and weighted metric graphs	14
5. The moduli space of tropical curves	19
References	24

## 1. INTRODUCTION

This paper is made of two distinct parts, with the second partially motivating the first. The second part studies the moduli space of tropical curves of genus  $g \geq 2$  as a topological space. In order to establish certain connectedness properties, we encounter some questions about graphs which are of interest in their own right; the solution of these graph theoretic problems occupies the first part of this paper.

Let us describe the two parts in some details. The first is concerned with classification of  $p$ -regular (every vertex has valency  $p$ ) connected graphs. It is quite easy to see that there exists a unique 1-regular graph, namely two vertices joined by a unique edge. Similarly, 2-regular graphs are classified by the number of their edges, indeed there exists a unique 2-regular graph with  $n$ -edges: the cycle on  $n$  vertices, and these are all the 2-regular graphs. As soon as  $p \geq 3$  the situation gets complicated; as a matter of fact, as far as we are aware of, the number of 3-regular graphs with fixed first Betti number is not known. And this number would be very interesting for several reasons; for instance, it counts the 0-dimensional combinatorial cycles in the moduli space of Deligne-Mumford stable curves,  $\overline{M}_g$ .

Our main result in the first part of the paper is Theorem 3.3. This states, first of all, that any two  $p$ -regular connected graphs  $\Gamma$  and  $\Gamma'$ , with the same first Betti number, are “linked”, i.e. they can be obtained one from the other with a finite sequence of alternating one-edge contractions as follows. There exists a finite

sequence

$$(1.1) \quad \begin{array}{ccccccc} \Gamma = \Gamma_1 & & \Gamma_3 & \cdots & \cdots & & \Gamma_{2h+1} = \Gamma' \\ & \searrow & \swarrow & \searrow & & & \swarrow \\ & & \Gamma_2 & \cdots & & & \Gamma_{2h} \end{array}$$

where every arrow is the map contracting precisely one edge and leaving everything else unchanged. Also, every odd-indexed graph in the diagram above is  $p$ -regular. Secondly, we prove that if  $\Gamma$  and  $\Gamma'$  are 3-edge-connected there exists a diagram as above where the graph  $\Gamma_i$  is 3-edge-connected, for every  $i = 1, \dots, 2h+1$ . This second part makes the proof seriously more complicated, but it does play an important role in the application of this result to the second part of the paper. We refer to this property as the “conservation of 3-edge-connectivity”.

In case  $p = 3$  the result, without the conservation of 3-edge-connectivity, is due to A.Hatcher and W.Thurston [HtT80], by a non combinatorial argument; a combinatorial proof valid for simple graphs is given by Y.Tsukui [T96].

Our proof is purely combinatorial. We first reduce it to hamiltonian graphs (in Section 2), and then show that every hamiltonian graph is linked to a special type of graph called the  $p$ -polygon (in Section 3).

Now we turn to the part concerning moduli of tropical curves, which begins from Section 4. In this paper we use the definition of tropical curve given by S. Brannetti, M. Melo and F. Viviani in [BMV], which is a generalization of the original one due to G. Mikhalkin (see[Mi07]), here called “pure tropical curve”. Albeit slightly more complicated, the generalized version has the advantage that the moduli space,  $M_g^{\text{trop}}$ , of tropical curves of genus  $g \geq 2$ , is closed under specialization, while the moduli space of pure tropical curves is not. See subsection 4.1 for more details. Notice however that essentially everything that we will prove for tropical curves holds for pure tropical curves. The crux of the matter is that, by [MiZ08], to every tropical curve is associated a metric graph (plus a weight function when using the generalized definition). In this perspective, the moduli space of tropical curves can be interpreted as a moduli space for metric graphs with valency at least 3.

A motivating observation for us was that the connectedness properties of  $M_g^{\text{trop}}$  and of related moduli spaces, such as the Schottky space,  $Sch_g^{\text{trop}}$  (i.e. the quotient of  $M_g^{\text{trop}}$  via the Torelli map, see subsection 5.6), are equivalent to the linkage properties of graphs studied in the first part of the paper. More precisely, it is quite easy to see that  $M_g^{\text{trop}}$  and  $Sch_g^{\text{trop}}$  are connected, however a stronger property holds. To explain this we need to give some structure to  $M_g^{\text{trop}}$ . In this paper we deal with  $M_g^{\text{trop}}$  as a topological space; this enables us to work with a slightly simpler definition than that the one used in [BMV]. Our definition makes the topological properties somewhat more evident; for example, we prove in Theorem 5.2 that  $M_g^{\text{trop}}$  is a Hausdorff topological space, a fact that was not known.

Although it is not known whether  $M_g^{\text{trop}}$  is a tropical variety, in [BMV] the authors show that it is a so-called “stacky fan”, something close to a tropical variety. See [Mi05], [GKM09], [MaS], [Mi] for more on this and related issues.

The space  $M_g^{\text{trop}}$  is endowed with a finite decomposition  $M_g^{\text{trop}} = \coprod_{i \in I} M_i$  where each  $M_i$  is the quotient of a connected topological manifold by a finite group, and  $M_g^{\text{trop}}$  is the closure of the union of those  $M_i$  having dimension  $3g - 3$ . Every  $M_i$  has a clear geometric interpretation, for example the above mentioned dense union

$$M_g^{\text{reg}} := \coprod_{i \in I: \dim M_i = 3g-3} M_i \subset M_g^{\text{trop}} = \coprod_{i \in I} M_i$$

parametrizes tropical curves whose underlying graph is 3-regular and has first Betti number equal to  $g$ . Moreover  $M_g^{\text{reg}}$  is open in  $M_g^{\text{trop}}$ . It is easy to see that  $M_g^{\text{reg}}$  is not connected.

As  $M_g^{\text{trop}}$  is connected one naturally asks: if we add to  $M_g^{\text{reg}}$  all strata  $M_i$  of codimension one (i.e. of dimension  $3g - 4$ ), do we get a connected space? Or, in modern terminology: is  $M_g^{\text{trop}}$  connected through codimension one?

This question is particularly interesting as tropical varieties are always connected through codimension one (see the Structure Thm. in [MaS, Ch. 3]). The answer to the question is yes, and this follows from the linkage property treated in the first part of the paper.

Next, by the tropical Torelli theorem of [CV09] the Schottky locus  $Sch_g^{\text{trop}}$  is the image via the Torelli map of the locus in  $M_g^{\text{trop}}$  parametrizing 3-edge-connected tropical curves. This motivates our interest in 3-edge-connected graphs. Indeed, the fact that graph linkage preserves 3-edge-connectivity (Theorem 3.3), implies that  $Sch_g^{\text{trop}}$  is connected through codimension one; see Proposition 5.9.

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**1.1. Terminology.** Throughout the paper, unless otherwise specified, we assume that  $g$  and  $p$  are two integers with  $g \geq 2$  and  $p \geq 3$ .

All graphs are assumed to be finite and connected. Throughout the paper  $\Gamma$  denotes a graph,  $E(\Gamma)$  the set of its edges and  $V(\Gamma)$  the set of its vertices. Every  $e \in E(\Gamma)$  joins two, possibly equal, vertices, which we call the *ends* of  $e$ . If the two ends of  $e$  coincide we say that  $e$  is a *loop*. The first Betti number of  $\Gamma$  is

$$b_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1.$$

Following the standard terminology (see [D97]) a connected graph  $\Gamma$  is called

- (1) *p-regular* if every vertex has valency (or degree) equal to  $p$ ;
- (2) a *path* if its first Betti number is equal to 0, and if it contains no vertex of valency  $\geq 3$ . A path  $\Gamma$  satisfies  $|V(\Gamma)| = |E(\Gamma)| + 1$ ; we shall say that  $|E(\Gamma)|$  is the length of the path.
- (3) a *cycle* if it is 2-regular. A cycle has  $b_1(\Gamma) = 1$ , and hence an equal number of edges and vertices; this number will be called its length.
- (4) *p-edge-connected* if  $|V(\Gamma)| \geq 1$  and if  $\Gamma \setminus F$  is connected for any  $F \subset E(\Gamma)$  with  $\#F < p$ .

**1.2. Linkage.** Let  $e \in E(\Gamma)$  be an edge of a graph  $\Gamma$ . Let  $\Gamma/e$  be the graph obtained by contracting  $e$  to a point and leaving everything else unchanged ([D97, sect I.1.7]). Then there is a natural surjective map  $\sigma : \Gamma \rightarrow \Gamma/e$ ; we call  $\sigma$  the *contraction* of  $e$ . More generally, if  $S \subset E(\Gamma)$  is a set of edges, we denote by  $\Gamma/S$  the contraction of every edge in  $S$ , so the map  $\Gamma \rightarrow \Gamma/S$  is the composition of all the contractions of the edges in  $S$ .

*Remark 1.3.* Let  $\sigma : \Gamma \rightarrow \Gamma/e$  be the contraction of  $e$  as above.

- (1) If  $e$  is not a loop  $b_1(\Gamma) = b_1(\Gamma/e)$  (otherwise  $b_1(\Gamma) = b_1(\Gamma/e) + 1$ ).
- (2) If  $\Gamma$  is  $p$ -edge-connected so is  $\Gamma/e$ .

**Definition 1.4.** (1) Let  $\Gamma_1$  and  $\Gamma_2$  be two graphs. We say that  $\Gamma_1$  and  $\Gamma_2$  are *strongly linked* if for  $i = 1, 2$  there exists a non-loop edge  $e_i \in E(\Gamma_i)$  such that the contraction of  $e_1$  and the contraction of  $e_2$  coincide, i.e.

$$\Gamma_1 \xrightarrow{\sigma_1} \Gamma_1/e_1 = \Gamma_2/e_2 \xleftarrow{\sigma_2} \Gamma_2,$$

and  $\sigma_1(e_1) = \sigma_2(e_2)$  (i.e.  $e_1$  and  $e_2$  are mapped into the same vertex).

- (2) Let  $\Gamma$  and  $\Gamma'$  be two graphs. We say that  $\Gamma$  and  $\Gamma'$  are *linked* if there exists a finite sequence of graphs

$$\Gamma = \Gamma_1, \Gamma_2, \dots, \Gamma_{n-1}, \Gamma_n = \Gamma'$$

such that  $\Gamma_i$  and  $\Gamma_{i-1}$  are strongly linked, for every  $i = 2, \dots, n$ .

Since we are particularly interested in 3-edge-connected graphs, we need the following variation on the previous definition.

**Definition 1.5.** Let  $\Gamma$  and  $\Gamma'$  be two 3-edge-connected graphs. We say that  $\Gamma$  and  $\Gamma'$  are *3-linked* if there exists a finite sequence of 3-edge-connected graphs

$$\Gamma = \Gamma_1, \Gamma_2, \dots, \Gamma_{n-1}, \Gamma_n = \Gamma'$$

such that  $\Gamma_i$  and  $\Gamma_{i-1}$  are strongly linked, for every  $i = 2, \dots, n$ .

*Remark 1.6.* Being linked, or 3-linked is an equivalence relation. Two linked graphs have the same number of edges and the same number of vertices.

**Example 1.7.** The picture below represents two strongly linked 3-regular graphs, with the graph  $\Gamma_1/e_1$  (not drawn) equal to  $\Gamma_2/e_2$ .  $\Gamma_1$  is a so called Petersen graph.

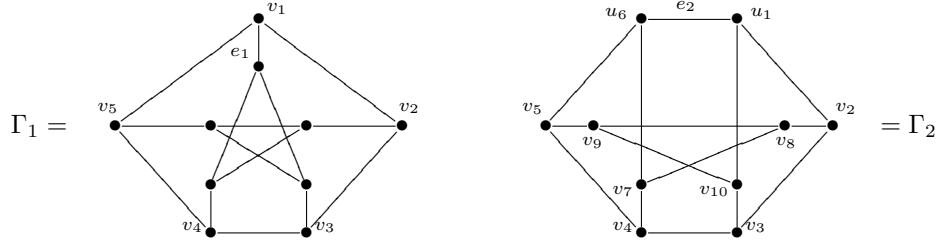


FIGURE 1. Petersen graph strongly linked to a hamiltonian graph.

*Remark 1.8.* Let  $\Gamma$  be a  $p$ -regular graph with  $p \geq 3$ ; set  $b = b_1(\Gamma)$ . We have  $|E(\Gamma)| = |V(\Gamma)|p/2$  hence

$$b = 1 + \frac{(p-2)|V(\Gamma)|}{2}, \quad |V(\Gamma)| = \frac{2b-2}{p-2} \quad \text{and} \quad |E(\Gamma)| = \frac{p(b-1)}{p-2}.$$

If  $\Gamma$  is 3-regular,  $|E(\Gamma)| = 3b - 3$  and  $|V(\Gamma)| = 2b - 2$ .

## 2. $p$ -REGULAR HAMILTONIAN GRAPHS

**Definition 2.1.** A graph  $\Gamma$  is called *hamiltonian* if  $|V(\Gamma)| \geq 2$  and if it contains a *hamiltonian cycle*, i.e. a cycle passing through every vertex. A  $p$ -regular hamiltonian graph free from loops will be called a  *$p$ -hamiltonian* graph.

Examples of  $p$ -hamiltonian graphs are all the graphs in figures 3 and 5. In Figure 1, the graph  $\Gamma_2$  (on the right) is hamiltonian, with  $b_1(\Gamma_2) = 5$ , whereas the graph  $\Gamma_1$  is not hamiltonian. So, Example 1.7 implies that the non hamiltonian graph  $\Gamma_1$  is linked to the 3-hamiltonian graph  $\Gamma_2$ . This is true in general, by the following Proposition 2.2.

In the next two proofs we will use the following not very standard terminology. To every edge  $e$  of a graph  $\Gamma$  we associate two *half-edges*,  $h$  and  $h'$ , defined as follows. Call  $v, v' \in V(\Gamma)$  the two (possibly equal) ends of  $e$ . Then  $h$  and  $h'$  are line

segments such that  $h \subsetneq e$ ,  $h' \subsetneq e$ ,  $e = h \cup h'$  and such that each of them contains precisely one end of  $e$ , so  $v \in h$  and  $v' \in h'$ .

There are obviously many possible choices for the half-edges of any edge; however everything we will say does not depend on such a choice. For example: the valency of any vertex  $v \in E(\Gamma)$  is equal to the number of half-edges of  $\Gamma$  touching  $v$ .

**Proposition 2.2.** *Every  $p$ -regular graph  $\Gamma$  is linked to a  $p$ -hamiltonian graph. Every  $p$ -regular, 3-edge-connected graph is 3-linked to a 3-edge-connected  $p$ -hamiltonian graph.*

*Proof.* Let  $\Gamma$  be our  $p$ -regular graph, and  $b = b_1(\Gamma)$ . Call  $\ell(\Gamma)$  the maximal length of a cycle contained in  $\Gamma$ ; by Remark 1.8  $\ell(\Gamma) \leq |V(\Gamma)| = \frac{2b-2}{p-2}$ . We shall use descending induction on  $\ell(\Gamma)$ . If  $\ell(\Gamma) = \frac{2b-2}{p-2}$  there is nothing to prove, so the basis of the induction is done.

Assume  $\ell(\Gamma) < \frac{2b-2}{p-2}$ ; let  $\Delta \subset \Gamma$  be a cycle of length  $\ell = \ell(\Gamma)$ .

We shall explicitly construct a  $p$ -regular graph,  $\Gamma_2$ , strongly linked to  $\Gamma$  (with the notation of Definition 1.4 we have  $\Gamma_1 = \Gamma$ ), and such that  $\ell(\Gamma_2) > \ell(\Gamma)$ . If  $\Gamma$  is 3-edge-connected so will be  $\Gamma_2$ . Using the induction hypothesis on  $\Gamma_2$  will suffice to complete the proof. Denote  $V(\Delta) = \{v_1, \dots, v_\ell\} \subsetneq V(\Gamma)$ .

The coming construction is pictured in Figure 2.

Pick a vertex  $v \in V(\Gamma)$  such that  $v \notin V(\Delta)$  and such that there is an edge  $e$  joining  $v$  to one of the vertices of  $\Delta$ ; obviously  $e \notin E(\Delta)$ . We can assume, with no loss of generality, that the ends of  $e$  are  $v_1$  and  $v$ . Let us call  $e_1$  and  $e_\ell$  the two edges of  $\Delta$  meeting at  $v_1$ .

Since  $v$  has valency  $p$ , there are  $p-1$  half-edges containing  $v$  and not contained in  $e$ ; let us call them  $h_1, \dots, h_{p-1}$ . Similarly as  $v_1$  has valency  $p$  there are  $p-3$  half-edges containing  $v_1$  and not contained in  $e$ ,  $e_1$  or  $e_\ell$ ; we call these  $h_p, \dots, h_{2p-4}$ . It is clear that no half-edge  $h_i$  lies in  $\Delta$ . Consider the contraction of  $e$

$$\sigma_1 : \Gamma_1 = \Gamma \longrightarrow \Gamma/e = \Gamma'$$

It is clear that  $w := \sigma_1(e)$  is a vertex of valency  $2p-2$ , indeed the images via  $\sigma_1$  of  $e_1, e_\ell, h_1, \dots, h_{2p-4}$  all touch  $w$ , and there is no other edge touching  $w$ .

Now we perform a valency reducing extension on  $w$  (cf. [CV09, A.2.2]); namely we introduce an edge contracting map from a new graph  $\Gamma_2$

$$\sigma_2 : \Gamma_2 \longrightarrow \Gamma'$$

such that  $\Gamma'$  is obtained from  $\Gamma_2$  as the contraction to  $w$  of a unique edge, which we call  $e_{\ell+1}$ ; hence  $\sigma_2(e_{\ell+1}) = w$  and  $\sigma_2$  leaves everything else unchanged. The two ends of  $e_{\ell+1}$  are two vertices of valency  $p$ , which we call  $u_{\ell+1}, u_1 \in V(\Gamma_2)$ . In  $\Gamma_2$  we distribute the  $2p-2$  half-edges  $h_1, \dots, h_{2p-4}$  so that  $p-1$  of them touch  $u_1$  and the remaining  $p-1$  touch  $u_{\ell+1}$ . Moreover we have the (old) edge  $e_1$  touching  $u_1$ , the old edge  $e_\ell$  touching  $u_{\ell+1}$ , and the new edge  $e_{\ell+1}$  joining  $u_1$  with  $u_{\ell+1}$ . Therefore the graph  $\Gamma_2$  is  $p$ -regular. Summarizing, we have

$$\Gamma_2 \xrightarrow{\sigma_2} \Gamma_2/e_{\ell+1} = \Gamma/e \xleftarrow{\sigma_1} \Gamma.$$

Therefore  $\Gamma$  and  $\Gamma_2$  are strongly linked. Now the given cycle  $\Delta \subset \Gamma$  is mapped to a cycle of the same length by  $\sigma_1$ , whereas  $\sigma_2^{-1}(\sigma_1(\Delta))$  is a cycle of length at least  $\ell+1$  (as it contains the vertices  $\{u_1, v_2, \dots, v_\ell, u_{\ell+1}\}$ ). Therefore  $\ell(\Gamma) < \ell(\Gamma_2)$ .

It is clear that by iterating this construction we arrive at a  $p$ -regular graph  $\widehat{\Gamma}$  with  $\ell(\widehat{\Gamma}) = 2b-2$ , so that  $\widehat{\Gamma}$  is hamiltonian graph. It is also clear that  $\widehat{\Gamma}$  and  $\Gamma$  are linked.

Now suppose that  $\Gamma$  is 3-edge-connected; then  $\Gamma'$  is also 3-edge-connected by Remark 1.3. To prove that  $\widehat{\Gamma}$  is 3-edge-connected we need to prove that the extension of  $w$  used during the proof may be constructed so as to yield a 3-edge-connected

graph  $\Gamma_2$ . This follows from the proof of [CV09, Prop. A.2.4], with trivial modifications.

Finally, the fact that we can take  $\widehat{\Gamma}$  free from loops follows from lemma 2.3. ■

The next picture illustrates the previous proof. We represent the relevant portions of  $\Gamma$  (on the left) of  $\Gamma'$  and of  $\Gamma_2$ . The vertices  $v_2$  and  $v_\ell$  belong to the cycle  $\Delta$ , hence they are joined by a path (not drawn) not intersecting  $h_1$  and  $h_2$ .

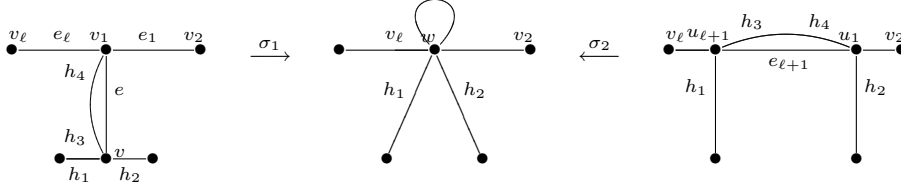


FIGURE 2. The procedure increasing the length of  $\Delta$  (proof of 2.2).

**Lemma 2.3.** *Every  $p$ -regular hamiltonian graph is linked to a  $p$ -hamiltonian graph. Every  $p$ -regular hamiltonian 3-edge-connected graph is 3-linked to a  $p$ -hamiltonian 3-edge-connected graph.*

*Proof.* It suffices to exhibit a procedure which decreases the number of loops, preserving the property of being hamiltonian,  $p$ -regular and 3-edge-connected.

Let  $\Delta \subset \Gamma$  be a fixed hamiltonian cycle; denote by  $E(\Delta) = \{e_1, \dots, e_t\}$  and  $V(\Gamma) = \{v_1, \dots, v_t\}$  with  $e_i$  joining  $v_i$  and  $v_{i+1}$  as usual. Suppose that  $\Gamma$  contains a loop  $\ell$ , and assume (with no loss of generality) that this loop is based at  $v_1$ . Let  $\ell_1$  and  $\ell_2$  be the half-edges of the loop (so  $\ell_1$  and  $\ell_2$  touch  $v_1$ ). We know that  $v_1$  is connected to  $v_2$  by the edge  $e_1$ . Since  $v_2$  has valency  $p \geq 3$  there is a half-edge  $h$  touching  $v_2$ , not contained in the hamiltonian cycle  $\Delta$ , and not contained in an edge touching  $v_1$  (for otherwise  $v_2$  would have valency less than that of  $v_1$ ). Let us consider  $\Gamma/e_1$ , and call  $w$  the vertex into which  $e_1$  is contracted. The valency of  $w$  is  $2p - 2$ .

Let  $\Gamma_2$  be the graph obtained from  $\Gamma$  by changing the loop  $\ell$  into an edge, called  $f_1$ , joining  $v_1$  with  $v_2$ , and by changing the half-edge  $h$  into a half-edge touching  $v_1$ . This operation does not create any new loop (as the edge of  $\Gamma$  containing  $h$  does not touch  $v_1$ ), and eliminates the loop  $\ell$ . So the number of loops of  $\Gamma_2$  is less than that of  $\Gamma$ . It is clear that  $\Gamma_2$  is  $p$ -regular (we added and removed a half-edge from  $v_1$  and  $v_2$ , and left everything else unchanged). The hamiltonian cycle  $\Delta$  is clearly contained in  $\Gamma_2$ , so  $\Gamma_2$  is hamiltonian. Finally,  $\Gamma/e_1 = \Gamma_2/e_1$ , so  $\Gamma$  and  $\Gamma_2$  are strongly linked.

It remains to show that if  $\Gamma$  is 3-edge-connected so is  $\Gamma_2$ . This follows from [CV09, Prop.A.2.4], in fact the extension of  $w$  given by  $\Gamma_2 \rightarrow \Gamma/e_1$  is the same as in Step 1 in the proof of that proposition (with obvious modifications). ■

**2.4. Normalized form of a  $p$ -hamiltonian graph.** Let  $\Gamma$  be a  $p$ -hamiltonian graph, with  $b := b_1(\Gamma)$ . We fix a hamiltonian cycle and call it  $\Delta$ ; let  $\gamma = |V(\Gamma)|$  be the length of  $\Delta$ . The choice of  $\Delta$  induces the *normalized form* of  $\Gamma$ , which consists of the following. The edges of  $\Gamma$  which do not lie in  $\Delta$  will be called *chords*. The number of chords of  $\Gamma$  is easily computed:

$$(2.1) \quad \text{Number of chords of } \Gamma = |E(\Gamma)| - \gamma = \frac{p(b-1)}{p-2} - \frac{2(b-1)}{p-2} = b-1.$$

The vertices of  $\Gamma$  will be ordered according to the cyclic structure of  $\Delta$ , i.e.  $V(\Gamma) = V(\Delta) = \{v_1, v_2, \dots, v_\gamma\}$  so that there exists an edge  $e_i \in E(\Delta) \subset E(\Gamma)$  joining  $v_i$  with  $v_{i+1}$  for every  $i = 1, \dots, \gamma$  (with the cyclic convention  $v_{\gamma+1} = v_1$ ); hence  $E(\Delta) = \{e_1, \dots, e_\gamma\}$ . The starting vertex  $v_1$  can be picked arbitrarily; furthermore, for any choice of  $v_1$ , there are two cyclic orderings of the vertices (corresponding to the two cyclic orientations of  $\Delta$ ). We call such a labeling of the vertices a *compatible labeling*. A normalized form will always be endowed with some compatible labeling, which we will be free to change.

**2.5. Chords of a hamiltonian graph in normalized form.** Let  $\Gamma$  be a  $p$ -hamiltonian graph in normalized form, as in subsection 2.4. Every chord has two distinct ends ( $\Gamma$  being free from loops). We shall denote by  $d_{i,j}$  a chord joining  $v_i$  with  $v_j$ , and always assume  $i < j$ . If  $p \geq 4$  there may be more than one chord joining  $v_i$  with  $v_j$ ; if we need to distinguish between them we will use superscripts, i.e. we denote  $\{d_{i,j}^\alpha, \alpha = 1, \dots, m\}$  the chords joining  $v_i$  and  $v_j$ ; notice that  $m \leq p - 2$ .

We also need a notation for a chord of which only one end is known. So, the chord having one end at the vertex  $v_j$  and the other end at some other vertex will be denoted  $d_{j,*}$ .

Let  $d_{i,j}$  be a chord as above. Then  $d_{i,j}$  determines two paths of the cycle  $\Delta$ , namely the two paths  $\Lambda$  and  $\Lambda'$  contained in  $\Delta$ , having extremes  $v_i$  and  $v_j$ . Hence  $\Lambda \cap \Lambda' = \{v_i, v_j\}$  and  $\Lambda \cup \Lambda' = \Delta$ . We call such two paths the *sides* of  $d_{i,j}$ . It is obvious that one of them has length  $j - i$  and the other has length  $\gamma - j + i$ . We define its *amplitude*,  $\alpha(d_{i,j})$ , as the minimum between these two lengths:

$$(2.2) \quad \alpha(d_{i,j}) := \min\{j - i, \gamma - j + i\}.$$

It is clear that  $\alpha(d_{i,j})$  does not depend on the choice of the (compatible) labeling.

**Lemma - Definition 2.6.** *Let  $\Gamma$  be a  $p$ -hamiltonian graph in normalized form. Set  $\gamma := |V(\Gamma)| = (2b_1(\Gamma) - 2)/(p - 2)$ .*

- (1) *For any chord  $d_{i,j}$  we have  $1 \leq \alpha(d_{i,j}) \leq \gamma/2$ . If  $\alpha(d_{i,j}) \leq \gamma/2 - 1$  we say that  $d_{i,j}$  is short.*
- (2) *Let  $d_{i,j}$  be a short chord. The side of  $d_{i,j}$  having length  $\alpha(d_{i,j})$  will be called the short side of  $d_{i,j}$ .*
- (3) *If  $\alpha(d_{i,j}) = \lfloor \gamma/2 \rfloor$  for every chord, or equivalently, if  $\Gamma$  has no short chords, then  $\Gamma$  is uniquely determined, it will be denoted by  $\Pi_\gamma^p$  and will be called the  $p$ -polygon with  $\gamma$  vertices (see Figures 3 and 4).*

*If  $\gamma$  is even the graph  $\Pi_\gamma^p$  has  $p - 2$  chords between  $v_i$  and  $v_{i+\gamma/2}$  for every  $i = 1, \dots, (\gamma - 1)/2$ , and no other chord.*

*If  $\gamma$  is odd then  $p$  is even. For every  $i = 1, \dots, (\gamma - 1)/2$ , the graph  $\Pi_\gamma^p$  has  $(p - 2)/2$  chords between  $v_i$  and  $v_{i+(\gamma-1)/2}$ ,  $(p - 2)/2$  chords between  $v_i$  and  $v_{i+(\gamma+1)/2}$ , and no other chord.*

*Proof.* Since  $\Gamma$  has no loops we have, for any chord  $d_{i,j}$ ,  $1 \leq \alpha(d_{i,j})$ . If  $\gamma$  is even (respectively, odd) the maximal amplitude of a chord is obviously  $\gamma/2$  (respectively,  $(\gamma - 1)/2$ ).

Now let  $\gamma$  be even. If there are no short chords, every chord is of type  $d_{i,i+\gamma/2}$  for  $i = 1, \dots, \gamma/2$ . Moreover, every pair of vertices  $v_i, v_{i+\gamma/2}$  is joined by exactly  $p - 2$  chords, because  $\Gamma$  is  $p$ -regular. This shows that  $\Gamma$  is uniquely determined.

Now suppose that  $\gamma$  is odd, and that  $\Gamma$  has no short chord. Then every chord is either of type  $d_{i,i+(\gamma-1)/2}$  or of type  $d_{i,i+(\gamma+1)/2}$ . Since  $|E(\Gamma)| = p\gamma/2$  we have that  $p$  is even; set  $r = (p - 2)/2$ . For every vertex there are  $2r$  chords touching it.

We claim that there are exactly  $r$  chords of type  $d_{i,i+(\gamma-1)/2}$  and  $r$  chords of type  $d_{i,i+(\gamma+1)/2}$  for every  $i = 1, \dots, (\gamma - 1)/2$ . By contradiction, suppose (with no loss of generality) that there are more than  $r$  chords joining  $v_1$  with  $v_{(\gamma+1)/2}$ ; hence

there are less than  $r$  chords joining  $v_1$  with  $v_{(\gamma+3)/2}$ . But then there are more than  $r$  chords joining  $v_2$  with  $v_{(\gamma+3)/2}$  and less than  $r$  chords joining  $v_2$  with  $v_{(\gamma+5)/2}$ . Continuing in this way we get that there are less than  $r$  chords joining  $v_{(\gamma-1)/2}$  with  $v_\gamma$ . The remaining chords touching  $v_\gamma$  are the ones touching also  $v_{(\gamma+1)/2}$ ; since there are already more than  $r$  chords of type  $d_{1,(\gamma+1)/2}$ , there can only be less than  $r$  chords of type  $d_{\gamma,(\gamma+1)/2}$ . We conclude that there are less than  $2r$  chords touching  $v_\gamma$ . A contradiction. This shows that  $\Gamma$  is uniquely determined. ■

**Example 2.7.** If  $p = 3$  we have  $\gamma = 2b_1(\Gamma) - 2$  and  $\Pi_\gamma^3$  has no multiple edge.

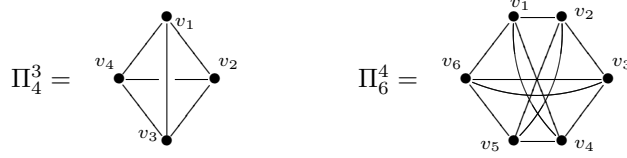


FIGURE 3. Some  $p$ -polygons with even number of vertices.

If  $\gamma$  is odd, then  $\Pi_\gamma^p$  has no multiple edges if and only if  $p \leq 4$ .

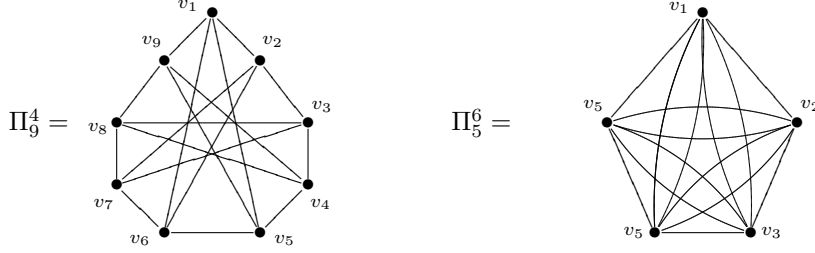


FIGURE 4. Some  $p$ -polygons with an odd number of vertices.

We need a criterion for 3-edge-connectivity.

**Lemma 2.8.** (1) Let  $\Gamma$  be a graph such that for every edge  $e$  there exist two distinct cycles  $\Delta_1$  and  $\Delta_2$  in  $\Gamma$  and such that  $E(\Delta_1) \cap E(\Delta_2) = \{e\}$ . Then  $\Gamma$  is 3-edge-connected.

(2) Let  $\Gamma_1$  be a 3-edge-connected graph and let  $\Gamma_2$  be a graph strongly linked to  $\Gamma_1$ , so that  $\Gamma_1/e_1 = \Gamma_2/e_2$  with  $e_i \in E(\Gamma_i)$  (notation in Def. 1.4). Then  $\Gamma_2$  is 3-edge-connected if it contains two cycles  $\Delta_1 \neq \Delta_2$  such that  $E(\Delta_1) \cap E(\Delta_2) = \{e_2\}$ .

(3) The  $p$ -polygon  $\Pi_\gamma^p$  is 3-edge-connected for every  $p \geq 3$ .

*Proof.* For part (1), we begin by noticing that  $\Gamma$  has no separating edges (a separating edge is not contained in any cycle). Suppose by contradiction that  $\Gamma$  is not 3-edge-connected; let  $(e, e')$  be a separating pair of edges of  $\Gamma$ . By [CV09, Lemma 2.3.2 (iv) and (iii)],  $(e, e')$  is a separating pair if and only if  $e$  and  $e'$  belong to the same cycles of  $\Gamma$ . By our assumption, this is clearly impossible.

Now part (2). The graph  $\Gamma_1/e_1 = \Gamma_2/e_2$  is 3-edge-connected as  $\Gamma_1$  is. Therefore any separating pair of edges of  $\Gamma_2$  must contain  $e_2$ . The proof of part (1) shows that our hypothesis implies that  $e_2$  is not contained in any separating pair of edges, hence we are done.

To prove part (3) we use again part (1). Pick a chord  $d_{i,j}$ ; then there obviously exist two cycles having only  $d_{i,j}$  as common edge: just take the two cycles obtained by adding to  $d_{i,j}$  one of its two sides (terminology in subsection diag). To prove that we can apply (1) on the remaining edges we need to distinguish two cases, according to the parity of  $\gamma$ .

Suppose  $\gamma$  even. By Lemma 2.6 in  $\Pi_\gamma^p$  there exists at least one chord  $d_{i,i+\gamma/2}$  joining  $v_i$  with  $v_{i+\gamma/2}$ , for every  $i = 1, \dots, \gamma/2$ . Pick an edge which is not a chord,  $e = e_1$ . Now  $\Pi_\gamma^p$  contains the chords  $d_{1,\gamma/2}$  and  $d_{2,\gamma/2+1}$ . Then  $\Delta_1 = (e_1, \dots, e_{\gamma/2-1}, d_{1,\gamma/2})$  and  $\Delta_2 = (e_1, e_\gamma, \dots, e_{\gamma/2}, d_{2,\gamma/2+1})$  are two cycles having only  $e$  as common edge. Therefore  $\Pi_\gamma^p$  is 3-edge-connected.

Now suppose that  $\gamma$  is odd; again we use (1). By Lemma 2.6 in  $\Pi_\gamma^p$  there exists at least one chord joining  $v_i$  with  $v_{i+(\gamma-1)/2}$ , and at least one chord joining  $v_i$  with  $v_{i+(\gamma+1)/2}$ . Let  $e = e_1$  be an edge which is not a chord. Let  $\Delta_1 = (e_1, d_{2,(\gamma+3)/2}, d_{1,(\gamma+3)/2})$  and  $\Delta_2 = (e_1, e_2, \dots, e_{(\gamma-1)/2}, d_{1,(\gamma+1)/2})$ ; these are two cycles whose only edge in common is  $e_1$ . Hence  $\Pi_\gamma^p$  is 3-edge-connected. ■

We say that two chords  $d_{i,j}$  and  $d_{k,l}$  do not cross each other if  $i < j < k < l$ .

**Lemma 2.9.** *Let  $\Gamma$  be a  $p$ -hamiltonian graph in normalized form. Let  $d_{i,j}$  be a short chord. Then there exists a short chord  $d_{k,l}$  with  $j < k$  (i.e.  $d_{i,j}$  and  $d_{k,l}$  do not cross each other) and such that the short side of  $d_{i,j}$  does not intersect the short side of  $d_{k,l}$ .*

*Proof.* We denote by  $\Delta$  the hamiltonian cycle of the normalized form. We may assume that  $i = 1$ , so that the given chord  $d_{1,j}$  has  $j \leq \gamma/2$  (i.e. the short side of  $d_{1,j}$  has vertices  $v_1, v_2, \dots, v_j$ ). We must prove that there exists a short chord  $d_{k,l}$  such that

- (a)  $j < k$  ( $d_{1,j}$  and  $d_{k,l}$  do not cross each other).
- (b)  $l - k < \lceil \gamma/2 \rceil$  (the short side of  $d_{k,l}$  has vertices  $v_k, v_{k+1}, \dots, v_{l-1}, v_l$ ).

Let us denote by  $D$  the set of chords satisfying (a); we begin by bounding  $\#D$  from below. Consider the  $j$  vertices  $v_1, v_2, \dots, v_j$ ; there are at most  $p - 2$  chords touching each of them. Therefore the total number of distinct chords touching these vertices is at most  $j(p - 2) - 1$  (to explain the “-1” notice that the chord  $d_{1,j}$  joins  $v_1$  with  $v_j$ , hence it must not be counted twice). Since  $\Gamma$  has  $b - 1$  chords, we get

$$(2.3) \quad \#D \geq b - 1 - j(p - 2) + 1 = b - j(p - 2).$$

In particular, since  $b = 1 + (p - 2)\gamma/2$  and  $j \leq \gamma/2$ , we have that  $\#D \geq 1$ , i.e.  $D$  is not empty. To prove that there exists at least one chord in  $D$  satisfying (b) we argue by contradiction. Suppose that every chord  $d_{k,l} \in D$  satisfies  $l - k \geq \gamma/2$ . This is to say that the path  $\Lambda \subset \Delta$  from  $v_{j+1}$  to  $v_\gamma$  contains a side of length at least  $\gamma/2$  for every non multiple chord in  $D$ . Let us restrict our attention to the subgraph  $\Gamma' = \Gamma \setminus \{v_1, \dots, v_j\}$ , obtained by removing the vertices  $\{v_1, \dots, v_j\}$  and all edges adjacent to them. So,  $\Gamma'$  is made of  $\Lambda$  together with every chord in  $D$ . Now, two vertices of  $\Lambda$  are joined by a chord of  $D$  only if they are separated by at least  $\gamma/2$  edges. Moreover, every two vertices can be joined by at most  $p - 2$  chords. Therefore the length of the path  $\Lambda$  satisfies, using (2.3),

$$\text{length}(\Lambda) \geq \frac{\gamma}{2} + \frac{\#D}{p - 2} - 1 \geq \frac{b - 1}{p - 2} + \frac{b}{p - 2} - j - 1 = \frac{2b - 1}{p - 2} - j - 1 > \gamma - j - 1$$

(since  $\gamma = \frac{2b-2}{p-2}$ ). On the other hand  $\Lambda$  is a path from  $v_{j+1}$  to  $v_\gamma$ , whose length is easily computed:

$$\text{length}(\Lambda) = \gamma - (j + 1) = \gamma - j - 1,$$

which is in contradiction with the above estimate on  $\text{length}(\Lambda)$ . ■

## 3. THE LINKAGE THEOREM

**3.1. Twisting pairs of chords in a  $p$ -hamiltonian graph.** Let  $\Gamma$  be a  $p$ -hamiltonian graph in normalized form, pick two chords  $d_{i,j}$  and  $d_{k,l}$ . In this subsection we momentarily drop the general convention  $i < j$  and  $k < l$  (which would be too restrictive). We introduce the graph  $\Gamma'$  obtained from  $\Gamma$  by swapping two ends of the above chords. So,  $\Gamma'$  is obtained from  $\Gamma$  by replacing the chord  $d_{i,j}$  with a new chord,  $d_{i,k}$ , joining  $v_i$  and  $v_k$ , and by replacing  $d_{k,l}$  with a chord  $d_{j,l}$ . Everything else is left unchanged. We shall say that  $\Gamma'$  is a *twist* of  $\Gamma$ , and that  $\Gamma'$  is obtained from  $\Gamma$  by *twisting* the pair of chords  $(d_{i,j}, d_{k,l})$  into the pair  $(d_{i,k}, d_{j,l})$ . We shall also say that we swapped the end points  $v_j$  and  $v_l$ .

The graph  $\Gamma'$  is obviously a  $p$ -regular hamiltonian graph; a normalized form will be naturally induced by the normalized form of  $\Gamma$ . So the vertices of  $\Gamma$  and  $\Gamma'$  will have the same names, and all the edges of  $\Gamma$  other than  $d_{1,j}$  and  $d_{k,l}$  correspond to edges of  $\Gamma'$  other than  $d_{1,k}$  and  $d_{j,l}$ .

The picture below represents two 3-hamiltonian graphs related by twisting a pair of chords (the dotted chords are the ones that are not changed).

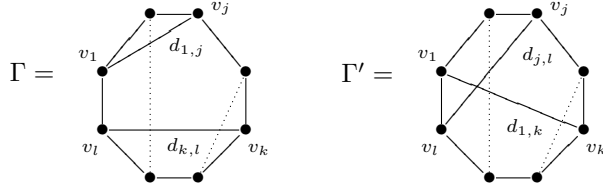


FIGURE 5. Twisting  $d_{1,j}$  and  $d_{k,l}$  into  $d_{1,k}$  and  $d_{j,l}$ .

The following technical lemma is used in the proof of Theorem 3.3.

**Lemma 3.2.** *Let  $\Gamma$  be a  $p$ -hamiltonian graph in normalized form.*

- (1) *If  $\Gamma'$  is a twist of  $\Gamma$ , then  $\Gamma$  and  $\Gamma'$  are linked.*
- (2) *Let  $\Gamma$  be 3-edge-connected and fix a chord  $d_{i,j}$  of  $\Gamma$ , with  $i < j$ . Let  $d_{j+1,*}$  be a chord of  $\Gamma$  starting at the vertex  $v_{j+1}$ ; suppose that either (a) or (b) below hold.*
  - (a)  *$d_{j+1,*} = d_{j+1,h}$  with  $j+1 < h$  (i.e.  $d_{j+1,*}$  does not cross  $d_{i,j}$ ).*
  - (b)  *$d_{j+1,*} = d_{h,j+1}$  with  $i < h < j$  and there exists a third chord  $d_{x,y}$  such that  $1 < x < j < y$  and  $h < x < j+1 < y$ .*

*Then the graph obtained by twisting  $(d_{i,j}, d_{j+1,*})$  into  $(d_{i,j+1}, d_{j,*})$  is 3-edge-connected and strongly linked to  $\Gamma$ .*

*Proof.* We can assume  $i = 1$  so that  $d_{i,j} = d_{1,j}$ . The edges of the hamiltonian cycle  $\Delta$  corresponding to the normalized form will be called, as usual,  $e_1, e_2, \dots, e_{2b-2}$  with  $e_i$  joining  $v_i$  with  $v_{i+1}$ . Let  $\Gamma'$  be a twist of  $\Gamma$ . We prove  $\Gamma'$  is linked to  $\Gamma$  by induction on  $k - j$  (i.e. on the distance along  $\Delta$  of the two swapped vertices). If  $k = j + 1$  let  $e$  be the edge of  $\Delta$  between  $v_j$  and  $v_{j+1}$ . Then the graph obtained from  $\Gamma$  by contracting  $e$  is the same as the graph obtained from  $\Gamma'$  by contracting  $e$ ; hence  $\Gamma$  and  $\Gamma'$  are strongly linked. Now assume  $k - j \geq 2$ . Let  $\Gamma_1$  be the graph obtained from  $\Gamma$  by twisting the chord  $d_{1,j}$  with the chord ending at  $v_{j+1}$ , denoted  $d_{j+1,*}$ . So, in  $\Gamma_1$  we have the chords  $d_{1,j+1}^1$  and  $d_{j,*}^1$ , where the superscript keeps track of the graph to which the chords belong. We already proved that  $\Gamma$  and  $\Gamma_1$  are linked. Now consider the graph  $\Gamma_2$  obtained from  $\Gamma_1$  by twisting  $d_{1,j+1}^1$  and  $d_{k,l}^1$ , replacing them with  $d_{1,k}^2$  and  $d_{j+1,l}^2$ ; since  $k - (j + 1) < k - j$ , by induction  $\Gamma_1$

and  $\Gamma_2$  are linked. Finally, let  $\Gamma_3$  be obtained from  $\Gamma_2$  by twisting  $d_{j+1,l}^2$  and  $d_{j,*}^2$ , replacing them with  $d_{j,l}^3$  and  $d_{j+1,*}^3$ . Again by induction (we are swapping  $v_j$  and  $v_{j+1}$ )  $\Gamma_3$  is linked to  $\Gamma_2$ , and therefore  $\Gamma_3$  is linked to  $\Gamma$ . It is obvious that  $\Gamma_3 = \Gamma'$ .

Let us prove the second part. Consider  $e_j$ , the edge between  $v_j$  and  $v_{j+1}$  (which, abusing notation as usual, is an edge of both  $\Gamma$  and  $\Gamma'$ ). It is easy to check that  $\Gamma$  and  $\Gamma'$  are strongly linked, as the graphs  $\bar{\Gamma}$  and  $\bar{\Gamma}'$  obtained from  $\Gamma$  and  $\Gamma'$  by contracting  $e_j$  are obviously isomorphic. We need to prove that if either (a) or (b) holds, then  $\Gamma'$  is 3-edge -connected if  $\Gamma$  is. By Lemma 2.8 (2), it is enough to show that the edge  $e_j$  belongs to two distinct cycles  $\Delta_1$  and  $\Delta_2$  of  $\Gamma'$ , such that  $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$ .

Suppose (a) holds, so we are twisting  $(d_{1,j}, d_{j+1,h})$  into  $(d_{1,j+1}, d_{j,h})$ , with  $h > j + 1$ . Then in  $\Gamma'$  we have the cycles  $\Delta_1$  and  $\Delta_2$  whose edge sets are

$$E(\Delta_1) = \{e_j, d_{1,j+1}, e_1, \dots, e_{j-1}\}$$

and

$$E(\Delta_2) = \{e_j, e_{j+1}, \dots, e_{h-1}, d_{j,h}\}.$$

It is clear that  $\Delta_1$  and  $\Delta_2$  are cycles and that  $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$ .

Now assume (b). We are twisting  $(d_{1,j}, d_{h,j+1})$  into  $(d_{1,j+1}, d_{h,j})$ . Let  $d_{x,y}$  be a chord crossing both  $d_{h,j}$  and  $d_{1,j}$ . We have

$$1 < h < x < j < j + 1 < y.$$

Now the edges of the two cycles containing  $e_j$  and sharing no other edge are

$$E(\Delta_1) = \{e_j, d_{1,j+1}, e_1, \dots, e_{h-1}, d_{h,j}\}$$

and

$$E(\Delta_2) = \{e_j, e_{j+1}, \dots, e_{y-1}, d_{x,y}, e_x, e_{x+1}, \dots, e_{j-1}\}.$$

Since  $1 < h < x$  we have  $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$ . ■

We are ready to prove the following linkage theorem.

**Theorem 3.3.** *Let  $\Gamma_1$  and  $\Gamma_2$  be two  $p$ -regular graphs with  $b_1(\Gamma_1) = b_1(\Gamma_2)$ .*

*Then  $\Gamma_1$  and  $\Gamma_2$  are linked.*

*If  $\Gamma_1$  and  $\Gamma_2$  are 3-edge-connected, then they are 3-linked.*

*Proof.* By Proposition 2.2 we can assume that  $\Gamma_1$  and  $\Gamma_2$  are  $p$ -hamiltonian.

We shall prove the theorem by showing that every  $p$ -hamiltonian graph  $\Gamma$  is linked to the  $p$ -polygon  $\Pi_\gamma^p$ , where  $\gamma = \frac{2b-2}{p-2}$  and  $b = b_1(\Gamma)$ . Moreover, if  $\Gamma$  is 3-edge-connected, we will prove that it is 3-linked to  $\Pi_\gamma^p$ , which is 3-edge-connected by Lemma 2.8.

Let us fix a normalized form of  $\Gamma$ , call  $\Delta$  its hamiltonian cycle, and use the notation of 2.4. Now set

$$(3.1) \quad \epsilon(\Gamma) := \sum \left( \lfloor \gamma/2 \rfloor - \alpha(d_{i,j}) \right)$$

where the sum is over all the chords of  $\Gamma$ . By Lemma 2.6 we have  $\epsilon(\Gamma) \geq 0$ , and  $\epsilon(\Gamma) = 0$  if and only if  $\Gamma$  has no short chord, if and only if  $\Gamma = \Pi_\gamma^p$ .

We will prove the theorem by induction on  $\epsilon(\Gamma)$ . By what we just observed, if  $\epsilon(\Gamma) = 0$  there is nothing to prove, so the induction basis is settled.

Assume now that  $\epsilon(\Gamma) > 0$  and let us pick a short chord; we may call it  $d_{1,j}$  and assume that  $j \leq \gamma/2$ . By Lemma 2.9 we have that there exist chords  $d_{k,l}$  satisfying

$$(3.2) \quad 1 < j < k < l \quad \text{and} \quad l - k < \lfloor \gamma/2 \rfloor.$$

We can assume (up to changing the labeling of the vertices) that there exists one of them such that the path (in  $\Delta$ ) from  $v_j$  to  $v_k$  is not longer than the path from  $v_l$  to  $v_1$ ; i.e. we can assume that

$$(3.3) \quad k - j \leq \gamma + 1 - l.$$

We shall pick the pair  $(d_{1,j}, d_{k,l})$  such that  $k - j$ , i.e. the length of the path in  $\Delta$  from  $v_j$  to  $v_k$ , is minimal with respect to all pairs satisfying (3.2) and (3.3); we shall refer to this as the “minimality property” of  $(d_{1,j}, d_{k,l})$ .

Now that we have fixed our two chords, we can assume, up to switching them and changing the labeling on the vertices, that

$$(3.4) \quad j - 1 = \alpha(d_{1,j}) \leq \alpha(d_{k,l}) = l - k.$$

With these settings, we have

$$(3.5) \quad k \leq \gamma/2 + 1, \quad \text{more exactly } k \leq \lfloor \gamma/2 \rfloor + 1.$$

By contradiction, suppose  $k \geq \lfloor \gamma/2 \rfloor + 2$ . Then (3.4) implies

$$\lfloor \gamma/2 \rfloor + 2 \leq k \leq l - j + 1.$$

Now, by (3.3), we have  $l - j + 1 \leq \gamma + 2 - k$ . Therefore  $\lfloor \gamma/2 \rfloor + 2 \leq \gamma + 2 - k$  and hence  $k \leq \lfloor \gamma/2 \rfloor + 1$ ; a contradiction.

**Claim 3.4.** *Let  $\Gamma'$  be the graph obtained from  $\Gamma$  by twisting the pair of chords  $(d_{1,j}, d_{k,l})$  into the pair  $(d_{1,k}, d_{j,l})$ . Then  $\epsilon(\Gamma') < \epsilon(\Gamma)$ .*

To prove the claim, consider a chord  $d'$  of  $\Gamma'$ . For notational clarity, we will denote by  $d'_{*,*}$  the chords of  $\Gamma'$ . If  $d'$  is not equal to  $d'_{1,k}$  or  $d'_{j,l}$ , then  $d'$  corresponds to a unique chord  $d$  of  $\Gamma$  such that  $\alpha(d) = \alpha(d')$ . Therefore we have

$$(3.6) \quad \epsilon(\Gamma) - \epsilon(\Gamma') = -\alpha(d_{1,j}) - \alpha(d_{k,l}) + \alpha(d'_{1,k}) + \alpha(d'_{j,l}).$$

We know that  $\alpha(d_{1,j}) = j - 1$  and  $\alpha(d_{k,l}) = l - k$ , by construction and by (3.2). Furthermore, by (3.5) we have  $\alpha(d'_{1,k}) = k - 1$ .

To compute the remaining term we need to distinguish two cases.

Case 1:  $l - j \leq \gamma/2$ . Then  $\alpha(d'_{j,l}) = l - j$ . Therefore

$$\epsilon(\Gamma) - \epsilon(\Gamma') = 1 - j + k - l + k - 1 + l - j = 2k - 2j \geq 2$$

by (3.2). So the claim is proved in this case.

Case 2:  $l - j \geq \gamma/2 + 1$ . Now  $\alpha(d'_{j,l}) = \gamma + j - l$ . Therefore

$$\epsilon(\Gamma) - \epsilon(\Gamma') = 1 - j + k - l + k - 1 + \gamma + j - l = \gamma + 2(k - l) \geq 2$$

as  $l - k < \lfloor \gamma/2 \rfloor$  by (3.2). The claim is proved.

Lemma 3.2 says that  $\Gamma$  and  $\Gamma'$  are linked. By the claim we may apply induction, getting that  $\Gamma'$  is linked to  $\Pi_\gamma^p$ ; hence the first part of the Theorem is proved.

Before continuing, we analyze the chords having one end at a vertex  $v_g$ , with  $j + 1 \leq g \leq k - 1$ . Let  $d_{g,*}$  be one such chord. We claim that with our choice of the pair  $(d_{1,j}, d_{k,l})$ , we have

$$(3.7) \quad d_{g,*} = d_{g,m}, \quad m \geq k.$$

By contradiction, suppose  $m < k$ . If  $m < g$  we have (as  $g \leq k - 1$  and  $m \geq 1$ )

$$g - m \leq k - 1 - 1 \leq \gamma/2 - 1$$

by (3.5). Therefore  $d_{m,g}$  satisfies the properties satisfied by  $d_{1,j}$ : it is a short chord whose short side does not intersect the short side of  $d_{k,l}$ , and it verifies (3.3), i.e. the path from  $v_g$  to  $v_k$  is not shorter than the path from  $v_l$  to  $v_m$ . Now, the path from  $v_g$  to  $v_k$  is obviously shorter than the path from  $v_j$  to  $v_k$ , contradicting the minimality property of  $(d_{1,j}, d_{k,l})$ .

Suppose now that  $g < m < k$ . Again,  $d_{g,m}$  satisfies (3.3) and  $v_m$  is closer to  $v_k$  than  $v_j$ . Therefore, in order to respect the minimality property of  $(d_{1,j}, d_{k,l})$ , we must have  $m - g \geq \gamma/2$ . This implies  $(m \leq k - 1 \leq \gamma/2$  by (3.5) and  $g \geq j + 1)$

$$\gamma/2 \leq m - g \leq \gamma/2 - j - 1$$

which is obviously impossible. (3.7) is proved.

To finish the proof of the theorem, it is enough to show that, if  $\Gamma$  is 3-edge-connected, then  $\Gamma'$  is 3-edge-connected and 3-linked to  $\Gamma$ . To do that we shall factor the twist of  $(d_{1,j}, d_{k,l})$  into  $(d_{1,k}, d_{j,l})$  by a series of twists swapping consecutive vertices, each of which preserves 3-edge-connectivity. We do that with two sets of twists. To define the first set, we make a choice of a chord  $d_{h+1,*}$  for every  $j \leq h \leq k - 1$ . It will be clear that such a choice is irrelevant.

- (I.1) Twist  $(d_{1,j}, d_{j+1,*})$  into  $(d_{1,j+1}, d_{j,*})$ .  
 (I.2) Twist  $(d_{1,j+1}, d_{j+2,*})$  into  $(d_{1,j+2}, d_{j+1,*})$ .  
 .....  
 (I.h+1-j) Twist  $(d_{1,h}, d_{h+1,*})$  into  $(d_{1,h+1}, d_{h,*})$ , with  $j \leq h \leq k - 1$ .  
 .....  
 (I.k-j) Twist  $(d_{1,k-1}, d_{k,l})$  into  $(d_{1,k}, d_{k-1,l})$

Observe that in each of the above twists, the two chords getting twisted,  $d_{1,h}$  and  $d_{h+1,*}$ , do not cross each other i.e.  $d_{h+1,*} = d_{h+1,m}$  with  $m > h + 1$ . This is obvious for the last step, (I.k - j), as  $1 < k - 1 < k < l$ . For the remaining steps, for which  $h \leq k - 2$ , we use (3.7), according to which every  $d_{h+1,*}$  is of type  $d_{h+1,m}$  with  $m \geq k$ . Hence  $1 < h < h + 1 \leq k - 1 < m$ , as claimed.

Therefore condition (a) of Lemma 3.2 holds, and we conclude that the graph  $\Gamma''$ , obtained after the above set of twists, is 3-edge-connected and 3-linked to  $\Gamma$ .

Notice that  $\Gamma''$  contains the chord  $d_{1,k}$  and the chord  $d_{k-1,l}$ . The second set of twists, starting from  $\Gamma''$  is the following.

- (II.1) Twist  $(d_{k-1,l}, d_{k-2,*})$  into  $(d_{k-2,l}, d_{k-1,*})$ .  
 (II.2) Twist  $(d_{k-2,l}, d_{k-3,*})$  into  $(d_{k-3,l}, d_{k-2,*})$ .  
 .....  
 (II.k-h) Twist  $(d_{h,l}, d_{h-1,*})$  into  $(d_{h-1,l}, d_{h,*})$ , where  $j + 1 \leq h \leq k - 1$ .  
 .....  
 (II.k-j-1) Twist  $(d_{j+1,l}, d_{j,*})$  into  $(d_{j,l}, d_{j+1,*})$

where the chords  $d_{h-1,*}$  are those chosen for the first set of twists. Observe that the chord  $d_{1,k}$  (which lies in every graph appearing in the above twists) crosses every chord  $d_{h,l}$  with  $j + 1 \leq h \leq k - 1$ . If  $d_{1,k}$  crosses also  $d_{h-1,*}$  Lemma 3.2 applies to the step (II.k-h) above (condition (b) of Lemma 3.2 holds), and hence 3-edge-connectivity is preserved (to fit in precisely with the notation of Lemma 3.2, one translates the starting vertex after  $v_h$ , sets  $h - 1 = j$  and  $h = j + 1$  so that  $d_{h-1,*}$  becomes  $d_{i,j}$  and  $d_{h,l}$  becomes  $d_{j+1,*}$ ).

What if  $d_{1,k}$  does not cross  $d_{h-1,*}$ ? Recall that by (3.7) we have  $d_{h-1,*} = d_{h-1,m}$  with  $m \geq k$ . Therefore  $d_{1,k}$  does not cross  $d_{h-1,m}$  only if  $k = m$ . Let us show that twisting  $(d_{h,l}, d_{h-1,k})$  into  $(d_{h-1,l}, d_{h,k})$  preserves 3-edge-connectivity; let  $\tilde{\Gamma}$  be the graph obtained after this twist. By Lemma 2.8(2) it suffices to show that  $\tilde{\Gamma}$  contains two cycles,  $\Delta_1$  and  $\Delta_2$ , whose only edge in common is the edge  $e_{h-1}$  (joining the two swapped vertices  $v_{h-1}$  and  $v_h$ ). Here are the two cycles

$$\Delta_1 = (e_{h-1}, d_{h,k}, d_{1,k}, e_1, \dots, e_{h-2})$$

and

$$\Delta_2 = (e_{h-1}, e_h, \dots, e_{l-1}, d_{h-1,l}).$$

Therefore the graph  $\Gamma'''$  obtained from  $\Gamma$  by our two sets of twists is 3-edge-connected and 3-linked to  $\Gamma$ . Let us check that  $\Gamma'''$  coincides with the  $\Gamma'$  of Claim 3.4. The chords  $d_{1,j}$  and  $d_{k,l}$  of  $\Gamma$  are twisted into  $d_{1,k}$  and  $d_{j,l}$  in  $\Gamma'$  and  $\Gamma'''$ . The remaining chords of  $\Gamma$  and  $\Gamma'$  are the same. The chord  $d_{h+1,m} \in E(\Gamma)$  with  $j \leq h \leq k-2$ , in the first set of twists, is changed into the chord  $d_{h,m} \in E(\Gamma')$ , which is changed back into  $d_{h+1,m} \in E(\Gamma''')$  by the second set of twists. All other chords of  $\Gamma$  are not touched by our twists. So  $\Gamma''' = \Gamma'$  and we are done.  $\blacksquare$

#### 4. TROPICAL CURVES AND WEIGHTED METRIC GRAPHS

**4.1. Context.** The original definition of a tropical curve can be given in terms of metric graphs (by [MiZ08]): a tropical curve is a metric graph  $(\Gamma, l)$  where  $\Gamma$  is a graph with every vertex of valency at least 3, and  $l : E(\Gamma) \rightarrow \mathbb{R}_{>0}^{|E(\Gamma)|}$  a *length* function on the edges. The genus of  $(\Gamma, l)$  is  $g(\Gamma, l) = b_1(\Gamma)$ .

This definition presents a problem when studying families of tropical curves. To explain why, let us fix the graph  $\Gamma$ , and consider the space of all metric graphs supported on it. Denote  $E(\Gamma) = \{e_1, \dots, e_{|E(\Gamma)|}\}$  and set  $\mathbb{R}(\Gamma) := \mathbb{R}_{>0}^{|E(\Gamma)|}$ . It is obvious that any  $(l_1, \dots, l_{|E(\Gamma)|}) \in \mathbb{R}(\Gamma)$  defines a unique metric graph with length function  $l$  defined by  $l(e_i) = l_i, \forall i$ . Therefore  $\mathbb{R}(\Gamma)$  surjects onto the space of all metric graphs supported on  $\Gamma$ , providing a nice ambient space in which to study families. Now,  $\mathbb{R}(\Gamma) \subset \mathbb{R}^{|E(\Gamma)|}$  is not closed, its closure is  $\overline{\mathbb{R}(\Gamma)} = \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$ . One is thus naturally led to try and interpret this closure in geometric terms. To do that, fix a point in  $\mathbb{R}(\Gamma)$  and let one of its entries,  $l_1$  say, tend to 0. This corresponds to considering a family of metric graphs, all supported on  $\Gamma$ , such that the length of  $e_1$  tends to 0. How do we describe the limit in geometric language? In tropical geometry this question has a simple answer. As  $l_1$  tends to zero,  $(\Gamma, l)$  specializes to a metric graph  $(\bar{\Gamma}, \bar{l})$  where  $\bar{\Gamma} = \Gamma/e_1$  is obtained by contracting  $e_1$  to a point (see 1.2); hence  $E(\bar{\Gamma}) = E(\Gamma) \setminus \{e_1\} = \{e_2, \dots, e_{|E(\Gamma)|}\}$  and  $\bar{l}(e_i) = l(e_i), \forall i \geq 2$ .

But there is a drawback with this limit: its genus may be smaller than  $g(\Gamma, l)$ . Indeed we have

$$g(\bar{\Gamma}, \bar{l}) = \begin{cases} g(\Gamma, l) - 1 & \text{if } e_1 \text{ is a loop} \\ g(\Gamma, l) & \text{otherwise.} \end{cases}$$

From a geometric perspective this is quite unpleasant. We would prefer the genus to remain constant under specialization. A solution to this problem is provided in [BMV]. The idea there is to extend the definition of a tropical curve by adding some structure to the original definition. A tropical curve should thus be a metric weighted graph, where a weight function on the set of vertices is the additional structure. Let us define that precisely.

**4.2. Weighted graphs and weighted contractions.** A *weighted graph* is a pair  $(\Gamma, w)$  where  $\Gamma$  is a graph and  $w : V(\Gamma) \rightarrow \mathbb{R}_{\geq 0}^{|V(\Gamma)|}$  a *weight* function on the vertices. The genus of a weighted graph is the number  $g(\Gamma, w)$  defined as follows:

$$(4.1) \quad g(\Gamma, w) = b_1(\Gamma) + \sum_{v \in V(\Gamma)} w(v) = b_1(\Gamma) + |w|.$$

$(\Gamma, w)$  is called *stable* if any vertex of weight 0 has valency at least 3.

Let  $S \subset E(\Gamma)$  be a set of edges of  $\Gamma$ ; we now define the weighted graph obtained by contracting  $S$ . This is the weighted graph  $(\Gamma/S, w/S)$  where  $\Gamma/S$  is the contraction of  $\Gamma$  obtained by contracting to a vertex the edges that are in  $S$ ; see subsection 1.2. The weight function  $w/S$  is defined as follows. We have a natural surjection  $\nu : V(\Gamma) \rightarrow V(\Gamma/S)$ , and a natural surjective map  $\sigma : \Gamma \rightarrow \Gamma/S$ . We set for every  $\bar{v} \in V(\Gamma/S)$

$$\begin{aligned}
 w/S(\bar{v}) &= \sum_{v \in \nu^{-1}(\bar{v})} w(v) + \#\{\text{loops contained in } \sigma^{-1}(\bar{v})\} = \\
 (4.2) \quad &= \sum_{v \in \nu^{-1}(\bar{v})} \left( w(v) + \#\{\text{loops in } S \text{ based at } v\} \right).
 \end{aligned}$$

With this definition we have, by Remark 1.3,

$$(4.3) \quad g(\Gamma, w) = g(\Gamma/S, w/S).$$

We say that  $(\Gamma/S, w/S)$  is a *weighted contraction* of  $(\Gamma, w)$ . Let  $(\Gamma', w')$  be a weighted graph. We denote.

$$(4.4) \quad (\Gamma, w) \geq (\Gamma', w') \text{ if and only if } (\Gamma', w') \text{ is a weighted contraction of } (\Gamma, w).$$

*Remark 4.3.* If  $(\Gamma, w)$  is stable and  $(\Gamma', w') \leq (\Gamma, w)$ , also  $(\Gamma', w')$  is stable.

**4.4. Metric weighted graphs and tropical curves.** Following [BMV], a tropical curve of genus  $g$  will be a stable metric weighted graph of genus  $g$ . A metric weighted graph is a triple  $(\Gamma, w, l)$  where  $(\Gamma, w)$  is a weighted graph and  $l : E(\Gamma) \rightarrow \mathbb{R}_{>0}^{|E(\Gamma)|}$  a length function on the edges. The genus of  $(\Gamma, w, l)$  is

$$(4.5) \quad g(\Gamma, w, l) := g(\Gamma, w) = b_1(\Gamma) + |w|.$$

$(\Gamma, w, l)$  is *stable* if so is  $(\Gamma, w)$  (any vertex of valency  $\leq 2$  has positive weight).

**Definition 4.5.** A stable metric weighted graph  $(\Gamma, w, l)$  will be called a *tropical curve* of genus  $g(\Gamma, w, l)$ .

If the weight function is constantly 0 (i.e.  $w(v) = 0$  for every  $v \in V(\Gamma)$ ) we say that the tropical curve is *pure* and write  $(\Gamma, \mathbf{0}, l)$ .

A tropical curve is called *regular* if it is pure and if  $\Gamma$  is a 3-regular graph.

This definition generalizes the original one, as unweighted metric graphs obviously correspond to pure tropical curves. The first part is the truly important one, and it is due to [BMV]. With this definition the genus of a tropical curve remains constant under specialization, by (4.3).

The remaining two notions, of pure and regular tropical curve, are ours. The notion of pure tropical curve is useful to keep track of the original definition and to connect to the pre-existing literature. The notion of regular tropical curve suggests that, in our view, pure tropical curves with a 3-regular graph play the role of regular (i.e. nonsingular) curves in the classical moduli theory of algebraic curves. In fact regular tropical curves have the following property. If a tropical curve  $X$  specializes to a regular curve, then  $X$  itself is regular. The same statement holds for nonsingular algebraic curves: if an algebraic curve specializes to a nonsingular one, it is nonsingular.

We say that  $(\Gamma_1, w_1, l_1)$  and  $(\Gamma_2, w_2, l_2)$  are isomorphic, writing  $(\Gamma_1, w_1, l_1) \cong (\Gamma_2, w_2, l_2)$ , if the following holds. There is an isomorphism  $\alpha = (\alpha_V, \alpha_E)$  between  $\Gamma_1$  and  $\Gamma_2$ , where  $\alpha_V : V(\Gamma_1) \rightarrow V(\Gamma_2)$  and  $\alpha_E : E(\Gamma_1) \rightarrow E(\Gamma_2)$  are bijections such that  $\alpha_E$  maps the endpoints of  $v \in V(\Gamma_1)$  to the endpoints of  $\alpha_V(v)$  for every  $v \in V(\Gamma_1)$  ([D97] p.3). Moreover  $\forall v \in V(\Gamma_1)$  and  $\forall e \in E(\Gamma_1)$  we have  $w_1(v) = w_2(\alpha_V(v))$  and  $l_1(e) = l_2(\alpha_E(e))$ .

In particular if  $(\Gamma, w)$  is a weighted graph its automorphism group  $\text{Aut}(\Gamma, w)$  is given by pairs  $\alpha = (\alpha_V, \alpha_E)$  where  $\alpha$  is an isomorphism of  $(\Gamma, w)$  with itself as in the previous definition, but ignoring the condition on the length.

*Remark 4.6.* The automorphism group of a weighted graph, and hence of a tropical curve, is finite.

Let us fix the weighted graph  $(\Gamma, w)$ , let  $g = g(\Gamma, w)$ , and let us consider the space  $M(\Gamma, w)$  of isomorphism classes of metric graphs supported on it.

In order to study that, we generalize what we did before for unweighted graphs, using the same notation. Denote

$$\mathbb{R}(\Gamma, w) := \mathbb{R}_{>0}^{|E(\Gamma)|}.$$

Again any element in  $\mathbb{R}(\Gamma, w)$  defines a unique metric graph supported on  $(\Gamma, w)$ . Therefore there is a natural surjection

$$(4.6) \quad \pi : \mathbb{R}(\Gamma, w) \longrightarrow M(\Gamma, w);$$

it is clear that  $\pi(\Gamma, w, l) = \pi(\Gamma, w, l')$  if and only if  $(\Gamma, w, l) \cong (\Gamma, w, l')$ . Notice in particular that  $\pi$  has finite fibers (Remark 4.6).

As before, the closure of  $\mathbb{R}(\Gamma, w)$  is

$$\overline{\mathbb{R}(\Gamma, w)} = \mathbb{R}_{\geq 0}^{|E(\Gamma)|} \subset \mathbb{R}^{|E(\Gamma)|}.$$

Now we ask again: is there a natural geometric interpretation of  $\overline{\mathbb{R}(\Gamma, w)}$ ?

Let  $p \in \overline{\mathbb{R}(\Gamma, w)} \setminus \mathbb{R}(\Gamma, w)$  and, to simplify the notation, suppose that

$$p = (x_1, \dots, x_n, 0, \dots, 0)$$

with  $x_i > 0$  for all  $i = 1, \dots, n$ , for some  $0 \leq n < |E(\Gamma)|$ . Let us show that there is a unique metric weighted graph of genus  $g = g(\Gamma, w)$  associated to  $p$ , denoted  $(\Gamma_p, w_p, l_p)$  defined as follows. Let  $S = \{e_{n+1}, \dots, e_{|E(\Gamma)|}\} \subset E(\Gamma)$ ; then  $(\Gamma_p, w_p) = (\Gamma/S, w/S)$ , i.e.  $(\Gamma_p, w_p)$  is the weighted contraction of  $(\Gamma, w)$  obtained by contracting  $S$ . Therefore we have a natural identification

$$E(\Gamma_p) = E(\Gamma) \setminus S = \{e_1, \dots, e_n\}.$$

The length function  $l_p$  is defined by setting  $l_p(e_i) = x_i$  for all  $e_i \in E(\Gamma_p)$ . As we said above in (4.3), we have  $g = g(\Gamma_p, w_p)$ .

Summarizing: we showed that the boundary points of  $\overline{\mathbb{R}(\Gamma, w)}$  parametrize metric weighted graphs of genus  $g$  whose underlying weighted graph is a contraction of  $(\Gamma, w)$ . More precisely, for any set of indices  $I \subset \{1, 2, \dots, |E(\Gamma)|\}$ , denote by  $\overline{\mathbb{R}(\Gamma, w)}_I \subset \overline{\mathbb{R}(\Gamma, w)} = \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$  the set

$$\overline{\mathbb{R}(\Gamma, w)}_I := \{(x_1, \dots, x_{|E(\Gamma)|}) \in \overline{\mathbb{R}(\Gamma, w)} : x_i = 0 \forall i \in I, x_i > 0 \forall i \notin I\}.$$

We have proved the following.

**Lemma 4.7.** *Notation as above. The decomposition*

$$\overline{\mathbb{R}(\Gamma, w)} = \coprod_{I \subset \{1, 2, \dots, |E(\Gamma)|\}} \overline{\mathbb{R}(\Gamma, w)}_I$$

is such that for every  $I$  there is a natural isomorphism  $\overline{\mathbb{R}(\Gamma, w)}_I \cong \mathbb{R}(\Gamma', w')$  where  $(\Gamma', w')$  is a uniquely determined weighted contraction of  $(\Gamma, w)$ .

If  $I = \emptyset$  then  $\overline{\mathbb{R}(\Gamma, w)}_I = \mathbb{R}(\Gamma, w)$ ; if  $I = E(\Gamma)$  then  $\overline{\mathbb{R}(\Gamma, w)}_I = \{0\}$ , corresponding to the graph with no edges and one vertex of weight  $g$ .

Let  $(\Gamma, w)$  be a stable weighted graph as above. Consider two points  $p_1, p_2 \in \overline{\mathbb{R}(\Gamma, w)}$ . By what we said there exist two metric weighted graphs associated to them, denoted  $(\Gamma_1, w_1, l_1)$  and  $(\Gamma_2, w_2, l_2)$  (with  $(\Gamma, w) \geq (\Gamma_i, w_i)$  for  $i = 1, 2$ ). We define an equivalence relation on  $\overline{\mathbb{R}(\Gamma, w)}$  as follows

$$(4.7) \quad p_1 \sim p_2 \text{ if and only if } (\Gamma_1, w_1, l_1) \cong (\Gamma_2, w_2, l_2)$$

(see subsection 4.4 for the definition of isomorphism). The quotient space with respect to this equivalence relation will be denoted

$$(4.8) \quad \overline{\pi} : \overline{\mathbb{R}(\Gamma, w)} \longrightarrow \overline{M(\Gamma, w)} := \overline{\mathbb{R}(\Gamma, w)} / \sim$$

and  $\overline{M(\Gamma, w)}$  is a topological space, with the quotient topology of the euclidean topology on  $\mathbb{R}(\Gamma, w) = \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$ .

Recall now that  $\text{Aut}(\Gamma, w)$  is made of (suitable) pairs  $\alpha = (\alpha_V, \alpha_E)$  of permutations on the vertices and on the edges. Therefore  $\text{Aut}(\Gamma, w)$  acts linearly on  $\mathbb{R}(\Gamma, w)$  by permuting the coordinates according to  $\alpha_E$ . It is clear that  $\text{Aut}(\Gamma, w)$  acts as group of isometries with respect to the Euclidean metric.

For any  $G$  acting on  $\mathbb{R}(\Gamma, w)$  and any subset  $Z \subset \mathbb{R}(\Gamma, w)$ , we denote by  $Z^G$  the set of  $G$ -orbits of the elements in  $Z$ .

**Proposition 4.8.** *Let  $(\Gamma, w)$  be a stable weighted graph.*

(1) *There is a canonical decomposition*

$$\overline{M(\Gamma, w)} = \coprod_{(\Gamma', w') \leq (\Gamma, w)} M(\Gamma', w'),$$

where  $M(\Gamma, w)$  is open and dense in  $\overline{M(\Gamma, w)}$ .

(2) *The quotient map  $\bar{\pi} : \mathbb{R}(\Gamma, w) \rightarrow \overline{M(\Gamma, w)}$  factors as follows:*

$$\bar{\pi} : \mathbb{R}(\Gamma, w) \xrightarrow{\tau} \overline{\mathbb{R}(\Gamma, w) / \text{Aut}(\Gamma, w)} \xrightarrow{\gamma} \overline{M(\Gamma, w)}.$$

Moreover  $\tau$  is an open map and  $\bar{\pi}$  has finite fibers.

(3)  *$\overline{M(\Gamma, w)}$  is a Hausdorff topological space.*

*Proof.* The existence of the decomposition is a straightforward consequence of the definition of  $\overline{M(\Gamma, w)}$  and of Lemma 4.7.

Recall from (4.6) that  $M(\Gamma, w) = \mathbb{R}(\Gamma, w) / \sim$ . Now  $\mathbb{R}(\Gamma, w)$  is open and dense in  $\mathbb{R}(\Gamma, w)$  and we have  $\bar{\pi}^{-1}(M(\Gamma, w)) = \mathbb{R}(\Gamma, w)$ . Therefore  $M(\Gamma, w)$  is open and dense in  $\overline{M(\Gamma, w)}$ .

Now let  $G := \text{Aut}(\Gamma, w)$ . The restriction of  $\bar{\pi}$  to  $\mathbb{R}(\Gamma, w)$  is the quotient  $\pi : \mathbb{R}(\Gamma, w) \rightarrow M(\Gamma, w) = \mathbb{R}(\Gamma, w) / G$  so in this case part (2) is proved.

Let now  $p \in \mathbb{R}(\Gamma, w)_I$  with  $I \neq \emptyset$  (cf. Lemma 4.7). Then  $p = (x_1, \dots, x_{|E(\Gamma)|})$  with  $x_i = 0$  for every  $i \in I$  and  $x_i > 0$  otherwise. Let  $\alpha \in G$ , then  $\alpha$  acts as a permutation on  $|E(\Gamma)|$  elements, hence, abusing notation, we have

$$\alpha(p) = (x_{\alpha(1)}, \dots, x_{\alpha(|E(\Gamma)|)}) \in \mathbb{R}(\Gamma, w)_{\alpha(I)},$$

where  $\alpha(I) = \{\alpha(i), \forall i \in I\} \subset \{1, 2, \dots, |E(\Gamma)|\}$ . Let  $S \subset E(\Gamma)$  and  $\alpha(S) \subset E(\Gamma)$  be the set of edges corresponding, respectively, to  $I$  and  $\alpha(I)$ . Lemma 4.7 yields

$$\overline{\mathbb{R}(\Gamma, w)}_I = \mathbb{R}(\Gamma/S, w/S), \quad \overline{\mathbb{R}(\Gamma, w)}_{\alpha(I)} = \mathbb{R}(\Gamma/\alpha(S), w/\alpha(S)).$$

The automorphism  $\alpha$  maps  $S$  into  $\alpha(S)$ , hence it induces an isomorphism

$$\bar{\alpha} : \Gamma/S \xrightarrow{\cong} \Gamma/\alpha(S).$$

Moreover,  $\alpha$  (as any automorphism of any graph) maps bijectively loops to loops, therefore  $\bar{\alpha}$  is actually an isomorphism of weighted graphs. We thus proved that  $(\Gamma/S, w/S) \cong (\Gamma/\alpha(S), w/\alpha(S))$ . Finally, the point  $p$  corresponds to a metric graph on  $(\Gamma/S, w/S)$  with length function  $l(e_i) = x_i$  for every  $i \notin I$  (we have  $E(\Gamma/S) = E(\Gamma) \setminus S$ ); on the other hand  $\alpha(p)$  corresponds to a metric graph on  $(\Gamma/S, w/S)$  with length  $\alpha(l)$  given by  $\alpha(l)(e_{\alpha(i)}) = x_{\alpha(i)}$  for every  $i \notin I$ . But then  $p$  and  $\alpha(p)$  parametrize isomorphic metric weighted graphs (and the isomorphism is given by  $\bar{\alpha}$ ). Therefore  $\bar{\pi}(p) = \bar{\pi}(\alpha(p))$  and the factorization of part (2) is proved.

The fact that  $\tau$  is open follows easily from  $G$  being a finite group of isometries of  $\mathbb{R}(\Gamma, w)$ . (Let  $A \subset \mathbb{R}(\Gamma, w)$  be open, then  $A^G$  is open and it is a finite union of copies of  $A$ , moreover  $\tau(A) = \tau(A^G)$  and  $\tau^{-1}(\tau(A^G)) = A^G$ : hence  $\tau(A)$  is open.)

This also easily implies that  $\overline{\mathbb{R}(\Gamma, w) / \text{Aut}(\Gamma, w)}$  is Hausdorff. (Let  $p, q$  be in  $\overline{\mathbb{R}(\Gamma, w) / \text{Aut}(\Gamma, w)}$ , let  $\tau^{-1}(p) = \{p_1, \dots, p_n\}$  and  $\tau^{-1}(q) = \{q_1, \dots, q_m\}$ ; then

there exists  $\epsilon \in \mathbb{R}_{>0}$  such that the ball of radius  $\epsilon$  centered at  $p_i$ ,  $B_{p_i}(\epsilon)$ , does not intersect  $B_{q_j}(\epsilon) \forall i, j$ . Set  $U_p := \cup B_{p_i}(\epsilon) \cap \overline{\mathbb{R}(\Gamma, w)}$  and  $U_q := \cup B_{q_i}(\epsilon) \cap \overline{\mathbb{R}(\Gamma, w)}$ . Then  $U_p \cap U_q = \emptyset$ ,  $U_p, U_q$  are open and invariant under  $\text{Aut}(\Gamma, w)$ . Hence  $\tau(U_p)$  and  $\tau(U_q)$  are disjoint open subsets separating  $p$  from  $q$ .

The fibers of  $\bar{\pi}$  are finite because the automorphism group of a tropical curve is finite, and because the decomposition of Lemma 4.7 is obviously finite.

Let us show that  $\overline{M(\Gamma, w)}$  is Hausdorff. By what we proved,  $\gamma$  induces a homeomorphism of the dense open subset  $M(\Gamma, w) \subset \overline{\mathbb{R}(\Gamma, w)}/\text{Aut}(\Gamma, w)$  with itself sitting in  $\overline{M(\Gamma, w)}$ . We will refer to the complement of  $M(\Gamma, w)$  in  $\overline{\mathbb{R}(\Gamma, w)}/\text{Aut}(\Gamma, w)$ , or in  $\overline{M(\Gamma, w)}$ , as the “boundary”, which is a closed subset.

Let  $\bar{p}$  and  $\bar{q}$  two points in  $\overline{M(\Gamma, w)}$ , let  $(\Gamma_{\bar{p}}, w_{\bar{p}})$  and  $(\Gamma_{\bar{q}}, w_{\bar{q}})$  be their weighted graphs. (If both  $\bar{p}$  and  $\bar{q}$  are in  $M(\Gamma, w)$  there is nothing to add.)

As  $\overline{\mathbb{R}(\Gamma, w)}/\text{Aut}(\Gamma, w)$  is Hausdorff it contains two disjoint open subsets  $U$  and  $V$  such that  $\gamma^{-1}(\bar{p}) \subset U$  and  $\gamma^{-1}(\bar{q}) \subset V$  (the fibers of  $\gamma$  are finite). We may assume that  $U$  intersects the boundary only in connected neighborhoods of the points in  $\gamma^{-1}(\bar{p})$ , parametrizing tropical curves supported on  $(\Gamma_{\bar{p}}, w_{\bar{p}})$ . Similarly for  $V$ . We denote  $\partial U$  and  $\partial V$  the intersection of  $U$  and  $V$  with the boundary. Notice that, by what we proved before, the restriction of  $\gamma$  over  $M(\Gamma_{\bar{p}}, w_{\bar{p}}) \subset \overline{M(\Gamma, w)}$  is of type

$$\gamma^{-1}(M(\Gamma_{\bar{p}}, w_{\bar{p}})) \cong \mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})/H \longrightarrow M(\Gamma_{\bar{p}}, w_{\bar{p}}) = \mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})/\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})$$

where  $H \subset \text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})$  is the subgroup of automorphisms which lift to automorphisms of  $(\Gamma, w)$ . Therefore  $\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})$  acts on  $\gamma^{-1}(M(\Gamma_{\bar{p}}, w_{\bar{p}}))$  by its quotient mod  $H$  and we have

$$\gamma^{-1}(\gamma(U)) = U \cup (\partial U)^{\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})}$$

( $\gamma$  is an isomorphism away from the boundary). Similarly for  $V$ . Therefore if  $(\partial U)^{\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})} \subset U$  we have that  $\gamma(U)$  is open. We will prove that we can choose  $U$  and  $V$  so that this condition holds.

To do that, let  $U' = \tau^{-1}(U)$  and  $V' = \tau^{-1}(V)$ . Let  $\bar{\pi}^{-1}(\bar{p}) = \{p_1, \dots, p_n\}$  and  $\bar{\pi}^{-1}(\bar{q}) = \{q_1, \dots, q_m\}$ . So  $U'$  and  $V'$  are disjoint open subsets of  $\overline{\mathbb{R}(\Gamma, w)}$  such that  $\{p_1, \dots, p_n\} \subset U'$  and  $\{q_1, \dots, q_m\} \subset V'$ . Moreover  $U'$  intersects the boundary only in connected open subsets isomorphic to  $\mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})$  (recall Lemma 4.7); each such intersection contains some  $p_i$ ; similarly for  $V'$  (replacing “ $p$ ” by “ $q$ ”). Now fix a boundary locus intersecting  $U'$ , suppose that  $p_1, \dots, p_l$  lie in it for some  $l \leq n$  and focus on the restriction of  $\bar{\pi}$ :

$$\overline{\mathbb{R}(\Gamma, w)} \supset \mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}}) \longrightarrow M(\Gamma_{\bar{p}}, w_{\bar{p}}) = \mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})/\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}}) \subset \overline{M(\Gamma, w)}.$$

We know that the above is a quotient by a finite group of isometries of  $\mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})$ . Therefore we can find open balls  $B_{p_i}(\epsilon) \subset U'$  for  $i = 1, \dots, l$ . Since  $\overline{\mathbb{R}(\Gamma, w)}$  contains finitely many subsets isomorphic to  $\mathbb{R}(\Gamma_{\bar{p}}, w_{\bar{p}})$  (using Lemma 4.7), we can find an  $\epsilon$  such that  $B_{p_i}(\epsilon) \subset U'$  for every  $i = 1, \dots, n$ . Now, up to shrinking  $U'$  near each  $p_i$  we can assume that its intersection with the boundary is the union of those balls. Note that this union is invariant under  $\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})$ , i.e.

$$(\cup_1^n B_{p_i}(\epsilon))^{\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})} = \cup_1^n B_{p_i}(\epsilon).$$

We do the same for  $V'$ . We have  $\tau^{-1}(\tau(U')) = U'$  and  $\tau^{-1}(\tau(V')) = V'$ . Therefore  $U := \tau(U')$  and  $V := \tau(V')$  are open. By construction  $\text{Aut}(\Gamma_{\bar{p}}, w_{\bar{p}})$  fixes  $\partial U$  and  $\text{Aut}(\Gamma_{\bar{q}}, w_{\bar{q}})$  fixes  $\partial V$ . Therefore  $\gamma(U)$  and  $\gamma(V)$  are open disjoint subsets with  $\bar{p} \in \gamma(U)$  and  $\bar{q} \in \gamma(V)$ . This shows that  $\overline{M(\Gamma, w)}$  is Hausdorff.  $\blacksquare$

Next, recall that  $\underline{0}$  denotes the weight function constantly 0,

**Lemma 4.9.** *Let  $(\Gamma, w)$  be a genus  $g$  stable weighted graph. Then  $|E(\Gamma)| \leq 3g - 3$  and equality holds if and only if  $\Gamma$  is a 3-regular graph with  $b_1(\Gamma) = g$ . Moreover in such a case  $w = \mathbf{0}$ .*

*Proof.* We have  $g = b_1(\Gamma) + |w| = |E(\Gamma)| - |V(\Gamma)| + 1 + |w|$ ; hence, as  $|w| \geq 0$ ,

$$|E(\Gamma)| = g - 1 - |w| + |V(\Gamma)| \leq g - 1 + |V(\Gamma)|$$

and the maximum is achieved for  $|w| = 0$ . In this case  $g = b_1(\Gamma)$  and every vertex of  $\Gamma$  must have valency at least 3 (as  $(\Gamma, w)$  is stable), therefore

$$g = |E(\Gamma)| - |V(\Gamma)| + 1 \geq 3|V(\Gamma)|/2 - |V(\Gamma)| + 1 = |V(\Gamma)|/2 + 1$$

hence  $|V(\Gamma)| \leq 2g - 2$  and equality holds if and only if  $\Gamma$  is 3-regular. In such a case we have  $|E(\Gamma)| = 3g - 3$  and  $b_1(\Gamma) = g$ .  $\blacksquare$

## 5. THE MODULI SPACE OF TROPICAL CURVES

We shall now introduce the moduli space of genus  $g$  tropical curves,  $M_g^{\text{trop}}$ , as a topological space. Our results about it are summarized in Theorem 5.2, whose most interesting part is, we believe, the fact that  $M_g^{\text{trop}}$  is Hausdorff. We set

$$(5.1) \quad M_g^{\text{trop}} := \left( \coprod_{\substack{\Gamma \text{ 3-regular} \\ b_1(\Gamma)=g}} \overline{M(\Gamma, \mathbf{0})} \right) / \equiv$$

where  $\equiv$  is the following equivalence relation. Let  $[(\Gamma_1, w_1, l_1)]$  and  $[(\Gamma_2, w_2, l_2)]$  be two points in  $\coprod \overline{M(\Gamma, \mathbf{0})}$ . We set

$$[(\Gamma_1, w_1, l_1)] \equiv [(\Gamma_2, w_2, l_2)] \text{ if and only if } (\Gamma_1, w_1, l_1) \cong (\Gamma_2, w_2, l_2).$$

*Remark 5.1.* Consider the quotient map

$$(5.2) \quad \pi_g : \coprod_{\substack{\Gamma \text{ 3-regular} \\ b_1(\Gamma)=g}} \overline{M(\Gamma, \mathbf{0})} \longrightarrow M_g^{\text{trop}}.$$

If  $\Gamma$  is a 3-regular graph with  $b_1(\Gamma) = g$ , then  $\pi_g^{-1}(\pi_g([(\Gamma, \mathbf{0}, l)]) = [(\Gamma, \mathbf{0}, l)]$ .

Comparing with (4.7) we have another description of  $M_g^{\text{trop}}$ :

$$(5.3) \quad M_g^{\text{trop}} = \left( \coprod_{\substack{\Gamma \text{ 3-regular} \\ b_1(\Gamma)=g}} \overline{\mathbb{R}(\Gamma, \mathbf{0})} \right) / \sim.$$

Since  $M_g^{\text{trop}}$  is the quotient space of a topological space, we endow it with the quotient topology (the two expressions of  $M_g^{\text{trop}}$  as a quotient yield the same topology).

**Theorem 5.2.** (1) *The points of  $M_g^{\text{trop}}$  bijectively parametrize isomorphism classes of tropical curves of genus  $g$ .*

(2) *Let  $M_g^{\text{reg}} \subset M_g^{\text{trop}}$  be the subset parametrizing regular curves. Then*

$$M_g^{\text{reg}} = \coprod_{\substack{\Gamma \text{ 3-regular} \\ b_1(\Gamma)=g}} M(\Gamma, \mathbf{0}) \subset M_g^{\text{trop}}.$$

*$M_g^{\text{reg}}$  is open and dense.*

(3) *Let  $M_g^{\text{pure}}$  be the subset parametrizing pure tropical curves. Then  $M_g^{\text{pure}}$  is open and dense.*

(4)  *$M_g^{\text{trop}}$  is a connected, Hausdorff topological space.*

*Proof.* Let  $(\Gamma, w, l)$  be a stable metric graph of genus  $g$ , and let us prove that its isomorphism class corresponds to a point in  $M_g^{\text{trop}}$ . This amounts to showing that  $(\Gamma, w, l)$  is in the closure of  $\mathbb{R}(\Gamma_0, \underline{0})$  for some 3-regular graph  $\Gamma_0$  with  $b_1(\Gamma_0) = g$ . In other words, by Lemma 4.7, that  $(\Gamma, w)$  is a weighted contraction of  $(\Gamma_0, \underline{0})$ , with  $\Gamma_0$  a 3-regular graph.

Suppose first that  $w = \underline{0}$ , i.e. the tropical curve is pure. A proof of this fact is the proof of [CV09, Prop. A.2.4] (that Proposition is concerned with 3-edge-connected curves, but the proof is easily seen to work for our case).

We continue by induction on  $|w|$ , the basis being the case  $w = \underline{0}$ , which we just did. Let us now assume that  $w(v_1) \geq 1$  for some  $v_1 \in V(\Gamma)$ . Let  $(\Gamma', w')$  be the genus  $g$  stable graph defined as follows:  $\Gamma'$  is obtained from  $\Gamma$  by just adding a loop,  $e_0$ , based at  $v_1$ . Therefore we have  $V(\Gamma') = V(\Gamma)$  and  $E(\Gamma') = E(\Gamma) \cup \{e_0\}$ . Now let

$$w'(v) = \begin{cases} w(v) & \text{if } v \neq v_1 \\ w(v_1) - 1 & \text{otherwise.} \end{cases}$$

It is clear that  $(\Gamma, w)$  is a weighted contraction of  $(\Gamma', w')$ , indeed

$$(\Gamma, w) = (\Gamma' / e_0, w' / e_0).$$

We can apply induction, as  $|w'| = |w| - 1$ ; hence  $(\Gamma', w')$  is a weighted contraction of  $(\Gamma_0, \underline{0})$  for some 3-regular graph  $\Gamma_0$ , i.e

$$(\Gamma', w') = (\Gamma_0 / S, \underline{0} / S)$$

for some  $S \subset E(\Gamma_0)$ . Now,  $e_0 \in E(\Gamma') \subset E(\Gamma_0)$ , hence, denoting  $S_0 := \{e_0\} \cup S$ , we have

$$(\Gamma, w) = (\Gamma_0 / S_0, \underline{0} / S_0).$$

The first part is proved.

Let us prove (2); fix a 3-regular graph  $\Gamma$ . By Proposition 4.8 we know that  $M(\Gamma, \underline{0})$  is open and dense in  $\overline{M(\Gamma, \underline{0})}$ . Now, consider the definition of  $M_g^{\text{trop}}$  given in (5.1). Pick a point  $[(\Gamma, \underline{0}, l)] \in M(\Gamma, \underline{0})$ ; the equivalence class (under  $\equiv$ ) of such a point contains only itself, therefore the quotient map  $\pi_g$  induces a homeomorphism of  $M(\Gamma, \underline{0})$  with its image such that

$$\pi_g^{-1}(\pi_g(M(\Gamma, \underline{0}))) = M(\Gamma, \underline{0}).$$

Therefore  $M(\Gamma, \underline{0})$  is open in  $M_g^{\text{trop}}$ . Moreover, the union of the  $M(\Gamma, \underline{0})$ , as  $\Gamma$  runs through all 3-regular graphs with  $b_1(\Gamma) = g$  is obviously dense in  $M_g^{\text{trop}}$ . Finally, there are only finitely many such graphs  $(\Gamma, \underline{0})$ , therefore the above union is open, as claimed.

Now (3). We have, of course

$$M_g^{\text{reg}} \subset M_g^{\text{pure}} \subset M_g^{\text{trop}}.$$

Hence  $M_g^{\text{pure}}$  is dense by part (2). Now recall that

$$M_g^{\text{pure}} = \{[(\Gamma, w, l)] : b_1(\Gamma) = g\} = \{[(\Gamma, w, l)] : |w| = 0\}.$$

Let us denote  $\mathbb{R}(\Gamma, w)^+ \subset \overline{\mathbb{R}(\Gamma, w)}$  the union of all loci corresponding to weighted graphs  $(\Gamma', w')$  such that  $b_1(\Gamma') < g$  (cf. Lemma 4.7). The set  $\mathbb{R}(\Gamma, w)^+$  is closed, as the first Betti number does not grow under edge contraction. Hence its complement, the locus in  $\mathbb{R}(\Gamma, w)$  parametrizing pure tropical curve, is open. Hence the locus in  $\overline{M(\Gamma, \underline{0})}$  corresponding to pure tropical curves is also open.

Let now  $\bar{p} \in M_g^{\text{pure}} \subset M_g^{\text{trop}}$  and let  $(\Gamma_p, \underline{0})$  be its supporting graph. For every 3-regular graph  $\Gamma$  with  $b_1(\Gamma) = g$  let  $p_\Gamma \in \overline{M(\Gamma, \underline{0})}$  be the preimage of  $\bar{p}$ . By what we just said  $p_\Gamma$  admits an open neighborhood  $U_\Gamma \subset \overline{M(\Gamma, \underline{0})}$  such that  $U_\Gamma$  parametrizes only pure tropical curves. Up to shrinking each  $U_\Gamma$  around  $p_\Gamma$  we can assume that

$\pi_g(U_\Gamma) = \pi_g(U_{\Gamma'})$  for all such  $\Gamma$ , and  $\Gamma'$  (we first assume that each  $U_\Gamma$  intersects the boundary only along the locus  $M(\Gamma_p, \underline{0})$  containing  $p_\Gamma$ ; then we use the fact that the restriction of  $\pi_g$  on any  $M(\Gamma_p, \underline{0})$  is a homeomorphism with its image). But then

$$\pi_g(\cup U_\Gamma) \subset M_g^{\text{pure}}, \quad \text{and} \quad \pi_g^{-1}(\pi_g(\cup U_\Gamma)) = \cup U_\Gamma$$

(where the union is over all 3-regular graph  $\Gamma$  with  $b_1(\Gamma) = g$ ). This implies that  $\pi_g(\cup U_\Gamma)$  is an open neighborhood of  $\bar{p}$  all contained in  $M_g^{\text{pure}}$ . Part (3) is proved.

To prove that  $M_g^{\text{trop}}$  is connected it suffices to observe that the closure of every open subset of type  $M(\Gamma, \underline{0})$  contains the point parametrizing the metric weighted graphs with no edges and one vertex of weight  $g$  (see Lemma 4.7).

Finally, let us show that  $M_g^{\text{trop}}$  is Hausdorff. The quotient map  $\pi_g$  (5.2) induces a bijection of  $\overline{M(\Gamma, \underline{0})}$  with its image. Therefore, by Proposition 4.8,  $M_g^{\text{trop}}$  is obtained by gluing together finitely many Hausdorff spaces. Let  $\bar{p}, \bar{q} \in M_g^{\text{trop}}$ ; for every 3-regular  $\Gamma$  with  $b_1(\Gamma) = g$  let  $p_\Gamma = \overline{M(\Gamma, \underline{0})} \cap \pi_g^{-1}(\bar{p})$  and  $q_\Gamma = \overline{M(\Gamma, \underline{0})} \cap \pi_g^{-1}(\bar{q})$  and pick, in the Hausdorff space  $\overline{M(\Gamma, \underline{0})}$ , disjoint open neighborhoods of the two points:  $p_\Gamma \in U_\Gamma, q_\Gamma \in V_\Gamma$ . If  $\pi_g^{-1}(\bar{p})$  doesn't intersect  $\overline{M(\Gamma, \underline{0})}$  we don't do anything, similarly for  $\bar{q}$ . Arguing as before, we can assume that  $\pi_g(U_\Gamma) = \pi_g(U_{\Gamma'})$  for all  $\Gamma$  and  $\Gamma'$  as above, hence  $\pi_g(U_\Gamma)$  is open; similarly for  $V_\Gamma$ . Then,  $\pi_g(U_\Gamma)$  and  $\pi_g(V_\Gamma)$  are open disjoint neighborhoods of  $\bar{p}$  and  $\bar{q}$  in  $M_g^{\text{trop}}$ . ■

The moduli space of tropical curves is denoted  $M_g^{\text{tr}}$  in [BMV]. Although we gave a different definition for our  $M_g^{\text{trop}}$  the two spaces coincide, see [BMV, Def. 3.2.1, Theorem 3.2.2] (what we here denote  $M(\Gamma, w)$  is there denoted  $C(\Gamma, w)$ ).

**5.3. Relating  $M_g^{\text{trop}}$  to the moduli space,  $\overline{M}_g$ , of Deligne-Mumford stable curves.** The material of this subsection is almost, if not all, known to the experts. A written account seems not to exist, so this part is included for completeness, as the analogies between the two moduli spaces appear quite interesting.

To begin with, recall that  $\overline{M}_g$  has a combinatorial decomposition by means of the dual graph of a stable curve as follows. Let  $X$  be a stable curve, the *weighted dual graph* of  $X$  is the weighted graph  $(\Gamma_X, w_X)$  such that  $\Gamma_X$  is the graph having a vertex for every irreducible component of  $X$  and an edge for every node of  $X$  joining the two (possibly equal) vertices corresponding to the components on which the node lie. The weight function  $w_X$  assigns to every vertex the geometric genus of the corresponding component. So the (arithmetic) genus of  $X$  is  $g = b_1(\Gamma_X) + \sum_{v \in V(\Gamma_X)} w_X(v)$ . To say that  $X$  is stable in the sense of Deligne-Mumford is to say that  $(\Gamma_X, w_X)$  is a stable weighted graph. Now, for any stable weighted graph  $(\Gamma, w)$  of genus  $g$  denote by  $C_g(\Gamma, w)$  the combinatorial locus of curves with  $(\Gamma, w)$  as dual graph:

$$C_g(\Gamma, w) := \{X \in \overline{M}_g : (\Gamma_X, w_X) = (\Gamma, w)\} \subset \overline{M}_g.$$

Then we have

$$(5.4) \quad \overline{M}_g = \coprod_{(\Gamma, w) \text{ stable of genus } g} C_g(\Gamma, w).$$

By Theorem 5.2 and Proposition 4.8 we have a very similar decomposition on  $M_g^{\text{trop}}$

$$(5.5) \quad M_g^{\text{trop}} = \coprod_{(\Gamma, w) \text{ stable of genus } g} M(\Gamma, w).$$

*Remark 5.4.* In the next statement, by  $\dim C_g(\Gamma, w)$  we mean the dimension as an algebraic scheme. By  $\dim M(\Gamma, w)$  we mean the dimension as an orbifold, i.e. the dimension of the topological manifold,  $\mathbb{R}(\Gamma, w)$ , of which it is a finite quotient.

More generally, if a topological space  $X$  contains a dense open subset which is an orbifold of dimension  $n$ , we will say that  $X$  has pure dimension  $n$ . For example,  $M_g^{\text{trop}}$  has pure dimension equal to  $3g - 3$ .

Part (1) of the next Proposition is well known. The second part may also be known, but a written proof seems to be lacking.

**Proposition 5.5.** *Fix  $g \geq 2$  and consider the decompositions (5.4) and (5.5). The bijection between their strata given by  $C_g(\Gamma, w) \mapsto M(\Gamma, w)$  has the following properties.*

- (1)  $\dim C_g(\Gamma, w) = \overline{\text{codim}} M(\Gamma, w) = 3g - 3 - |E(\Gamma)|$ .
- (2)  $C_g(\Gamma', w') \subset \overline{C_g(\Gamma, w)} \Leftrightarrow M(\Gamma, w) \subset \overline{M(\Gamma', w')} \Leftrightarrow (\Gamma', w') \geq (\Gamma, w)$

*Proof.* We prove only (2); the statement on the closure of  $M(\Gamma, w)$  follows from Proposition 4.8. Let now  $X' \in C_g(\Gamma', w')$  be a stable curve; as  $X'$  lies in the closure of  $C_g(\Gamma, w)$  there is a family of curves with dual graph  $(\Gamma, w)$  specializing to  $X'$ . We can assume that this is a family over a one dimensional base  $B$  with a marked point  $b_0$ , i.e. we have a morphism  $f : \mathcal{X} \rightarrow B$  whose fiber  $f^{-1}(b)$  is  $X_b \in C_g(\Gamma, w)$  if  $b \neq b_0$  and  $X'$  if  $b = b_0$ .

Under such a specialization every node of  $X_b$  specializes to a node of  $X'$ , and distinct nodes of  $X_b$  specialize to distinct nodes of  $X'$ . This enables us to pick a well determined subset of nodes of  $X'$ , i.e. a well determined  $T \subset E(\Gamma')$ , namely  $T$  corresponds to the set of nodes of  $X'$  that are specializations of nodes of  $X_b$ . Let  $S = E(\Gamma') \setminus T$ , so  $S$  corresponds to the nodes of  $X'$  that do not come from nodes of  $X_b$ . Consider the graph  $(\Gamma'/S, w'/S)$ ; of course  $(\Gamma', w') \geq (\Gamma'/S, w'/S)$ .

We claim that  $(\Gamma'/S, w'/S) = (\Gamma, w)$ , which would conclude the proof.

We already have a bijection between  $E(\Gamma)$  and  $E(\Gamma')$  by construction, mapping an edge of  $\Gamma$ , i.e. a node of  $X$ , to the node in  $X'$ , i.e. to the edge in  $T$ , to which it specializes. The total space  $\mathcal{X}$  of our family of curves is singular along the nodes of the fibers  $X_b$ , so when we desingularize  $\mathcal{X}$  at such loci (exactly  $\#T$  of those) we obtain a new family  $\mathcal{Y} \rightarrow B$  whose fiber over  $b \neq b_0$  is the normalization of  $X_b$ . The fiber over  $b_0$  is the partial normalization  $Y'$  of  $X'$  at  $T$ ; so the dual graph of  $Y'$  is  $\Gamma' \setminus T$ . Therefore  $\mathcal{Y} \rightarrow B$  is a union of families parametrized by the irreducible components of  $X_b$ , i.e. by the vertices of  $\Gamma$ ; let us denote these families  $\mathcal{X}^v \rightarrow B$ . So if  $b \neq b_0$  the fiber  $X_b^v$  is the irreducible component corresponding to the vertex  $v \in V(\Gamma)$ . The fiber over  $b_0$  of  $\mathcal{X}^v \rightarrow B$  is a partial normalization of a subcurve of  $X'$ ; hence it uniquely determines a set of vertices of  $\Gamma'$  (the vertices corresponding to its components). Obviously, two different vertices  $v_1$  and  $v_2$  of  $\Gamma$  determine in this way disjoint sets of vertices of  $\Gamma'$ . Therefore we have a surjection  $V(\Gamma') \rightarrow V(\Gamma)$  mapping each vertex  $v'$  to the vertex  $v$  such that the component corresponding to  $v'$  is in the special fiber of the family  $\mathcal{X}^v \rightarrow B$ . Let us denote by  $\nu : V(\Gamma') \rightarrow V(\Gamma)$  this surjection. So  $\nu$  identifies two vertices of  $\Gamma'$  if and only if they belong to the same connected component of  $\Gamma' \setminus T$ . Therefore  $\nu$  is the same map as the canonical map  $V(\Gamma') \rightarrow V(\Gamma'/S)$ . This shows that  $V(\Gamma)$  and  $V(\Gamma'/S)$  are in natural bijection. Finally, since the arithmetic genus of a family of algebraic curves is constant, we have, for any  $v \in V(\Gamma)$ , that the geometric genus of the component corresponding to  $v$ , i.e.  $w(v)$ , is equal to the arithmetic genus of the limiting curve. Therefore

$$w(v) = \sum_{v' \in \nu^{-1}(v)} w'(v') + \#\{\text{loops in } S \text{ based at } v'\}.$$

Therefore (cf. (4.2)) the weight function  $w$  coincides with  $w/S$ ; so we are done.  $\blacksquare$

**5.6. Connectedness properties of tropical moduli spaces.** In this last subsection we apply our linkage theorem 3.3 to the geometry of some moduli spaces of tropical curves. The next definition is adapted from [MaS, Definition 3.3.2].

**Definition 5.7.** Let  $X$  be a topological space of pure dimension  $n$ ; see 5.4. Assume that  $X$  is endowed with a decomposition  $X = \coprod_{i \in I} X_i$ , where every  $X_i$  is a connected orbifold (locally the quotient of a topological manifold by a finite group). We say that  $X$  is *connected through codimension one* if the subset

$$\coprod_{i \in I: \dim X_i \geq n-1} X_i \subset X$$

is connected. Notice that if  $X$  is pure dimensional and connected through codimension one, then  $X$  is connected.

**Example 5.8.** The space  $M_g^{\text{trop}}$  is connected through codimension one. We already noticed that  $M_g^{\text{trop}}$  is of pure dimension  $3g - 3$ ; by Theorem 5.2 it is clear that

$$M_g^{\text{trop}} = \coprod_{(\Gamma, w) \text{ stable of genus } g} M(\Gamma, w).$$

To prove that it is connected through codimension one it suffices to show that any two 3-regular graphs are linked in the sense of Definition 1.4. This follows from [HtT80, Prop. page 236], which is a remarkable and already known special case of our Theorem 3.3.

We can now prove that the property of being connected through codimension one holds for some other tropical moduli spaces.

In analogy with the classical situation we have a tropical Torelli map  $t_g^{\text{trop}} : M_g^{\text{trop}} \rightarrow A_g^{\text{trop}}$  to the moduli space of tropical Abelian varieties, mapping a curve to its tropical Jacobian (see [MiZ08], [CV09] and [BMV] for details). We denote by  $Sch_g^{\text{trop}}$  the image of  $t_g^{\text{trop}}$ , and refer to it, as it is customary, as the tropical Schottky locus in  $A_g^{\text{trop}}$ . For our purposes  $Sch_g^{\text{trop}}$  can be identified with the topological quotient

$$M_g^{\text{trop}} \longrightarrow Sch_g^{\text{trop}} := M_g^{\text{trop}} / \equiv_{t_g^{\text{trop}}}$$

where  $[(\Gamma, l, w)] \equiv_{t_g^{\text{trop}}} [(\Gamma, l, w)] \Leftrightarrow t_g^{\text{trop}}([(\Gamma, l, w)]) = t_g^{\text{trop}}([(\Gamma', l', w')])$ . For more structure on  $A_g^{\text{trop}}$  and  $Sch_g^{\text{trop}}$  we refer to [BMV].

As proved in [CV09, Thm 4.1.9], and generalized by [BMV, Thm 5.3.3], the Torelli map identifies curves having the same so-called “3-edge-connected class”. In particular, let us denote by  $M_g^{\text{trop}}[3]$  the locus of tropical curves with 3-edge-connected graph:

$$M_g^{\text{trop}} \supset M_g^{\text{trop}}[3] := \{[(\Gamma, l, w)] : \Gamma \text{ is 3-edge-connected}\}.$$

Then we have

$$t_g^{\text{trop}}(M_g^{\text{trop}}[3]) = t_g^{\text{trop}}(M_g^{\text{trop}}) \cong Sch_g^{\text{trop}} \subset A_g^{\text{trop}}.$$

Furthermore, the restriction of  $t_g^{\text{trop}}$  to  $M_g^{\text{trop}}[3]$  is injective on every subspace  $M(\Gamma, w) \subset M_g^{\text{trop}}[3]$ , and it identifies two such spaces,  $M(\Gamma, w)$  and  $M(\Gamma', w')$ , only if the graphs  $\Gamma$  and  $\Gamma'$  are cyclically equivalent.

The conservation of 3-edge-connectivity under linkage, proved in Theorem 3.3, enables us to obtain the following result.

**Proposition 5.9.** *The spaces  $M_g^{\text{trop}}[3]$  and  $Sch_g^{\text{trop}}$  have pure dimension  $3g - 3$  and are connected through codimension one.*

*Proof.* We introduce the locus of regular, 3-edge-connected curves

$$M_g^{\text{reg}}[3] \subset M_g^{\text{reg}} \subset M_g^{\text{trop}}.$$

We have that the closure in  $M_g^{\text{trop}}$  of regular, 3-edge-connected curves is the locus of all 3-edge-connected curves, i.e.  $\overline{M_g^{\text{reg}}[3]} = M_g^{\text{trop}}[3]$ . This follows from [CV09, Prop A.2.4], whose proof (stated there only for pure regular curves) works word for

word in our setting (i.e. for all tropical curves). It is clear that  $M_g^{\text{reg}}[3]$  has pure dimension  $3g - 3$ . We conclude that  $M_g^{\text{trop}}[3]$  has pure dimension  $3g - 3$ .

Now the connectedness through codimension one follows from Theorem 3.3, observing that if  $(\Gamma', w')$  is obtained from  $(\Gamma, w)$  by contracting only one edge, then  $\dim M(\Gamma, w) = \dim M(\Gamma', w') + 1$  (cf. Proposition 5.5(1)). This proves that  $M_g^{\text{trop}}[3]$  is connected through codimension one.

Now, by what we said before there is a surjection with finite fibers

$$t_g^{\text{trop}}[3] : M_g^{\text{trop}}[3] \longrightarrow \text{Sch}_g^{\text{trop}}$$

obtained by restricting the Torelli map. This surjection induces a homeomorphism with its image of every subspace  $M(\Gamma, \underline{0}) \subset M_g^{\text{trop}}[3]$ . This implies that  $\text{Sch}_g^{\text{trop}}$  has pure dimension  $3g - 3$ . Furthermore, as  $t_g^{\text{trop}}[3]$  is injective on every  $M(\Gamma, w)$ , it preserves the dimension of these subsets; therefore  $\text{Sch}_g^{\text{trop}}$  is connected through codimension one, because so is  $M_g^{\text{trop}}[3]$ . ■

*Remark 5.10.* What are the consequences on tropical moduli spaces of the linkage theorem when  $p \geq 4$ ? Consider the subset

$$M_g^{p\text{-reg}} := \coprod_{\substack{\Gamma \text{ } p\text{-regular} \\ b_1(\Gamma) = g}} M(\Gamma, \underline{0}) \subset M_g^{\text{pure}}$$

and assume it is not empty. By a proof similar to that of Proposition 5.9 one obtains that the closure of  $M_g^{p\text{-reg}}$  is of pure dimension equal to  $p(g - 1)/(p - 2)$ , by Remark 1.8 (this number is an integer by the non-emptiness assumption), and connected through codimension one.

The same holds if the above disjoint union is restricted to all 3-edge-connected and  $p$ -regular graphs with  $b_1(\Gamma) = g$ . That is, with self-explanatory notation, the closure of  $M_g^{p\text{-reg}}[3]$  is of pure dimension  $p(g - 1)/(p - 2)$  and connected through codimension one.

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