# ENERGY GAP ESTIMATES IN XXZ FERROMAGNETS AND STOCHASTIC PARTICLE SYSTEMS

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Abstract. This expository article is a survey of recent results [8, 9] on the energy gap above the interface ground states of XXZ ferromagnets. Main ideas and techniques are reviewed with special emphasis on the equivalence between the quantum spin models and classical stochastic particle systems.

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## 1. Introduction

Given $J \in \frac{1}{2}\mathbb{N}$, the spin–$J$ XXZ ferromagnet is described by a Hamiltonian of the form

$$\mathcal{H} = -\sum_{x\sim y} \Big\{ \frac{1}{\Delta}\big( S_x^1 S_y^1 + S_x^2 S_y^2 \big) + S_x^3 S_y^3 \Big\} \\ + \text{ boundary conditions},$$

where the sum is over adjacent vertices of a suitable graph, and each vertex $x$ is equipped with the $(2J+1)$–dimensional representation of $SU(2)$ given by the usual spin–$J$ operators $S_x^i$, $i = 1, 2, 3$. Here $\Delta \geqslant 1$ is a parameter measuring the anisotropy: $\Delta = 1$ is the isotropic model also known as XXX model; the limit $\Delta \to \infty$ gives the classical Ising model. For an appropriate geometry of the underlying graph and for suitable boundary conditions the system is known [3] to have ground states describing domain walls, or interfaces. Since the discovery of such interface ground states there has been growing interest in the XXZ ferromagnets. Several studies have been devoted to the properties of the low–lying spectrum of these models, see e.g. [18, 15, 5, 16, 12] and other contributions to this volume. In this paper I mostly review recent results [8, 9] obtained in collaboration with F. Martinelli. The material is presented in an introductory fashion, with emphasis on main ideas, examples and possible developments. Besides simple and instructive arguments full proofs are often omitted.

A recurrent theme of this paper is that in some cases the spectrum of the quantum spin Hamiltonian $\mathcal{H}$ coincides with the spectrum of the Markov generator of an exclusion–type dynamics. This correspondence is illustrated by means of several examples and, in addition to the models considered in [8, 9], I include here also a preliminary discussion of the XXZ model in the presence of a transverse external field, where, on top of the (conservative) exchange dynamics one has a (dissipative) birth and death process.

Relations between classical stochastic particle systems and quantum spin models have been known for a long time and continue to play an important role in many theoretical developments, see e.g. [2, 4, 21, 22] for a partial list of references related to the models we consider here. This analogy lies at the heart of our results on energy gap estimates, which ultimately rely on techniques developed to bound the relaxation time to equilibrium of interacting particle systems. The recursive scheme needed to derive bounds on the gap which give the correct scaling with the size of the system is based on the so–called martingale approach introduced in [17]. We refer also to [1] for an earlier use of similar techniques directly in the framework of quantum spin chains with frustration-free ground states. Besides the martingale approach, some profound ideas recently introduced in [10] play an important role in our analysis.

We start our discussion, in section 2, with the simplest model, namely the one–dimensional isotropic model, or XXX chain. Then, in section 3, we turn to the one–dimensional anisotropic model, or XXZ chain. Generalizations to higher dimensions are reviewed in section 4. Finally, in section 5 we derive an expression for the equivalent stochastic process in the presence of transverse external fields.

## 2. XXX chains

Here we introduce the basic elements of our approach, namely the ground state transformation, the exclusion processes and the relevant spectral gap estimates. We start with the case $J = 1/2$ and later show how the general spin–$J$ case can be in a sense reduced to that of spin–1/2.

### 2.1. Spin–$\frac{1}{2}$ XXX chain.

Given a positive integer $L$, the XXX chain on sites $\{1, \dots, L\}$ is described by the Hamiltonian

$$\mathcal{H}_L = \sum_{j=1}^{L-1} \mathcal{H}_{j,j+1}\,, \tag{2.1}$$

$$\mathcal{H}_{j,j+1} = -\big( S_j^1 S_{j+1}^1 + S_j^2 S_{j+1}^2 + S_j^3 S_{j+1}^3 \big) + \frac{1}{4}\,. \tag{2.2}$$

Here $S_j^k$, $k = 1, 2, 3$ are the usual spin–1/2 operators at site $j$ and the constant 1/4 has been added to have zero energy ground states. The operator $\mathcal{H}_L$ acts on the tensor product Hilbert space $\mathfrak{H}_L := \otimes_{j=1}^L \mathbb{C}^2$. We take the natural orthonormal basis in $\mathfrak{H}_L$ corresponding to eigenvectors $|\alpha_j\rangle$ of the third component of the spin $S_j^3$ with the convention that $\alpha_j = 1$ stands for spin "up" and $\alpha_j = 0$ stands for spin "down":

$$|\alpha\rangle = \otimes_j |\alpha_j\rangle\,, \quad S_j^3 |\alpha\rangle = (\alpha_j - 1/2)|\alpha\rangle\,.$$

The variables $\alpha$ will be interpreted as occupation numbers, i.e. $\alpha_j = 1$ stands for the presence of a particle at $j$ while $\alpha_j = 0$ means that site $j$ is empty. We write $\Omega_L = \{0, 1\}^L$ for the space of possible configurations $\alpha$. A generic vector $|\psi\rangle$ in $\mathfrak{H}_L$ is often identified with the function $\psi : \Omega_L \to \mathbb{C}$ defining its coordinates:

$$|\psi\rangle = \sum_\alpha \psi(\alpha)|\alpha\rangle\,, \qquad \psi(\alpha) = \langle\alpha|\psi\rangle\ . \tag{2.3}$$

The operator (2.1) corresponds to the choice of free boundary conditions but everything we say in this section applies without modification to the case of a periodic chain, i.e. when the extra coupling $\mathcal{H}_{L,1}$ is added to the sum in (2.1).

*The particle system.* We recall the action of raising and lowering operators $S_j^\pm := S_j^1 \pm i S_j^2$ on basis vectors:

$$S_j^+|\alpha\rangle = (1 - \alpha_j)|\alpha^{(j)}\rangle\,, \qquad S_j^-|\alpha\rangle = \alpha_j|\alpha^{(j)}\rangle\,, \tag{2.4}$$

where $\alpha^{(j)}$ is the configuration "flipped" at site $j$:

$$\alpha_k^{(j)} = \begin{cases} \alpha_k & k \neq j \\ 1-\alpha_j & k = j \end{cases}$$

In particular,

$$(S_j^+ + S_j^-)|\alpha\rangle = |\alpha^{(j)}\rangle\,. \tag{2.5}$$

Similarly,

$$(S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-)|\alpha\rangle = |\alpha^{j,j+1}\rangle\,, \tag{2.6}$$

where $\alpha^{j,j+1}$ is the "exchanged" configuration

$$(\alpha^{j,j+1})_k = \begin{cases} \alpha_k & k \neq j, j+1 \\ \alpha_{j+1} & k = j \\ \alpha_j & k = j+1 \end{cases} \tag{2.7}$$

We can then write

$$\big[\mathcal{H}_{j,j+1}\psi\big](\alpha) = \langle\alpha|\,\mathcal{H}_{j,j+1}|\psi\rangle = -\frac{1}{2}\big[\psi(\alpha^{j,j+1}) - \psi(\alpha)\big]\,. \tag{2.8}$$

This shows in particular that the system conserves the total third component of the spin $S^3_{\text{tot}} := \sum_j S_j^3$. This *conservation law*, usually expressed as the vanishing of the commutator $[\mathcal{H}_L, S^3_{\text{tot}}] = 0$, allows to cut the space $\mathfrak{H}_L$ into $L+1$ "non–communicating" sectors $\mathfrak{H}_n$, $n = 0, 1, \ldots, L$, each given by the span of vectors $|\alpha\rangle$ such that

$$\sum_{j=1}^{L} \alpha_j = n\,, \qquad S^3_{\text{tot}}|\alpha\rangle = (n - L/2)|\alpha\rangle\,.$$

The identity (2.8) also shows that in each sector $\mathfrak{H}_n$ we have a zero energy state given by the constant function

$$\phi_n(\alpha) = \mathbf{1}_{\{\sum_j \alpha_j = n\}}(\alpha) = \begin{cases} 1 & \sum_j \alpha_j = n \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

Since $\mathcal{H}_L \geqslant 0$ the vectors $\phi_n$ are ground states. Moreover, $\mathfrak{H}_n$ is unitarily equivalent to $L^2(\Omega_L, \nu_{L,n})$, where $\nu_{L,n}$ is the uniform probability measure on all configurations $\alpha$ satisfying $\sum_j \alpha_j = n$. Then (2.8) says that $\mathcal{H}_L$ is equivalent to the Markov generator of the *simple exclusion process*:

$$\mathcal{H}_L \;\simeq\; -\frac{1}{2}\,\mathcal{L}\,, \qquad \mathcal{L}\varphi = \sum_{j=1}^{L-1} \nabla_{j,j+1}\varphi\,. \tag{2.10}$$

Here $\nabla_{j,j+1}\varphi(\alpha) = \varphi(\alpha^{j,j+1}) - \varphi(\alpha)$ is the exchange gradient along the bond $(j, j+1)$. The dynamics generated by $\mathcal{L}$ is the continuous time Markov process in which each bond is exchanged with rate 1 independently of all others. Since particles are not distinguishable, the interpretation is that particles try to jump to neighboring sites, their attempt being suppressed if the arrival site is already occupied. The operator $\mathcal{L}$ is non–positive and *ergodic* in $L^2(\Omega_L, \nu_{L,n})$. Namely, if $\mathcal{E}_{L,n}(\varphi)$, $\varphi : \Omega_L \to \mathbb{R}$, denotes the Dirichlet form

$$\mathcal{E}_{L,n}(\varphi) = \nu_{L,n}\big[\varphi(-\mathcal{L})\varphi\big] = \frac{1}{2}\sum_{j=1}^{L-1} \nu_{L,n}\big[\big(\nabla_{j,j+1}\varphi\big)^2\big]\,, \tag{2.11}$$

then $\mathcal{E}_{L,n}(\varphi) = 0$ implies that $\varphi$ is constant in $L^2(\Omega_L, \nu_{L,n})$. This also shows that in each sector the ground state $\phi_n$ is unique (up to multiplication).

*Energy gap.* The energy gap is the first excited level of $\mathcal{H}_L$ and is given by the variational formula

$$\text{gap}(\mathcal{H}_L) = \inf_{\psi \in (\text{Ker}\mathcal{H}_L)^\perp} \frac{\langle \psi | \, \mathcal{H}_L | \psi \rangle}{\langle \psi | \psi \rangle} ,$$

where Ker$\mathcal{H}_L$ is the vector space spanned by $\phi_n$, $n = 0, 1, \dots, L$. Equivalently we may write $\text{gap}(\mathcal{H}_L) = \min_{n \in \{1,\dots,L-1\}} \text{gap}_n(\mathcal{H}_L)$, where

$$\text{gap}_n(\mathcal{H}_L) = \inf_{\substack{\psi \in \mathfrak{H}_n : \\ \psi \perp \phi_n}} \frac{\langle \psi | \, \mathcal{H}_L | \psi \rangle}{\langle \psi | \psi \rangle} . \tag{2.12}$$

By the unitary equivalence (2.10) we see that $\text{gap}_n(\mathcal{H}_L)$ coincides with $1/(2\gamma_n(L))$, where $\gamma_n(L)$ is the best constant $\gamma$ in the Poincaré inequality

$$\text{Var}_{L,n}(\varphi) \leqslant \gamma \, \mathcal{E}_{L,n}(\varphi) , \qquad \varphi \, : \, \Omega_L \to \mathbb{R}. \tag{2.13}$$

Here $\text{Var}_{L,n}(\varphi) = \nu_{L,n}(\varphi^2) - \nu_{L,n}(\varphi)^2$ denotes the variance of $\varphi$ w.r.t. $\nu_{L,n}$. It is well known that the Poincaré constant $\gamma_n(L)$ for the symmetric simple exclusion process scales diffusively, i.e.

$$\delta \, L^2 \leqslant \gamma_n(L) \leqslant \delta^{-1} \, L^2 , \tag{2.14}$$

for some $\delta \in (0, 1)$, uniformly in $n = 1, \dots, L - 1$. The lower bound in (2.14) is obtained immediately by plugging in (2.13) any spin–wave function $\varphi$ of the form

$$\varphi(\alpha) = \sum_{j=1}^{L} g(j/L) \, \alpha_j , \tag{2.15}$$

with $g : [0, 1] \to \mathbb{R}$ a smooth function such that $\int g = 0$, $\int g^2 = 1$ and $\int (g')^2$ is bounded. For the upper bound there are several approaches [20, 17, 6]. Perhaps the simplest way is to use an apriori comparison bound [20, 13] of the form

$$\mathcal{E}_{L,n}(\varphi) \geqslant \delta \, L^{-2} \, \bar{\mathcal{E}}_{L,n}(\varphi) , \qquad \bar{\mathcal{E}}_{L,n}(\varphi) := \frac{1}{L} \sum_{j,k=1}^{L} \nu_{L,n} \big[ \big( \nabla_{j,k} \varphi \big)^2 \big] , \tag{2.16}$$

and then prove that the Poincaré constant $\bar{\gamma}_n(L)$ associated to the complete graph Dirichlet form $\bar{\mathcal{E}}_{L,n}$ is uniformly bounded. The proof of (2.16) only uses a telescopic decomposition for the gradient $\varphi(\alpha^{j,k}) - \varphi(\alpha)$ in terms of nearest neighbor exchanges and the Schwarz' inequality. The dynamics defined by the Dirichlet form $\bar{\mathcal{E}}_{L,n}$ is a variant of the so–called Bernoulli–Laplace diffusion model, see e.g. [14]. From (2.8) its Markov generator

$$\bar{\mathcal{L}} \varphi = \frac{1}{L} \sum_{j,k=1}^{L} \nabla_{j,k} \varphi ,$$

is equivalent to a *mean–field* Heisenberg Hamiltonian, as already discussed in [21]. In the next paragraph we give a proof of the uniform upper bound on the Poincaré constant $\bar{\gamma}_n(L)$ following a simple recursive argument borrowed from [11, 9].

*A first recursive argument.* Here we prove that $\bar\gamma_n(L) \leqslant 1/4$, where

$$\bar\gamma_n(L) = \sup_\varphi \frac{\mathrm{Var}_{L,n}(\varphi)}{\bar{\mathcal{E}}_{L,n}(\varphi)}\,, \tag{2.17}$$

with the convention $\bar\gamma_0(L) = \bar\gamma_L(L) = 0$. Consider the non–negative operator $P : L^2(\Omega_L, \nu_{L,n}) \to L^2(\Omega_L, \nu_{L,n})$ defined by

$$P\varphi(\alpha) = \frac{1}{L}\sum_{k=1}^{L} \nu_{L,n}(\varphi\,|\,\alpha_k)\,. \tag{2.18}$$

The notation $\nu_{L,n}(\cdot\,|\,\alpha_k)$ stands for conditional expectation given the value of the $k$–th variable $\alpha_k$. We first show that

$$\nu_{L,n}(\varphi P\varphi) \leqslant \frac{1}{L-1}\,\nu_{L,n}(\varphi^2)\,, \tag{2.19}$$

where $\varphi$ is an arbitrary real function on $\Omega_L$ such that $\nu_{L,n}(\varphi) = 0$. To prove (2.19) observe that for every $k$ we have $\nu_{L,n}(\varphi\,|\,\alpha_k) = a_k\bar\alpha_k$ with $\bar\alpha_k := \alpha_k - n/L$ and some $a_k \in \mathbb{R}$. Let $\mathcal{X} \subset L^2(\Omega_L, \nu_{L,n})$ be the span of functions of the form $\sum_{k=1}^L a_k\bar\alpha_k$, with arbitrary $a \in \mathbb{R}^L$. Since $P\varphi \in \mathcal{X}$, for every $\varphi$ satisfying $\nu_{L,n}(\varphi) = 0$, we may restrict to $\varphi \in \mathcal{X}$ to prove (2.19). On the other hand a simple computation shows that any $\varphi \in \mathcal{X}$ satisfies $P\varphi = \frac{1}{L-1}\varphi$. This proves (2.19).

The elementary decomposition

$$\nu_{L,n}(\varphi^2) = \nu_{L,n}\big[\mathrm{Var}_{L,n}(\varphi\,|\,\alpha_k)\big] + \nu_{L,n}\big[\varphi\,\nu_{L,n}(\varphi\,|\,\alpha_k)\big]\,, \qquad k = 1,\dots,L$$

yields

$$\nu_{L,n}\big[\varphi(1-P)\varphi\big] = \frac{1}{L}\sum_{k=1}^{L}\nu_{L,n}\big[\mathrm{Var}_{L,n}(\varphi\,|\,\alpha_k)\big]\,. \tag{2.20}$$

We are using the notation $\mathrm{Var}_{L,n}(\cdot\,|\,\alpha_k)$ for the variance w.r.t. conditional probability $\nu_{L,n}(\cdot\,|\,\alpha_k)$. Note that $\nu_{L,n}(\varphi\,|\,\alpha_k) = \nu_{L-1,n-\alpha_k}(\varphi^k)$, where $\varphi^k$ is obtained from the function $\varphi$ by freezing the $k$–th variable $\alpha_k$. Set $\bar\gamma(L) = \sup_n \bar\gamma_n(L)$. By definition of the constant (2.17), the r.h.s. of (2.20) is bounded above by

$$\bar\gamma(L-1)\,\frac{1}{L}\sum_{k=1}^{L}\nu_{L,n}\big[\bar{\mathcal{E}}_{L-1,n-\alpha_k}(\varphi^k)\big] = \bar\gamma(L-1)\,\frac{L-2}{L-1}\,\bar{\mathcal{E}}_{L,n}(\varphi)\,.$$

From (2.20) and the bound (2.19) we obtain

$$\nu_{L,n}(\varphi^2) \leqslant \bar\gamma(L-1)\,\bar{\mathcal{E}}_{L,n}(\varphi)\,,$$

for every real $\varphi$ with mean zero. This implies $\bar\gamma(L) \leqslant \bar\gamma(L-1)$. Iterating we arrive at $\bar\gamma(L) \leqslant \bar\gamma(2) = \bar\gamma_1(2) = 1/4$, where the last identity follows from a straightforward computation. This completes the proof.

2.2. **Spin–$J$ XXX chain.** Given $J \in \frac{1}{2}\mathbb{N}$ we consider the Hamiltonian (2.1) with

$$\mathcal{H}_{j,j+1} = -\big(S^1_jS^1_{j+1} + S^2_jS^2_{j+1} + S^3_jS^3_{j+1}\big) + J^2\,.$$

Now the $S^k_j$, $k = 1,2,3$ are the spin–$J$ operators at site $j$ and the Hilbert space is $\mathfrak{H}_L = \otimes_{j=1}^L\mathbb{C}^{2J+1}$. The orthonormal basis is labeled by configurations $\omega \in \Omega^J_L := \{0,\dots,2J\}^L$, corresponding to eigenvalues of the third component of the spin $S^3_j$:

$$|\omega\rangle = \otimes_j|\omega_j\rangle\,, \quad S^3_j|\omega\rangle = (\omega_j - J)|\omega\rangle\,,$$

As usual we identify a vector $|\psi\rangle$ in $\mathfrak{H}_L$ with the function $\psi(\omega) = \langle\omega|\psi\rangle$. It will be convenient to think of $\omega_j$ as the height of a nonnegative profile or as the number of particles sitting at $j$.

*The profile dynamics.* As in the case $J = 1/2$ the Hamiltonian can be interpreted as the generator of a continuous time Markov process. To see this we are going to generalize to arbitrary values of $J$ the discussion of the previous subsection. Let us first recall the action of raising and lowering operators $S_j^\pm = S_j^1 \pm iS_j^2$ on the basis vectors:

$$\begin{aligned} S_j^+ S_{j+1}^- |\omega\rangle &= \sqrt{(2J-\omega_j)(\omega_j+1)(2J-\omega_{j+1}+1)\omega_{j+1}}\,|\omega_+^{j,j+1}\rangle\,, \\ S_j^- S_{j+1}^+ |\omega\rangle &= \sqrt{(2J-\omega_{j+1})(\omega_{j+1}+1)(2J-\omega_j+1)\omega_j}\,|\omega_-^{j,j+1}\rangle\,, \end{aligned} \tag{2.21}$$

with the notation

$$(\omega_\pm^{j,j+1})_k = \begin{cases} \omega_j \pm 1 & k = j \\ \omega_{j+1} \mp 1 & k = j+1 \\ \omega_k & k \neq j, j+1 \end{cases}$$

If we introduce the functions

$$\phi_n(\omega) = \mathbf{1}_{\{\sum_j \omega_j = n\}}(\omega) \prod_{j=1}^L \sqrt{\binom{2J}{\omega_j}}\,, \tag{2.22}$$

then a simple computation shows that for every $n \in \{0, \ldots, 2JL\}$ and every $\omega$ such that $\sum_k \omega_k = n$ we have

$$\begin{aligned} \big[\mathcal{H}_{j,j+1}\psi\big](\omega) = &-\frac{1}{2}\,r_+^{j,j+1}(\omega)\left[\frac{\phi_n(\omega)}{\phi_n(\omega_+^{j,j+1})}\,\psi(\omega_+^{j,j+1}) \,-\, \psi(\omega)\right] \\ &-\frac{1}{2}\,r_-^{j,j+1}(\omega)\left[\frac{\phi_n(\omega)}{\phi_n(\omega_-^{j,j+1})}\,\psi(\omega_-^{j,j+1}) \,-\, \psi(\omega)\right]\,, \end{aligned} \tag{2.23}$$

where

$$r_+^{j,j+1}(\omega) = (2J-\omega_j)\omega_{j+1}, \qquad r_-^{j,j+1}(\omega) = (2J-\omega_{j+1})\omega_j\,. \tag{2.24}$$

The identity (2.23) tells us several things. First, it expresses the conservation law, $[\mathcal{H}_L, S^3_{\text{tot}}] = 0$. Moreover, it shows that for each $n$ the function (2.22) defines a ground state of $\mathcal{H}_L$.

As in the case $J = 1/2$ we can represent the Hilbert space $\mathfrak{H}_L$ as the direct sum of sectors $\mathfrak{H}_n$, $n \in \{0, \ldots, 2JL\}$, characterized by $n = \sum_k \omega_k$. Then $\mathcal{U}_n\psi := \psi/\phi_n$ maps unitarily $\mathfrak{H}_n$ into $L^2(\Omega_L^J, \nu_{L,n}^J)$ with the probability measure

$$\nu_{L,n}^J(\omega) = \frac{|\phi_n(\omega)|^2}{Z_{L,n}^J}\,, \qquad Z_{L,n}^J = \sum_{\omega\in\Omega_L^J} |\phi_n(\omega)|^2\,. \tag{2.25}$$

From (2.23) we have the equivalence

$$\mathcal{U}_n\,\mathcal{H}_L\,\mathcal{U}_n^{-1} = -\frac{1}{2}\,\mathcal{L}\,, \qquad \mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-\,, \tag{2.26}$$

$$\mathcal{L}_\pm\psi(\omega) = \sum_{j=1}^{L-1} r_\pm^{j,j+1}(\omega)\Big[\psi(\omega_\pm^{j,j+1}) - \psi(\omega)\Big]\,. \tag{2.27}$$

The operator $\mathcal{L}$ above is the Markov generator of the conservative dynamics described as follows. At each bond $(j, j+1)$ two independent Poisson clocks $\tau_{\pm}^{j,j+1}$ ring with rate $(2J)^2$. When $\tau_{+}^{j,j+1}$ rings, the current configuration $\omega$ is updated to $\omega_{+}^{j,j+1}$ with probability $r_{+}^{j,j+1}(\omega)/(2J)^2$; similarly for the opposite transition $\omega \to \omega_{-}^{j,j+1}$. Clearly, the total area $n$ under the profile $\omega$ is conserved. The process is reversible w.r.t. $\nu_{L,n}^{J}$, i.e. the detailed balance equations are satisfied. It is ergodic for $n = 1, \dots, 2JL-1$, therefore $\phi_n^J$ is the only ground state of $\mathcal{H}_L$ in each sector $\mathfrak{H}_n$.

*Example.* When $J = 1$ we obtain the isotropic version of the diffusion limited chemical reaction model considered by Alcaraz [4]. Namely, interpret $\omega_j = 0$ as the presence of a red particle, $\omega_j = 2$ as the presence of a blue particle and $\omega_j = 1$ as the absence of particles. Particles diffuse and creation/annihilation reactions appear when particles of different colors occupy neighboring sites.

*Energy gap.* The first excited level of $\mathcal{H}_L$, denoted $\mathrm{gap}(\mathcal{H}_L)$ is obtained by minimizing $\mathrm{gap}_n(\mathcal{H}_L)$, with $\mathrm{gap}_n(\mathcal{H}_L)$ given by (2.12). From the unitary equivalence (2.26), in turn, $\mathrm{gap}_n(\mathcal{H}_L)$ is the spectral gap of the profile dynamics in $L^2(\Omega_L^J, \nu_{L,n}^J)$. Using this representation we are able to give a simple proof of the following generalization of the estimates in the previous subsection.

**Theorem 2.1.** *There exists $\delta \in (0,1)$ such that for any $J \in \frac{1}{2}\mathbb{N}$ and any $L \geqslant 2$*

$$\delta\, J\, L^{-2} \leqslant \mathrm{gap}(\mathcal{H}_L) \leqslant \delta^{-1}\, J\, L^{-2} . \tag{2.28}$$

*Proof.* We start by formulating the problem in terms of Poincaré inequalities. We write $\mathrm{Var}_{L,n}^J$ for the variance w.r.t. $\nu_{L,n}^J$, and $\mathcal{E}_{L,n}^J$ for the Dirichlet form

$$\mathcal{E}_{L,n}^J(\varphi) = \nu_{L,n}^J\big[\varphi(-\mathcal{L})\varphi\big] = \frac{1}{2}\sum_{j=1}^{L-1} \nu_{L,n}^J\big[r_{+}^{j,j+1}(\nabla_{j,j+1}^{+}\varphi)^2 + r_{-}^{j,j+1}(\nabla_{j,j+1}^{-}\varphi)^2\big] ,$$

with the notation $\nabla_{j,j+1}^{\pm}\varphi = \varphi(\omega_{\pm}^{j,j+1}) - \varphi(\omega)$. Let us call $\gamma_n^J(L)$ the corresponding Poincaré constant, see (2.13). By the unitary equivalence the claim (2.28) is equivalent to uniform upper and lower bounds on $JL^{-2}\gamma_n^J(L)$.

As usual a lower bound on $\gamma_n^J(L)$ is obtained simply from the Ansatz $\varphi(\omega) := \sum_{j=1}^{L} g(j/L)\,\omega_j$, with $g$ as in (2.15). In this case the ratio $\mathcal{E}_{L,n}^J(\varphi)/\mathrm{Var}_{L,n}^J(\varphi)$ is seen to be of order $JL^{-2}$ by direct computation.

We turn to the upper bound on $\gamma_n^J(L)$. Here the main observation is that the profile dynamics with generator $\mathcal{L}$ is a suitable (Markovian) projection of an elementary exlcusion process. Namely, let $\Omega_{2JL} = \{0,1\}^{2JL}$ be the space of possible configurations $\alpha$ of an exclusion process on $2JL$ sites. It is convenient to picture the $2JL$ sites as the vertices of a $2J \times L$ rectangle $\Lambda$ in $\mathbb{Z}^2$, and $\alpha \in \Omega_{2JL}$ as a collection $\{\alpha_{(k,j)}\}$, with $k = 1, \dots, 2J$, $j = 1, \dots, L$. We call $\nu_{\Lambda,n}$ the uniform probability measure on all $\alpha \in \Omega_{2JL}$ such that $\sum_{(k,j)\in\Lambda} \alpha_{(k,j)} = n$. A straightforward computation shows that $\nu_{L,n}^J$ is simply the marginal of $\nu_{\Lambda,n}$ on the horizontal sums

$$\omega_j = \sum_{k=1}^{2J} \alpha_{(k,j)} . \tag{2.29}$$

Writing $\varphi(\omega) = \tilde\varphi(\alpha)$ we have, in particular, $\mathrm{Var}^J_{L,n}(\varphi) = \mathrm{Var}_{\Lambda,n}(\tilde\varphi)$. Similarly, a direct computation shows that

$$\mathcal{E}^J_{L,n}(\varphi) = \frac{1}{2}\sum_{k,\ell=1}^{2J}\sum_{j=1}^{L-1}\nu_{\Lambda,n}\big[(\nabla_{(k,j),(\ell,j+1)}\tilde\varphi)^2\big]\,, \tag{2.30}$$

where $\nabla_{(k,j),(\ell,j+1)}\tilde\varphi(\alpha) = \tilde\varphi(\alpha^{(k,j),(\ell,j+1)}) - \tilde\varphi(\alpha)$ stands for the exchange gradient between sites $(k,j)$ and $(\ell,j+1)$. The profile dynamics is thus a projection on horizontal sums of the elementary exclusion process in $\Lambda$ where each bond $(k,j)$–$(\ell,j+1)$, $k,\ell = 1,\dots,2J$; $j = 1,\dots,L-1$, is exchanged with rate 1.

Consider now the Bernoulli–Laplace diffusion with $2JL$ sites and $n$ particles, with Dirichlet form

$$\bar{\mathcal{E}}_{\Lambda,n}(\tilde\varphi) = \frac{1}{2JL}\sum_{k,\ell=1}^{2J}\sum_{j,h=1}^{L}\nu_{\Lambda,n}\big[(\nabla_{(k,j),(\ell,h)}\tilde\varphi)^2\big]\,.$$

From (2.30), a simple bound like (2.16) gives

$$\mathcal{E}^J_{L,n}(\varphi) \geqslant \delta J L^{-2}\bar{\mathcal{E}}_{\Lambda,n}(\tilde\varphi)\,. \tag{2.31}$$

The conclusion now follows from the recursive argument given in the previous subsection, which shows that $\bar{\mathcal{E}}_{\Lambda,n}(\tilde\varphi) \geqslant 4\mathrm{Var}_{\Lambda,n}(\tilde\varphi)$. ☐

## 3. XXZ chains

In this section we consider the anisotropic chain described by the Hamiltonian (2.1) when the bond interaction is given by

$$\mathcal{H}_{j,j+1} = -\Delta^{-1}\left(S^1_jS^1_{j+1} + S^2_jS^2_{j+1}\right) - S^3_jS^3_{j+1} + A(\Delta)\,\left(S^3_{h+1} - S^3_h\right) + J^2\,. \tag{3.1}$$

Here $S^k_j$, $k = 1,2,3$ are the spin–$J$ operators at site $j$, $\Delta > 1$ is the anisotropy parameter. The terms $A(\Delta)(S^3_{h+1} - S^3_h)$ sum up to a boundary condition favoring a discrepancy between the first and the last spin of the chain. Following [3] (see also [5] and other contributions to this volume) we make the choice

$$A(\Delta) = J\sqrt{1-\Delta^{-2}}\,,$$

which gives a set of ground states describing sharply localized domain walls. We review below some of the known facts about these ground states and the energy gap above them from the point of view of the equivalent asymmetric exclusion process.

### 3.1. Spin–1/2 XXZ chain.

We start with the case $J = 1/2$. The notation is that of the previous section. It is standard to introduce the parameter $q \in (0,1)$ solution of

$$\Delta = \frac{1}{2}\left(q + q^{-1}\right).$$

When $\Delta = q = 1$ we are back to the isotropic case studied before. The anisotropic version of the functions (2.9) is

$$\phi^q_n(\alpha) = \mathbf{1}_{\{\sum_j \alpha_j = n\}}(\alpha)\prod_{j=1}^{L} q^{j\alpha_j}\,. \tag{3.2}$$

By direct computation we find the analog of (2.8): for every $n$, every $\alpha \in \Omega_L$ such that $\sum_j \alpha_j = n$, we have

$$\left[\mathcal{H}_{j,j+1}\psi\right](\alpha) = \langle\alpha|\,\mathcal{H}_{j,j+1}|\psi\rangle = -\frac{1}{2\Delta}\,c_j(\alpha)\,\left[\frac{\phi_n^q(\alpha)}{\phi_n^q(\alpha^{j,j+1})}\,\psi(\alpha^{j,j+1}) - \psi(\alpha)\right], \tag{3.3}$$

where $c_j(\alpha) := q^{\alpha_j - \alpha_{j+1}}$.

The identity (3.3) shows that in each sector $\mathfrak{H}_n$ we have the ground state $\phi_n^q$ and the map $\mathcal{U}_n\psi = \psi/\phi_n^q$ gives the unitary equivalence

$$\mathcal{U}_n\,\mathcal{H}_L\,\mathcal{U}_n^{-1} = -\frac{1}{2\Delta}\,\mathcal{L}_q\,, \qquad \mathcal{L}_q\psi = \sum_{j=1}^{L-1} c_j\,\nabla_{j,j+1}\psi\,. \tag{3.4}$$

Here $\mathcal{L}_q$ is the generator of the *asymmetric exclusion process* where particles jump to the right with rate $q$ and to the left with rate $q^{-1}$. Clearly, the process is reversible w.r.t. the probability measure $\nu_{L,n}^q = (Z_{L,n}^q)^{-1}|\phi_n^q|^2$. Moreover, it shares the so–called $U_q[SU(2)]$ quantum–group symmetry with the Hamiltonian $\mathcal{H}_L$. This latter fact was exploited in [22] to obtain interesting duality relations.

The energy gap above the ground states $\phi_n^q$ was computed by Koma and Nachtergaele [15] using explicitly the quantum–group symmetry. Their result implies that for $J = 1/2$, independently of $n$, $\mathrm{gap}_n(\mathcal{H}_L) \to 1 - \Delta^{-1}$, as $L \to \infty$. That for $\Delta > 1$ the gap must be positive uniformly in $L$, can be seen by means of a recursive argument using the particle system representation, see [8], Theorem 4.3. The interest of an alternative method is apparent when we move to the general $J > 1/2$ case, where the quantum symmetry is lost.

3.2. **Spin–$J$ XXZ chain.** The model here is again (3.1). In analogy with (3.2) and (2.22) we introduce the functions

$$\phi_n^q(\omega) = \mathbf{1}_{\{\sum_j \omega_j = n\}}(\omega)\prod_{j=1}^{L} q^{j\omega_j}\sqrt{\binom{2J}{\omega_j}}\,. \tag{3.5}$$

The identities (2.21) allow to compute, for each $\omega \in \Omega_L^J$ with $\sum_j \omega_j = n$:

$$\begin{aligned}\left[\mathcal{H}_{j,j+1}\psi\right](\omega) = &-\frac{1}{2\Delta}\,r_{+,q}^{j,j+1}(\omega)\left[\frac{\phi_n^q(\omega)}{\phi_n^q(\omega_+^{j,j+1})}\,\psi(\omega_+^{j,j+1}) \,-\, \psi(\omega)\right]\\ &-\frac{1}{2\Delta}\,r_{-,q}^{j,j+1}(\omega)\left[\frac{\phi_n^q(\omega)}{\phi_n^q(\omega_-^{j,j+1})}\,\psi(\omega_-^{j,j+1}) \,-\, \psi(\omega)\right],\end{aligned} \tag{3.6}$$

where

$$r_{+,q}^{j,j+1}(\omega) = q^{-1}\,(2J-\omega_j)\omega_{j+1}, \qquad r_{-,q}^{j,j+1}(\omega) = q\,(2J-\omega_{j+1})\omega_j\,. \tag{3.7}$$

From (3.6) we see that $\phi_n^q$ is a ground state for each $n$, as first discussed in [3]. As usual (3.6) allows to derive the unitary equivalence through the map $\mathcal{U}_n : \psi \to \psi/\phi_n^q$,

$$\mathcal{U}_n\,\mathcal{H}_L\,\mathcal{U}_n^{-1} = -\frac{1}{2\Delta}\,\mathcal{L}_q\,, \qquad \mathcal{L}_q = \mathcal{L}_{+,q} + \mathcal{L}_{-,q}\,, \tag{3.8}$$

where $\mathcal{L}_q$ is the generator of the asymmetric profile dynamics

$$\mathcal{L}_{\pm,q}\psi(\omega) = \sum_{j=1}^{L-1} r_{\pm,q}^{j,j+1}(\omega)\left[\psi(\omega_\pm^{j,j+1}) - \psi(\omega)\right]. \tag{3.9}$$

The difference with the process in (2.27) is only the asymmetry in the rates (3.7) which favors high profiles on the left and low profiles on the right. The process is reversible w.r.t. the probability $\nu^{J,q}_{L,n} = (Z^{J,q}_{L,n})^{-1}|\phi^q_n|^2$. Ergodicity and uniqueness of the ground states follow immediately as before. When $J=1$ we obtain the asymmetric counterpart of the diffusion limited chemical reaction model mentioned in the previous section.

One of our main results in [9] is that the spectral gap of the anisotropic spin–$J$ chain grows linearly with $J$, in the sense of the following theorem. The precise asymptotics for large $J$ was predicted in [16], with the help of numerical data.

**Theorem 3.1.** *For every $\Delta > 1$, there exists $\delta \in (0,1)$ such that for any $L \geqslant 2$ and any $J \in \frac{1}{2}\mathbb{N}$*

$$\delta\, J \leqslant \mathrm{gap}(\mathcal{H}_L) \leqslant \delta^{-1}\, J$$

In the isotropic case of Theorem 2.1 we have seen that diffusive scaling was a direct consequence of (the local nature of the interaction and) translation invariance. Here the profile process is asymmetric and has a finite (of order $J^{-1}$, independently of $L$) relaxation time to equilibrium.

The proof is not as easy as that of Theorem 2.1 and we refer to [9] for the details. In the next paragraph, however, we sketch the main ideas involved in the lower bound.

*Ideas for the proof of Theorem 3.1.* As in the proof of Theorem 2.1 we have to bound the Poincaré constant $\gamma^{J,q}_n(L)$ of the Dirichlet form

$$\mathcal{E}^{J,q}_{L,n}(\varphi) = \nu^{J,q}_{L,n}\big[\varphi(-\mathcal{L}_q)\varphi\big] = \frac{1}{2}\sum_{j=1}^{L-1}\nu^{J,q}_{L,n}\big[r^{j,j+1}_{+,q}(\nabla^{+}_{j,j+1}\varphi)^2 + r^{j,j+1}_{-,q}(\nabla^{-}_{j,j+1}\varphi)^2\big]\,. \tag{3.10}$$

We write again $\Lambda$ for the $2J \times L$ rectangle in $\mathbb{Z}^2$. The measure $\nu^{J,q}_{L,n}$ on $\omega$'s is now seen to be the marginal on the horizontal sums (2.29) of the probability $\nu^q_{\Lambda,n}$ on $\alpha \in \{0,1\}^\Lambda$ given by

$$\nu^q_{\Lambda,n}(\alpha) = \frac{1}{Z^q_{\Lambda,n}}\,\mathbf{1}_{\{\sum_j \alpha_j = n\}}(\omega)\prod_{k=1}^{2J}\prod_{j=1}^{L} q^{2j\alpha_{(k,j)}}\,. \tag{3.11}$$

Note that the asymmetry is only in the vertical direction. With $\varphi(\omega) = \tilde\varphi(\alpha)$, we have

$$\mathcal{E}^{J,q}_{L,n}(\varphi) = \frac{1}{2}\sum_{k,\ell=1}^{2J}\sum_{j=1}^{L-1}\nu^q_{\Lambda,n}\big[c^{k,\ell}_j(\nabla_{(k,j),(\ell,j+1)}\tilde\varphi)^2\big]\,, \tag{3.12}$$

where $c^{k,\ell}_j(\alpha) = q^{\alpha_{(k,j)} - \alpha_{(\ell,j+1)}}$. The identity (3.12) says that the asymmetric profile dynamics with generator $\mathcal{L}_q$ can be computed by projecting on horizontal sums $\omega$ the dynamics of particles in the box $\Lambda$ which take arbitrary jumps in the horizontal direction but only unit jumps in the vertical one, with particles jumping to unoccupied sites, upwards (downwards) with rate $q$ ($q^{-1}$).

At this point there does not seem to be a straightforward comparison argument to bound (3.12) from below as in (2.31) and we turn to a finer recursive analysis which is roughly described as follows. We first set $N = 2J$ and rewrite (3.12) as

$$\mathcal{E}^{J,q}_{L,n}(\varphi) = J\,\hat{\mathcal{E}}_{N,n}(\tilde\varphi)\,,\quad \hat{\mathcal{E}}_{N,n}(\tilde\varphi) = \frac{1}{N}\sum_{k,\ell=1}^{N} D_{k,\ell}(\tilde\varphi)$$

$$D_{k,\ell}(\tilde\varphi) := \frac{1}{2}\sum_{j=1}^{L-1}\nu^q_{\Lambda,n}\big[c^{k,\ell}_j(\nabla_{(k,j),(\ell,j+1)}\tilde\varphi)^2\big]\,.$$

Here the idea is to isolate the $J$–dependence so that the final claim would follow from a uniform (in $N, L, n$) upper bound on the Poincaré constant of the Dirichlet form $\hat{\mathcal{E}}_{N,n}$. The argument for the latter estimate is sketched below.

For each $k = 1,\dots,N$, consider the variables $\eta_k := \{\alpha_{(k,j)},\ j = 1,\dots,L\}$. Let $P$ denote the non–negative operator

$$P\tilde\varphi(\alpha) = \frac{1}{N}\sum_{k=1}^{N}\nu^q_{\Lambda,n}(\tilde\varphi\,|\,\eta_k)\,.$$

As in (2.20), when $\tilde\varphi$ has mean zero we have

$$\nu^q_{\Lambda,n}\big[\tilde\varphi(1-P)\tilde\varphi\big] = \frac{1}{N}\sum_{k=1}^{N}\nu^q_{\Lambda,n}\big[\mathrm{Var}^q_{\Lambda,n}(\tilde\varphi\,|\,\eta_k)\big]\,. \tag{3.13}$$

Letting $\gamma_n(N,L)$ denote the Poincaré constant of the $\hat{\mathcal{E}}_{N,n}$–dynamics and writing $\gamma(N,L) = \sup_n \gamma_n(N,L)$ we have that the r.h.s. in (3.13) is bounded from above by

$$\gamma(N-1,L)\,\frac{N-2}{N-1}\,\hat{\mathcal{E}}_{N,n}(\tilde\varphi)\,.$$

The above bound is actually only correct if all the diagonal terms $D_{k,k}(\tilde\varphi)$ vanish. This problem, however, is easily solved by slightly modifying the Dirichlet form from the beginning. Once we have this, the main claim is that there exist uniform constants $C < \infty$ and $\zeta > 0$ such that for any mean zero $\tilde\varphi$ we have

$$\nu^q_{\Lambda,n}\big(\tilde\varphi P\tilde\varphi\big) \leqslant \frac{1}{N-1}\big(1 + CN^{-\zeta}\big)\,\nu^q_{\Lambda,n}(\tilde\varphi^2)\,. \tag{3.14}$$

If (3.14) holds then (3.13) implies

$$\gamma(N,L) \leqslant (1 + C'N^{-1-\zeta})\,\gamma(N-1,L)\,,$$

with a finite constant $C'$. This estimate can be iterated down to some fixed $N_0$, so that $\gamma(N,L) \leqslant C''\gamma(N_0,L)$ for all $N > N_0$ with another constant $C''$. The uniform estimate

$$\sup_{L\geqslant 2}\gamma(N_0,L) < \infty\,,$$

can be obtained by the martingale technique combined with the uniform upper bound on $\gamma(1,L)$ as in [8], Theorem 4.1 and Theorem 4.3. This shows that (3.14) is sufficient for our purposes.

Note that (3.14) only refers to a property of the measure, independent of the original dynamics. Following [10], the main idea behind the proof of (3.14) is to exploit exchangeability of the variables $\eta$ (due to horizontal symmetry in the measure $\nu^q_{\Lambda,n}$) to reduce (3.14) to a suitable one–dimensional problem, i.e. an estimate involving functions of a single variable, say $\eta_1$. Let $h = h(\eta_1)$ be such a function, and assume $h$ is orthogonal to both the constant and the number of particles, i.e. $\nu^q_{\Lambda,n}(h) = 0$ and

$$\nu^q_{\Lambda,n}\big[h(\eta_1)\xi(\eta_1)\big] = 0\,, \qquad \xi(\eta_1) = \sum_{j=1}^{L} \alpha_{(1,j)}\,.$$

The required estimate can be formulated as follows [9]: There exist uniform constants $C < \infty$, $\zeta > 0$ such that any function $h$ with the above properties satifies

$$\big|\nu^q_{\Lambda,n}\big[h(\eta_1)h(\eta_2)\big]\big| \leqslant C\,N^{-1-\zeta}\,\nu^q_{\Lambda,n}\big[h(\eta_1)^2\big]\,. \tag{3.15}$$

We shall not describe the problem (3.15) but only note that the main tools used here are some local expansions related to the central limit theorem. Indeed, the point of (3.15) is to estimate how far the measures $\nu^q_{\Lambda,n}$ are from product measures on the random variables $\eta_1, \eta_2$. This equivalence of ensembles problem is strictly related to the central limit theorem. We refer to [8, 9, 7] for further details.

## 4. Higher dimension: 111 interfaces

The mapping between quantum Heisenberg models and particle systems can be discussed, in principle, on general graphs. As we have seen, however, boundary conditions play a crucial role. In the isotropic case, for instance, when free or periodic boundary conditions are considered, a straightforward modification of the arguments in the proof of Theorem 2.1 shows that the gap scales like $JL^{-2}$ on any rectangular box in $\mathbb{Z}^d$ whose longest side is $L$, for any $d \geqslant 1$. In this section we focus on some higher dimensional version of the anisotropic model which recently attracted many investigations in connection with the problem of stability of quantum domain walls, see e.g. [19] for an introduction to this fascinating subject.

Following [5] we study the case of boundary conditions forcing a diagonal interface in the system. For simplicity we discuss only the 2–dimensional case, but the results below can be shown to hold in any dimension $d \geqslant 2$.

Consider a tilted rectangle $\Gamma = \Gamma_{R,L}$ with height along the 11 direction

$$\Gamma = \big\{x \in \mathbb{Z}^2 :\; -R \leqslant x_1 - x_2 \leqslant R\,,\; 1 \leqslant x_1 + x_2 \leqslant L\big\}$$

where $R, L$ are two positive integers. We write $\ell_x = x_1 + x_2$ for the distance of a site $x = (x_1, x_2)$ from the line $x_1 = -x_2$. We call $\mathcal{B}$ the set of (ordered) pairs $(x, y) \in \Gamma \times \Gamma$ such that $|x_1 - y_1| + |x_2 - y_2| = 1$ and $\ell_y = \ell_x + 1$. For any $J \in \frac{1}{2}\mathbb{N}$ the anisotropic spin–$J$ Hamiltonian in the region $\Gamma$ with 11 boundary conditions is defined by

$$\mathcal{H}_{R,L} = \sum_{(x,y)\in\mathcal{B}} \mathcal{H}_{(x,y)}\,, \tag{4.1}$$

$$\mathcal{H}_{(x,y)} = -\Delta^{-1}\left(S^1_x S^1_y + S^2_x S^2_y\right) - S^3_x S^3_y + J\sqrt{1-\Delta^{-2}}\left(S^3_y - S^3_x\right) + J^2\,.$$

This corresponds to classical boundary conditions favoring high (low) values of $S^3$ on the south–west (north–east) side of $\Gamma$. The analog of the ground states (3.5) are given here by the functions

$$\phi^q_{\Gamma,n}(\omega) = \mathbf{1}_{\{\sum_{x\in\Gamma}\omega_x = n\}}(\omega)\prod_{x\in\Gamma} q^{\ell_x\omega_x}\sqrt{\binom{2J}{\omega_x}}\,. \tag{4.2}$$

As usual the Hilbert space $\mathfrak{H}_\Gamma = \otimes_{x\in\Gamma}\mathbb{C}^{2J+1}$ is divided into sectors $\mathfrak{H}_n$, $n = 0,\dots,2J|\Gamma|$, and $\phi^q_{\Gamma,n}$ is the unique ground state in each $\mathfrak{H}_n$. The derivation of the unitary equivalence (3.8) can be repeated here without modification and the resulting process is described by

$$\mathcal{L}_q = \mathcal{L}_{+,q} + \mathcal{L}_{-,q}\,, \qquad \mathcal{L}_{\pm,q}\psi(\omega) = \sum_{(x,y)\in\mathcal{B}} r^{x,y}_{\pm,q}(\omega)\Big[\psi(\omega^{x,y}_{\pm}) - \psi(\omega)\Big]\,. \tag{4.3}$$

Note that the asymmetry is now along the 11 direction while there is essentially translation invariance in the orthogonal direction. To remove fluctuations in the orthogonal direction it can be convenient to study the modified model defined as follows. Consider a new graph with vertices the sites of $\Gamma$ but with edge set given by

$$\bar{\mathcal{B}} = \{(x,y)\in\Gamma\times\Gamma\,:\;\ell_y = \ell_x + 1\}\,.$$

Define the Hamiltonian

$$\bar{\mathcal{H}}_{R,L} = \frac{1}{R}\sum_{(x,y)\in\bar{\mathcal{B}}}\mathcal{H}_{(x,y)}\,, \tag{4.4}$$

with the bond operator $\mathcal{H}_{(x,y)}$ as in (4.1). $\bar{\mathcal{H}}_{R,L}$ models a mean–field type interaction in the orthogonal direction while keeping a local interaction in the 11 direction. It is not difficult to check that the ground states of $\bar{\mathcal{H}}_{R,L}$ are again given by the functions $\phi^q_{\Gamma,n}$ in (4.2). Therefore the associated particle system has the generator $R^{-1}\bar{\mathcal{L}}_q$, with $\bar{\mathcal{L}}_q$ given by (4.3) with the only difference that now the edge set $\mathcal{B}$ is replaced by $\bar{\mathcal{B}}$. Combining the techniques of Theorem 2.1 and Theorem 3.1 we can derive the following result, see [9].

**Theorem 4.1.** *For every $\Delta > 1$, there exists $\delta > 0$ such that for all $J, R$ and $L$*

$$\delta\, J \;\leqslant\; \mathrm{gap}(\bar{\mathcal{H}}_{R,L}) \;\leqslant\; \delta^{-1}\, J\,.$$

A uniform lower bound on the gap like the one in Theorem 4.1 might be of help in establishing results on the stability of domain walls at positive temperature.

Going back to the local interaction (4.1) we restore fluctuations in the orthogonal direction and produce the associated low–lying (diffusive) excitations. Using simple comparison arguments as in (2.16) we have the following bounds.

**Corollary 4.2.** *For every $\Delta > 1$, there exists $\delta > 0$ such that for all $J, R$ and $L$*

$$\delta\, J\, R^{-2} \;\leqslant\; \mathrm{gap}(\mathcal{H}_{R,L}) \;\leqslant\; \delta^{-1}\, J\, R^{-2}\,.$$

That the system described by (4.1) should have gapless spectrum in the limit $R\to\infty$ was first observed in [18] as a consequence of a continuous symmetry breaking. Rather precise upper bounds on $\mathrm{gap}(\mathcal{H}_{R,L})$ were then obtained in [5] by a careful choice of the test function in the variational principle. The estimates in Corollary 4.2 were first derived in [8] for spin $J = 1/2$, and then in [9] for arbitrary values of $J$.

## 5. On the influence of an external field

In this last section we are interested in the effect of an external field located somewhere in the system. For simplicity we restrict to the one–dimensional case. Following [12] we consider a magnetic field $B = (B_1, B_2, B_3)$ acting at a site $y \in \{1,\dots,L\}$. We then have the Hamiltonian

$$\mathcal{H}^B_{L,y} = \mathcal{H}_L + B\cdot S_y\,,$$

where $\mathcal{H}_L$ is the anisotropic spin–$J$ chain obtained from (3.1) and $B\cdot S_y = \sum_{i=1}^3 B_i S_y^i$. We shall see that this model has a simple representation in terms of non–conservative particle systems.

Recall the set of ground states $\phi_n^q$ of $\mathcal{H}_L$, given in (3.5), such that $\mathcal{H}_L\phi_n^q = 0$. As observed in [12], if the transversal component is nonzero, i.e. if $B_1^2 + B_2^2 > 0$, then the unique ground state of $\mathcal{H}_{L,y}^B$ is an appropriate mixture of the states $\phi_n^q$ above. This is interpreted as a *pinning* of the interface. To be precise, consider the grand–canonical states

$$\psi^z = \sum_n z^{-n}\phi_n^q\,, \tag{5.1}$$

where $z\in\mathbb{C}\setminus\{0\}$ is a fugacity parameter. Then $\psi^z$ is a product state with

$$\psi^z(\omega) = \prod_{j=1}^L \psi_j^z(\omega_j)\,, \qquad \psi_j^z(\omega_j) = (zq^{-j})^{-\omega_j}\sqrt{\binom{2J}{\omega_j}}\,. \tag{5.2}$$

Proposition II.1 in [12] says that, if $B_1^2 + B_2^2 > 0$, the ground state of $\mathcal{H}_{L,y}^B$ is given by $\psi^z$ with the choice

$$z = z(B,y) := -\,q^y\,\frac{\|B\| + B_3}{B_1 - iB_2}\,, \tag{5.3}$$

where $\|B\|^2 = \sum_i B_i^2$. It also shows that $\mathcal{H}_{L,y}^B\psi^z = -J\|B\|$. These results can be recovered from the following computation. We use the notation $\omega^{\pm,y}$ for the configuration

$$(\omega^{\pm,y})_j = \begin{cases}\omega_j & y\neq j\\ \omega_y\pm 1 & y = j\end{cases}$$

**Lemma 5.1.** *For any $J\in\frac12\mathbb{N}$ and any $B=(B_1,B_2,B_3)$ such that $B_1^2+B_2^2>0$, we have, for every function $\varphi$*

$$\begin{aligned}\langle\omega|\,\mathcal{H}_y^B|\varphi\rangle = \left[\mathcal{H}_y^B\varphi\right](\omega) &= -w^{+,y}(\omega)\left[\frac{\psi^z(\omega)}{\psi^z(\omega^{+,y})}\,\varphi(\omega^{+,y}) - \varphi(\omega)\right]\\ &\quad - w^{-,y}(\omega)\left[\frac{\psi^z(\omega)}{\psi^z(\omega^{-,y})}\,\varphi(\omega^{-,y}) - \varphi(\omega)\right]\,,\end{aligned} \tag{5.4}$$

*where $\mathcal{H}_y^B := B\cdot S_y + J\|B\|$, $z = z(B,y)$ is given in (5.3) and*

$$w^{+,y}(\omega) = \frac12\,(\|B\| - B_3)\,(2J-\omega_y)\,,\quad w^{-,y}(\omega) = \frac12\,(\|B\|+B_3)\,\omega_y\,. \tag{5.5}$$

*Proof.* Observe that for any $z$, if $\omega_y < 2J$

$$\frac{\psi^z(\omega)}{\psi^z(\omega^{+,y})} = (zq^{-y})\sqrt{(\omega_y+1)/(2J-\omega_y)}\,,$$

and, if $\omega_y > 0$

$$\frac{\psi^z(\omega)}{\psi^z(\omega^{-,y})} = (zq^{-y})^{-1}\sqrt{(2J-\omega_y+1)/\omega_y}\,.$$

On the other hand, recall that raising and lowering operators $S_y^{\pm} = S_y^1 \pm iS_y^2$ satisfy

$$S_y^+|\omega\rangle = \sqrt{(2J-\omega_y)(\omega_y+1)}\,|\omega^{+,y}\rangle\,,\quad S_y^-|\omega\rangle = \sqrt{(2J-\omega_y+1)\omega_y}\,|\omega^{-,y}\rangle\,. \tag{5.6}$$

Using these expressions and $zq^{-y} = -(\|B\| + B_3)/(B_1 - iB_2)$ we see that

$$\big[(B_1 S_y^1 + B_2 S_y^2)\varphi\big](\omega) = -w^{+,y}(\omega)\,\frac{\psi^z(\omega)}{\psi^z(\omega^{+,y})}\,\varphi(\omega^{+,y}) \;-\; w^{-,y}(\omega)\,\frac{\psi^z(\omega)}{\psi^z(\omega^{-,y})}\,\varphi(\omega^{-,y})\,.$$

Finally, (5.4) follows from

$$B_3(\omega_y - J) + J\|B\| = w^{+,y} + w^{-,y}\,.$$

□

5.1. **Kawasaki + Glauber.** We now use the identity of Lemma 5.1 to describe the equivalent stochastic process. Recall the basic relations (3.6). Since $\psi^z(\omega)/\psi^z(\omega_\pm^{j,j+1}) = \phi_n^q(\omega)/\phi_n^q(\omega_\pm^{j,j+1})$ whenever $\sum_j \omega_j = n$, we have the following unitary equivalence.

Let $\mu_z$ denote the probability measure $\mu_z = (Z_z)^{-1}|\psi^z|^2$, $z = z(B, y)$. Then $\mathcal{U}_z\varphi = \varphi/\psi^z$ maps unitarily $\mathfrak{H}_L$ into $L^2(\Omega_L^J, \mu_z)$ and from Lemma 5.1 and (3.6)

$$\mathcal{U}_z\left[\mathcal{H}_L + \mathcal{H}_y^B\right]\mathcal{U}_z^{-1} \;=\; -\frac{1}{2\Delta}\,\mathcal{L}_q \;-\; \mathcal{L}_y^G \tag{5.7}$$

where $\mathcal{L}_q$ is the generator of the *conservative* profile dynamics described in (3.8), while $\mathcal{L}_y^G$ is the *dissipative* term given by

$$\mathcal{L}_y^G\varphi(\omega) = w^{+,y}(\omega)\big[\varphi(\omega^{+,y}) - \varphi(\omega)\big] + w^{-,y}(\omega)\big[\varphi(\omega^{-,y}) - \varphi(\omega)\big]\,.$$

As before the measure $\mu_z$ can be viewed as the marginal on horizontal sums (2.29) of variables $\alpha \in \{0,1\}^\Lambda$, with the difference that now the $\alpha$'s are distributed according to independent Bernoulli measures with

$$\mu_z(\alpha_{(\ell,j)} = 1) = \frac{|zq^{-j}|^{-2}}{1 + |zq^{-j}|^{-2}}\,, \qquad |zq^{-j}|^{-2} = \frac{\|B\| - B_3}{\|B\| + B_3}\,q^{2(j-y)}\,.$$

It is also easy to see that the dynamics generated by $\mathcal{L} = 1/(2\Delta)\mathcal{L}_q + \mathcal{L}_y^G$ coincides with the projection of the process on $\alpha$–variables, which we may call Kawasaki+Glauber dynamics, described as follows. On one hand we have the usual exchange (Kawasaki) dynamics with generator

$$\frac{1}{2\Delta}\sum_{\ell,\ell'=1}^{2J}\sum_{j=1}^{L-1} q^{\alpha_{(\ell,j)} - \alpha_{(\ell',j+1)}}\left[f(\alpha^{(\ell,j);(\ell',j+1)}) - f(\alpha)\right].$$

On the other hand we have the single–site Glauber dynamics at row $y$ with generator

$$\sum_{\ell=1}^{2J} g_{(\ell,y)}(\alpha)\left[f(\alpha^{\{(\ell,y)\}}) - f(\alpha)\right],$$

where $g_{(\ell,y)}(\alpha) = \frac{1}{2}(\|B\| - B_3)\alpha_{(\ell,y)} + \frac{1}{2}(\|B\| + B_3)(1 - \alpha_{(\ell,y)})$, and $\alpha^{\{(\ell,y)\}}$ denotes the configuration "flipped" at $(\ell, y)$.

Once the equivalent representation is established, a number of questions for the pinned interface can be asked and, possibly, answered in the framework of the Kawasaki+Glauber dynamics just described. This stochastic process seems to be a natural alternative tool to understand and extend the results obtained in [12] such as the existence of a uniformly positive spectral gap above the unique ground state $\psi^z$. We hope to come back to this and related problems in future work.

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