SPECTRAL GAP INEQUALITIES IN PRODUCT SPACES WITH CONSERVATION LAWS

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ABSTRACT. Following an idea introduced by Carlen, Carvalho and Loss [7] we propose a general strategy to prove Poincaré inequalities in product spaces with one or more conservation laws. The method is shown to yield alternative proofs of well known results, such as the diffusive bounds for the spectral gap of generalized exclusion and zero range processes. Other models are also discussed, including anisotropic exclusion processes, simple exclusion with site–disorder and Ginzburg–Landau processes, where this approach provides sharp spectral gap estimates apparently inaccessible by previously known techniques.

1. INTRODUCTION

The problem of determining the speed of convergence to equilibrium of conservative stochastic dynamics has motivated many investigations in recent years. In the context of reversible processes the simplest way to attack this question is by estimating the spectral gap of the corresponding Markov generators or – equivalently – by proving a Poincaré inequality. In this direction an important achievement are the diffusive estimates established for Kawasaki dynamics in high temperature lattice gases by Lu and Yau [21] and by Cancrini and Martinelli [3]. In this paper we confine ourselves to systems whose underlying equilibrium measure is *product* and the only remaining interaction is due to the global *conservation law*. Although this is certainly a radical simplification, we shall see that already in this class one finds interesting models for which traditional techniques apparently fail to give optimal spectral gap bounds.

The simplest model in this class is the *simple exclusion* process, for which sharp spectral gap estimates are well known, at least since the work of Quastel, [22]. Other conservative dynamics sharing the product–property are the so–called *generalized simple exclusion* processes and the *zero range* process. For these models the martingale approach of [21] was successfully applied by Landim, Sethuraman and Varadhan [20] to show that the spectral gap scales diffusively with the size of the system, uniformly in the conserved parameter. A rather complete picture of decay to equilibrium for the zero range process was then obtained by Janvresse, Landim, Quastel and Yau [14].

As already noted in [22], when the system is of product type it is natural to drop all geometrical constraints in the dynamics and consider processes where exchanges are performed along the edges of a complete graph rather than only along nearest neighbors edges. As we shall see in all the examples treated in this note, once one has a Poincaré inequality for this complete graph (mean–field) dynamics a straightforward comparison argument allows to derive diffusive scaling bounds for the local exchange dynamics.

An example of complete graph dynamics is the model proposed by Kac [15] to study trend to equilibrium for the Boltzmann equation. Spectral gap estimates for this process

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were investigated by Diaconis and Saloff-Coste [10], and by Janvresse [13]. The latter work catches the right shrinking–rate of the spectral gap by adapting the martingale approach of [21]. Recent remarkable work of Carlen, Carvalho and Loss [7, 8] however shows that spectral gap estimates for the Kac model can be sharpened considerably if one–site Poincaré inequalities in the martingale approach are replaced by a fine analysis of the spectrum of an auxiliary Markov process.

As observed in [8] their approach can be generalized to treat a broader class of models than just the Kac model. Our aim in this paper is to show that in principle some of the ideas of [7] apply to all conservative systems of product type. In the case of Kac and related models considered in [8] the spectrum of the auxiliary process can be computed rather explicitly in view of the special form of the probability measures involved. This is in general not the case for the models discussed here and the main technical ingredient in our estimates are uniform local expansions related to the central limit theorem.

Here is a plan of the paper. In section 2 we discuss the auxiliary dynamics introduced in [7] and outline a general strategy to prove uniform spectral gap estimates in product spaces with one or more conservation laws. Here we present explicit sufficient conditions to be checked in specific models. The known results on generalized exclusion and zero range processes mentioned above are re-derived in a compact way in section 3 and section 4, respectively. A simple instance of a model with many conservation laws is considered in section 5. Recent results on anisotropic exclusion and Ginzburg–Landau processes appearing in [5] and [4] are reviewed in section 6 and section 7, respectively. Finally in section 8 we prove a new estimate for the simple exclusion process with site disorder.

2. A general strategy

Consider a generic probability space $(X, \mathcal{F}, \mu)$. In the applications to be discussed below we shall choose $X = \mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$ depending on the specific model. For every $N \in \mathbb{N}$ denote by Ω_N the N-fold product of X, $\Omega_N = X^N$ and by $\mu_N = \mu^{\otimes N}$ the associated product measure. The conservation law is expressed in terms of a given measurable function $\xi : X \to \mathbb{R}$, with $\xi \in L^2(\mu)$. Namely, given a parameter $\rho \in \mathbb{R}$ to play the role of a density, we shall look at configurations $\eta = \{\eta_k\}_{k=1}^N \in \Omega_N$ such that $\sum_{k=1}^N \xi(\eta_k) = \rho N$. If we define $\xi_\rho = \xi - \rho$, we consider the measurable set

$$\Theta_{N,\rho} := \{\eta \in \Omega_N : \sum_{k=1}^N \xi_\rho(\eta_k) = 0\} . \tag{1}$$

Whenever it makes sense we define the canonical probability measure by conditioning on the event $\Theta_{N,\rho}$:

$$\nu_{N,\rho} = \mu_N\big(\ \cdot \ \mid \Theta_{N,\rho}\big) . \tag{2}$$

The complete graph dynamics will be described by a Dirichlet form of the type

$$\mathcal{E}_{N,\rho}(f) = \frac{1}{N} \sum_{k=1}^N \sum_{\ell=1}^N \nu_{N,\rho}\big[(v_{k,\ell} f)^2\big] , \tag{3}$$

where $v_{k,\ell}$ are generic exchange operators to be specified in each model. For the moment we only require that $v_{k,k} = 0$, $k = 1, 2, \dots, N$. To carry a concrete example in mind we recall that the complete graph exclusion process is recovered in the case $X = \{0, 1\}$, $\mu = \mathrm{Be}(p)$, any $p \in (0, 1)$; $\xi(\eta_k) = \eta_k$, $[v_{k,\ell} f](\eta) = f(\eta^{k,\ell}) - f(\eta)$, with $\eta^{k,\ell}$ denoting the configuration η where η_k and η_ℓ have been exchanged.

We denote by $\mathrm{Var}_{N,\rho}(f)$ the usual variance of $f \in L^2(\Omega_N, \nu_{N,\rho})$ with respect to $\nu_{N,\rho}$. The Poincaré constant for fixed N and ρ is defined by

$$\gamma(N,\rho) = \sup_f \frac{\mathrm{Var}_{N,\rho}(f)}{\mathcal{E}_{N,\rho}(f)}, \tag{4}$$

with the supremum ranging over functions f in the domain of the Dirichlet form $\mathcal{E}_{N,\rho}$. Definition (4) is meaningful for all ergodic processes, i.e. when $\mathrm{Var}_{N,\rho}(f) > 0$ implies $\mathcal{E}_{N,\rho}(f) > 0$, and we set by convention $\gamma(N,\rho) = 0$ in all degenerate cases, i.e. when $\mathrm{Var}_{N,\rho}(f) = \mathcal{E}_{N,\rho}(f) = 0$ for all f, such as e.g. the exclusion process with $\rho \in \{0,1\}$. We say that $\nu_{N,\rho}$ satisfies a *uniform Poincaré inequality* if $\sup_N \sup_\rho \gamma(N,\rho) < \infty$.

2.1. **The auxiliary process.** Let $\mathcal{F}_k$ denote the $\sigma-$algebra generated by the one–site variables η_k, $k = 1, \dots,, N$. Following [7, 8] we consider the nonnegative stochastic operator $\mathcal{P} : L^2(\nu_{N,\rho}) \to L^2(\nu_{N,\rho})$ defined by

$$\mathcal{P}f = \frac{1}{N}\sum_{k=1}^N \nu_{N,\rho}(f \,|\, \mathcal{F}_k) . \tag{5}$$

Then $1 - \mathcal{P}$ can be interpreted as the generator of a new Markov process with reversible invariant measure $\nu_{N,\rho}$. This is completely independent of the actual dynamics defined by (3), but we will see in a moment that an estimate on the spectral gap of this process produces useful recursive bounds on the constants $\gamma(N,\rho)$. To gain some insight observe that by symmetry

$$\nu_{N,\rho}(\xi(\eta_k)|\eta_j) = \rho_{\eta_j} := \rho + \frac{\rho - \xi(\eta_j)}{N-1} \tag{6}$$

whenever $k \neq j$, so that

$$\nu_{N,\rho}(\xi_\rho(\eta_k) \,|\, \eta_j) = -\frac{1}{N-1}\,\xi_\rho(\eta_j), \qquad k \neq j . \tag{7}$$

Here and in what follows we often write (with slight abuse) $\nu(f \,|\, \eta_j)$ for the function $\nu(f \,|\, \mathcal{F}_j)(\eta)$. It follows that any function of the form

$$f_\xi(\eta) = \sum_{k=1}^N \alpha_k \xi_\rho(\eta_k), \qquad \alpha \in \mathbb{R}^N \tag{8}$$

satisfies

$$\mathcal{P}f_\xi = \frac{1}{N-1}\, f_\xi, \qquad (1-\mathcal{P})f_\xi = \frac{N-2}{N-1}\, f_\xi. \tag{9}$$

We formulate the needed spectral gap inequality as follows. We say that property (SGP) holds if there exists $C < \infty$, $\delta > 0$ such that for every $N \geqslant 3$, $\rho \in \mathbb{R}$ and $f \in L^2(\nu_{N,\rho})$ with $\nu_{N,\rho}(f) = 0$:

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) \geqslant \frac{N-2}{N-1}\big[1 - CN^{-1-\delta}\big]\,\nu_{N,\rho}\big(f^2\big) . \tag{\textbf{SGP}}$$

We now turn to the implications of such a bound. A useful criterium to check the bound (SGP) in specific models will be developed in the next subsection. We define the constant

$$\gamma(N) := \sup_\rho \gamma(N,\rho) . \tag{10}$$

Proposition 2.1. *Assume $\gamma(N) < \infty$ for every $N \in \mathbb{N}$. If (SGP) holds then we have the uniform Poincaré inequality*

$$\sup_N \gamma(N) < \infty \tag{11}$$

Proof. It is sufficient to show that (SGP) implies a bound of the form

$$\gamma(N) \leqslant \left[1 + CN^{-1-\delta}\right] \gamma(N-1)\,, \tag{12}$$

with $C < \infty$ and $\delta > 0$ independent of ρ and N.

Take an arbitrary function[1] $f \in L^2(\nu_{N,\rho})$ with $\nu_{N,\rho}(f) = 0$. The conditional expectation $\nu_{N,\rho}(f \,|\, \eta_k)$ is identified with the average $\nu_{N-1,\rho_{\eta_k}}(f)$, where ρ_{η_k} is given in (6). For each k we then have the decomposition

$$\nu_{N,\rho}(f^2) = \nu_{N,\rho}\left[\mathrm{Var}_{N-1,\rho_{\eta_k}}(f)\right] + \nu_{N,\rho}\left[\nu_{N,\rho}(f \,|\, \eta_k)^2\right].$$

Averaging over k:

$$\nu_{N,\rho}(f^2) = \frac{1}{N}\sum_{k=1}^{N} \nu_{N,\rho}\left[\mathrm{Var}_{N-1,\rho_{\eta_k}}(f)\right] + \nu_{N,\rho}\left[f\mathcal{P}f\right] \tag{13}$$

with the operator $\mathcal{P}$ defined in (5). By definition of the constants (10):

$$\begin{aligned}\mathrm{Var}_{N-1,\rho_{\eta_k}}(f) &\leqslant \gamma(N-1)\,\mathcal{E}_{N-1,\rho_{\eta_k}}(f)\\ &= \frac{\gamma(N-1)}{N-1}\sum_{j\neq k}\sum_{\ell\neq k} \nu_{N,\rho}\left[(v_{j,\ell}f)^2 \,|\, \mathcal{F}_k\right]\end{aligned} \tag{14}$$

From (13)–(14) and the identity

$$\frac{1}{N}\sum_{k=1}^{N} \nu_{N,\rho}\left[\mathcal{E}_{N-1,\rho_{\eta_k}}(f)\right] = \frac{N-2}{N-1}\,\mathcal{E}_{N,\rho}(f)$$

we obtain the estimate

$$\nu_{N,\rho}\left[f(1-\mathcal{P})f\right] \leqslant \frac{N-2}{N-1}\,\gamma(N-1)\,\mathcal{E}_{N,\rho}(f)\,. \tag{15}$$

Now (12) follows from (15) and the hypothesis (SGP). ☐

2.2. **Reduction to one–dimensional process.** As in [8] the spectrum of $\mathcal{P}$ can be studied in terms of the spectrum of a one–dimensional operator $\mathcal{K}$, see (16) below. Here we show that the estimate (SGP) is implied by a suitable spectral estimate on $\mathcal{K}$, see (SGK) below.

Let π_k be the canonical projection of Ω_N onto X given by $\pi_k\eta = \eta_k$. We call $\nu^1_{N,\rho}$ the one–site marginal of $\nu_{N,\rho}$, i.e. $\nu^1_{N,\rho} = \nu_{N,\rho}\circ\pi_1^{-1}$ is the distribution of η_1 under $\nu_{N,\rho}$. By permutation symmetry all one–site marginals coincide. Let $\mathcal{H}$ denote the Hilbert space $L^2(X, \nu^1_{N,\rho})$ and use $\langle\cdot,\cdot\rangle$ for the corresponding scalar product. Write also $\langle g\rangle$ for the mean of a function $g \in \mathcal{H}$ w.r.t. $\nu^1_{N,\rho}$. We write $\mathcal{H}_0$ for the subspace of $g \in \mathcal{H}$ such that $\langle g\rangle = 0$. We define the stochastic self–adjoint operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ by the bilinear form:

$$\langle g, \mathcal{K}h\rangle = \nu_{N,\rho}\left[(g\circ\pi_1)(h\circ\pi_2)\right], \qquad g,h \in \mathcal{H}\,. \tag{16}$$

[1] In this proof we shall not be careful about questions of domains of the Dirichlet forms $\mathcal{E}_{N,\rho}$. It is however straightforward to settle these issues in all the following applications.

The identity (7) shows that

$$\mathcal{K}\xi_\rho = -\frac{1}{N-1}\,\xi_\rho \tag{17}$$

for every ρ. Thus the spectrum of $\mathcal{K}$ always contains the eigenvalues $-\frac{1}{N-1}$ and 1. We say that property (SGK) holds if the rest of the spectrum of $\mathcal{K}$ is confined around zero within a neighborhood of radius $O(N^{-1-\delta})$ for some $\delta > 0$ uniformly in N, ρ, i.e. if there exist constants $C < \infty, \delta > 0$ such that for every N and ρ, for every $g \in \mathcal{H}_0$ satisfying $\langle g, \xi_\rho\rangle = 0$ one has

$$\big|\langle g, \mathcal{K}g\rangle\big| \leqslant C\,N^{-1-\delta}\,\langle g, g\rangle\,. \tag{\textbf{SGK}}$$

Lemma 2.2. *(SGK) implies (SGP).*

Proof. We define the closed subspace Γ of $L^2(\nu_{N,\rho})$ consisting of sums of mean–zero functions of a single variable:

$$\Gamma = \Big\{ f \in L^2(\nu_{N,\rho}) : \; f = \sum_{k=1}^{N} g_k \circ \pi_k \,; \quad g_1, \dots, g_N \in \mathcal{H}_0\,, \Big\} \tag{18}$$

We first observe that $\mathcal{P}f \in \Gamma$ for every $f \in L^2(\nu_{N,\rho})$ with $\nu_{N,\rho}(f) = 0$. Therefore $\mathcal{P}f = 0$ whenever $f \in \Gamma^\perp$, f with mean zero. In particular we may restrict to $f \in \Gamma$ to prove (SGP).

Given $f \in \Gamma$, $f = \sum_k g_k \circ \pi_k$, we define $\varphi_f = \sum_k g_k$, a function in $\mathcal{H}_0$. A simple computation shows that

$$\nu_{N,\rho}\big(f^2\big) = \langle \varphi_f, \mathcal{K}\varphi_f\rangle + \sum_k \langle g_k, (1-\mathcal{K})g_k\rangle\,, \tag{19}$$

where $\mathcal{K}$ is the operator defined in (16). Similarly one computes

$$\begin{aligned}\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) = {} & \frac{N-2}{N}\,\langle \varphi_f, \mathcal{K}(1-\mathcal{K})\varphi_f\rangle \\ & + \frac{1}{N}\sum_k \langle g_k, (1-\mathcal{K})[(N-1)+\mathcal{K}]g_k\rangle\,.\end{aligned} \tag{20}$$

Consider now the subspace $\mathcal{S} \subset \Gamma$ of symmetric functions:

$$\mathcal{S} = \Big\{ f \in L^2(\nu_{N,\rho}) : \; f = \sum_{k=1}^{N} g \circ \pi_k\,, \;\; g \in \mathcal{H}_0 \Big\}\,. \tag{21}$$

Since $\mathcal{S}$ is invariant for $\mathcal{P}$, i.e. $\mathcal{P}\mathcal{S} \subset \mathcal{S}$ we may consider separately the cases $f \in \mathcal{S}$ and $f \in \mathcal{S}^\perp$, with $\mathcal{S}^\perp$ denoting the orthogonal complement in Γ. When $f \in \mathcal{S}$, $f = \sum_{k=1}^N g \circ \pi_k$ we have $\varphi_f = Ng$ and rearranging terms in (19) and (20) we obtain

$$\nu_{N,\rho}\big(f^2\big) = N(N-1)\,\langle g, [\mathcal{K} + \frac{1}{N-1}]g\rangle \tag{22}$$

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) = (N-1)^2\langle g, [1-\mathcal{K}][\mathcal{K} + \frac{1}{N-1}]g\rangle \tag{23}$$

From (SGK) we see that $\mathcal{K} + \frac{1}{N-1}$ is nonnegative on the whole subspace $\mathcal{H}_0$. Moreover, since $f = 0$ when g is a multiple of ξ_ρ, we may then restrict to the case $\langle g, \xi_\rho\rangle = 0$. Writing

$\tilde g = [\mathcal{K} + \frac{1}{N-1}]^{\frac12} g$ and observing that $\langle \tilde g\rangle = 0$ and $\langle \tilde g, \xi_\rho\rangle = 0$, the assumption (SGK) implies

$$\begin{aligned}\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) &\geqslant (N-1)^2\big[1 - CN^{-1-\delta}\big]\,\langle \tilde g, \tilde g\rangle\\ &\geqslant \frac{N-2}{N-1}\big[1 - CN^{-1-\delta}\big]\,\nu_{N,\rho}\big(f^2\big)\,, \qquad f \in \mathcal{S}\,. \end{aligned} \tag{24}$$

We turn to study the case $f \in \mathcal{S}^\perp$. Let us first observe that one can assume without loss that $f \in \Gamma$ is such that $\langle \varphi_f, \xi_\rho\rangle = \sum_k \langle g_k, \xi_\rho\rangle = 0$. Indeed if $c = (N\langle \xi_\rho, \xi_\rho\rangle)^{-1}\sum_k\langle g_k, \xi_\rho\rangle$ and $\tilde g_k = g_k - c\xi_\rho$, we have $\sum_k \tilde g_k \circ \pi_k = \sum_k g_k \circ \pi_k$ in $L^2(\nu_{N,\rho})$ since by the conservation law $\sum_k \xi_\rho \circ \pi_k = 0$. Now, for every $u \in \mathcal{S}$, $u = \sum_k u_0 \circ \pi_k$, with $u_0 \in \mathcal{H}_0$ one has

$$\nu_{N,\rho}(uf) = (N-1)\,\langle \varphi_f, [\mathcal{K} + \frac{1}{N-1}]u_0\rangle\,.$$

Thus $f \in \mathcal{S}^\perp$ implies that $[\mathcal{K} + \frac{1}{N-1}]\varphi_f$ is a constant in $\mathcal{H}$. Since $\langle\varphi_f\rangle = 0$ and $\langle \varphi_f, \xi_\rho\rangle = 0$, (SGK) implies $\varphi_f = 0$. Writing $\hat g_k = (1-\mathcal{K})^{\frac12} g_k$, then (19) and (20) imply

$$\nu_{N,\rho}\big(f^2\big) = \sum_k \langle \hat g_k, \hat g_k\rangle \tag{25}$$

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) = \frac1N \sum_k \langle \hat g_k, [(N-1) + \mathcal{K}]\hat g_k\rangle \tag{26}$$

Since $\langle \hat g_k\rangle = 0$ for all k we use (SGK) to estimate

$$\langle \hat g_k, \mathcal{K}\hat g_k\rangle \geqslant -\frac{1}{N-1}\langle \hat g_k, \hat g_k\rangle\,.$$

From (25) and (26) we obtain

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) \geqslant \frac{N-2}{N-1}\nu_{N,\rho}\big(f^2\big)\,, \qquad f \in \mathcal{S}^\perp\,. \tag{27}$$

From (24) and (27) we obtain (SGP) and the proof is completed. □

2.3. **Several conservation laws.** In the case of more than one conservation law we are given an r–dimensional vector $\bar\xi = (\xi^1, \dots, \xi^r)$ of measurable functions $\xi^j : X \to \mathbb{R}$, for some positive integer r, and we require that

$$\sum_{k=1}^N \xi^j(\eta_k) = \rho^j N\,, \qquad j = 1, \dots, r$$

with $\bar\rho := (\rho^1, \dots, \rho^r)$ an assigned density vector. If we denote $\Theta_{N,\bar\rho}$ the event realizing simultaneously all the constraints above we then define the conditional probability measure

$$\nu_{N,\bar\rho} = \mu_N\big(\ \cdot\ \mid \Theta_{N,\bar\rho}\big)\,. \tag{28}$$

With these notations the argument of Proposition 2.1 carries over with no change provided we replace ρ with $\bar\rho$. We observe that (17) now holds for every $\xi^j_{\rho^j}$, $j = 1, \dots, r$. Moreover, as in Lemma 2.2 one proves that (SGP) can be obtained as a consequence of (SGK), provided the latter condition is modified by requiring the spectral estimate for any $g \in \mathcal{H}_0$ which is orthogonal to all functions $\xi^j_{\rho^j}$ simultaneously. As a simple example of a system with several conservation laws we will discuss the colored exclusion process in section 5.

2.4. **From complete graph to local exchanges.** In many applications it is interesting to consider local versions of the conservative dynamics. In analogy with (3) we describe such local dynamics by the Dirichlet form

$$\mathcal{D}_{N,\rho}(f) = \sum_{k=1}^{N-1} \nu_{N,\rho}\big[(v_{k,k+1}f)^2\big]\,. \tag{29}$$

The standard tool to compare the forms $\mathcal{D}_{N,\rho}$ and $\mathcal{E}_{N,\rho}$ is what is often called (for obvious reasons) the moving–particle lemma. In this general setting we may state this as follows. We say that a moving–particle lemma holds, or simply that (MP) holds if there exists a constant $C < \infty$ such that for every N and ρ, every integer $n \leqslant N$ and every f one has

$$\nu_{N,\rho}\big[(v_{1,n}f)^2\big] \leqslant C\, n \sum_{k=1}^{n-1} \nu_{N,\rho}\big[(v_{k,k+1}f)^2\big]\,. \tag{\textbf{MP}}$$

A simple consequence of (MP) is the comparison estimate

$$\mathcal{E}_{N,\rho}(f) \leqslant C\, N^2\, \mathcal{D}_{N,\rho}(f)\,. \tag{30}$$

Thus, if we are able to prove the uniform Poincaré inequality (11) and (MP) holds we can infer uniform diffusive estimates for the local dynamics. These arguments can be generalized in a straightforward way to treat local dynamics in which particles are located at the sites of a box in a d–dimensional lattice $\mathbb{Z}^d$, any $d \geqslant 1$. Suppose for instance $N = L^d$, for some $L \in \mathbb{N}$, is the cardinality of the hypercube $\Lambda_L = \{1, \ldots, L\}^d \subset \mathbb{Z}^d$ and we are interested in a process defined by the Dirichlet form

$$\widetilde{\mathcal{D}}_{L,\rho}(f) = \sum_{\substack{x,y\in\Lambda_L:\\ |x-y|=1}} \nu_{N,\rho}\big[(v_{x,y}f)^2\big]\,, \tag{31}$$

where $|x| := \sum_{i=1}^d |x_i|$, $x \in \mathbb{Z}^d$. Then, assuming (MP), a straightforward path–counting argument gives the diffusive bound

$$\mathcal{E}_{N,\rho}(f) \leqslant C\, L^2\, \widetilde{\mathcal{D}}_{L,\rho}(f)\,. \tag{32}$$

We shall see that all the examples we consider hereafter do satisfy the (MP) property.

3. Generalized exclusion

Here we take $X = \{0, 1, \ldots, R\}$, R a given integer, and μ a probability measure on X such that $p(n) := \mu(\eta_1 = n) > 0$ for all $n = 0, 1, \ldots, R$. Ω_N is the space of configurations $\eta = (\eta_k)$, with the interpretation that η_k is the number of particles at site k. Here $\xi(\eta_k) = \eta_k$ and the total number of particles is conserved. For any $\rho \in I_{R,N} := \{0, \frac{1}{N}, \frac{2}{N}, \ldots, R - \frac{1}{N}, R\}$ we have the canonical measure $\nu_{N,\rho}$ defined by (2).

The generalized exclusion process on the complete graph $\{1, 2, \ldots, N\}$ can be loosely described as follows. At each site a Poisson clock rings with rate 1. When site k rings we choose uniformly one of the sites, say j. If $k \neq j$, if site k contains at least one particle (i.e. $\eta_k > 0$) and site j is not saturated (i.e. $\eta_j < R$), a particle is moved from k to j with rate $c(\eta_k, \eta_j)$, otherwise nothing happens. The rates $c(\cdot, \cdot)$ are chosen in such a way that the resulting process is reversible w.r.t. $\nu_{N,\rho}$. A possible choice is for instance $c(\eta_j, \eta_k) = 1/[p(\eta_j)p(\eta_k)]$. In any case, assuming a uniform bound from above and below on the rates $c(\cdot,\cdot)$, the resulting Dirichlet form is controlled (up to multiplicative constants) in terms of the quadratic form

$$\mathcal{E}_{N,\rho}(f) = \frac{1}{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}\nu_{N,\rho}\big[(v_{k,\ell}f)^2\big]\,, \qquad v_{k,\ell}f = f\circ T_{k,\ell} - f \tag{33}$$

where f is any real function on Ω_N and

$$\big(T_{k,\ell}\eta\big)_j = \begin{cases}\eta_k - 1 & \text{if } j = k\,,\ \eta_k > 0 \text{ and } \eta_\ell < R\\ \eta_\ell + 1 & \text{if } j = \ell\,,\ \eta_k > 0 \text{ and } \eta_\ell < R\\ \eta_j & \text{otherwise}\,.\end{cases}$$

As in Lemma A.2.8 of [16] (p.392) it is not difficult to prove that property (MP) holds for this model. In particular, by (30)–(32) the estimate of Theorem 3.1 below immediately implies the well known diffusive scaling estimate (as given e.g. in [16], Theorem A.2.1).

Theorem 3.1. *For every $R\in\mathbb{N}$ there exists $C<\infty$ such that*

$$\sup_{N\geqslant 2}\ \sup_{\rho\in I_{R,N}}\gamma(N,\rho)\ \leqslant\ C\,.$$

The proof of Theorem 3.1 is based on Proposition 2.1. We thus have to check that $\sup_\rho\gamma(N,\rho)$ is finite for all N and that property (SGP) holds.

The first requirement is easily seen to be satisfied. Namely for every fixed N and $\rho\in I_{R,N}$, $\rho\neq 0, R$, the process is ergodic, i.e. whenever $f\in L^2(\nu_{N,\rho})$ is such that $\mathcal{D}_{N,\rho}(f)=0$ then f is constant over $\Theta_{N,\rho}$. This implies that $\gamma(N,\rho)<\infty$. Since ρ can take only a finite number of values we have $\gamma(N)=\sup_\rho\gamma(N,\rho)<\infty$ for every fixed N.

To prove (SGP) we rely on Lemma 2.2. In this setting the operator $\mathcal{K}$ defined in (16) is a $(R+1)\times(R+1)$–matrix with entries

$$\mathcal{K}(n,m) = \nu_{N,\rho}(\eta_2 = m\,|\,\eta_1 = n)\,.$$

In order to simplify the notation we adopt the following shortcuts:

$$\nu(n) := \nu_{N,\rho}(\eta_1 = n)\,,\quad \nu(n,m) := \nu_{N,\rho}(\eta_1 = n, \eta_2 = m) \tag{34}$$

For any function $\varphi\in\mathcal{H}_0$ we have

$$\langle\varphi,\mathcal{K}\varphi\rangle = \sum_{n=0}^{R}\sum_{m=0}^{R}\nu(n)\nu(m)Q(n,m)\varphi(n)\varphi(m) \tag{35}$$

where we introduce the kernel

$$Q(n,m) = \frac{\nu(n,m)-\nu(n)\nu(m)}{\nu(n)\nu(m)} \tag{36}$$

The proof of (SGK) will be obtained by a careful examination of the kernel Q. If $\varphi\in\mathcal{H}_0$ is such that $\langle\varphi,\xi_\rho\rangle = 0$ as in the hypothesis of (SGK), then from (35) we have

$$\langle\varphi,\mathcal{K}\varphi\rangle = \sum_{n=0}^{R}\sum_{m=0}^{R}\nu(n)\nu(m)\Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big]\varphi(n)\varphi(m)\,,$$

where σ_ρ^2 refers to the grand–canonical variance at density ρ, see (39) below. Therefore (SGK) follows from the Schwarz' inequality and Proposition 3.2 below, which we prove in the next subsection.

Proposition 3.2. *For every $R \in \mathbb{N}$ there exists $C < \infty$ and $\delta > 0$ such that*

$$\sum_{n=0}^{R} \sum_{m=0}^{R} \nu(n)\nu(m) \Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N} \Big]^2 \leqslant C\, N^{-2-\delta} \tag{37}$$

3.1. **Proof of Proposition 3.2.** We start with some preliminaries. Let $\bar\mu_\alpha$, $\alpha > 0$, be the probability measure on X defined by

$$\bar\mu_\alpha(\eta_1 = k) = \frac{p(k)\alpha^k}{\bar Z_\alpha}, \qquad \bar Z_\alpha = \sum_{j=0}^{R} p(j)\alpha^j\,. \tag{38}$$

Let $\rho = \rho(\alpha)$ be the average number of particles according to $\bar\mu_\alpha$:

$$\rho = \frac{1}{\bar Z_\alpha} \sum_{k=1}^{R} k p(k)\alpha^k$$

Since the function $\rho : [0,\infty] \to [0,R]$ is strictly increasing, with $\rho'(\alpha) = \alpha^{-1}\mathrm{Var}_{\bar\mu_\alpha}(\eta_1)$, we can invert it to to find the function $\alpha(\rho) : [0,R] \to [0,\infty]$. From now on we shall write μ_ρ for the measure $\bar\mu_{\alpha(\rho)}$. We call σ_ρ^2 the variance

$$\sigma_\rho^2 = \mathrm{Var}_{\mu_\rho}(\eta_1) \tag{39}$$

Clearly $\sigma_\rho^2 \leqslant R^2/2$, and $\sigma_\rho^2 \to 0$ when $\rho \to 0$ or $\rho \to R$. Define $p_\rho(k) := \mu_\rho(\eta_1 = k)$. It is simple to check the following estimates, to be used for small density ρ:

$$p_\rho(0) = 1 - \rho + O(\rho^2)\,, \quad p_\rho(1) = \rho + O(\rho^2)\,, \quad p_\rho(k) = O(\rho^k)\,,\ k \geqslant 2\,. \tag{40}$$

In particular, $\sigma_\rho^2 = \rho + O(\rho^2)$, as $\rho \to 0$. By duality the same estimate holds with ρ replaced by $R - \rho$ when $\rho \to R$. The characteristic function of the rescaled variable ξ_ρ/σ_ρ is defined by

$$v_\rho(\zeta) = \mu_\rho\big(\exp\left(i\zeta\xi_\rho/\sigma_\rho\right)\big) \tag{41}$$

Lemma 3.3. *There exists $a = a(R) > 0$ such that for every $\rho \in (0,R)$*

$$|v_\rho(\zeta)| \leqslant e^{-a\zeta^2}\,, \qquad \zeta \in [-\pi\sigma_\rho, \pi\sigma_\rho]\,.$$

Proof. Observe that by the trigonometric identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$:

$$\begin{aligned}|v_\rho(\zeta)|^2 &= \mu_\rho\big[\cos(\zeta\xi_\rho/\sigma_\rho)\big]^2 + \mu_\rho\big[\sin(\zeta\xi_\rho/\sigma_\rho)\big]^2 \\ &= \sum_{k=0}^{R}\sum_{j=0}^{R} p_\rho(k)p_\rho(j)\cos[\zeta(k-j)/\sigma_\rho]\,.\end{aligned}$$

Now estimate

$$\cos[\zeta(k-j)/\sigma_\rho] \leqslant \begin{cases} 1 & \text{if } |k-j| \neq 1 \\ 1 - \frac{2\zeta^2}{\pi^2\sigma_\rho^2} & \text{if } |k-j| = 1 \end{cases} \qquad |\zeta| \leqslant \pi\sigma_\rho$$

It follows that

$$|v_\rho(\zeta)|^2 \leqslant 1 - \frac{4\zeta^2}{\pi^2\sigma_\rho^2} \sum_{k=0}^{R-1} p_\rho(k)p_\rho(k+1)\,.$$

Using (40) it is easy to check that there exists $\delta = \delta(R) > 0$ such that uniformly in $\rho \in (0,R)$

$$\sum_{k=0}^{R-1} p_\rho(k)p_\rho(k+1) \geqslant \delta\,\sigma_\rho^2\,.$$

We have shown that $|v_\rho(\zeta)|^2-1\leqslant -2a\zeta^2$, with $a=2\delta/\pi^2$. The lemma then follows from the elementary inequality $x\leqslant e^{\frac{1}{2}(x^2-1)}, x\in[0,1]$ applied to $x=|v_\rho(\zeta)|$. $\square$

We now start the proof of Proposition 3.2. By particle–hole duality we may restrict to densities ρ satisfying $\rho\leqslant R/2$. It is convenient to consider separately two regimes of density.

The case $\frac{R}{2}\geqslant\rho\geqslant N^{-\frac{3}{4}}$. Denote by $\mu_{N,\rho}$ the product measure $\mu_\rho^{\otimes N}$ and recall the event $\Theta_{N,\rho}$ that the sum of the η's is ρN. Set $\tilde v_\rho(\zeta)=v_\rho(\zeta/\sqrt{N})$. By elementary Fourier transform we have

$$2\pi\sigma_\rho\sqrt{N}\mu_{N,\rho}(\Theta_{N,\rho})=\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^N\,. \tag{42}$$

Here and in the rest of this proof all the integrals are over the interval $[-\pi\sigma_\rho\sqrt{N},\pi\sigma_\rho\sqrt{N}]$. Similarly

$$\nu(n)=\frac{p_\rho(n)}{2\pi\sigma_\rho\sqrt{N}\mu_{N,\rho}(\Theta_{N,\rho})}\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^{N-1}e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}\bar n} \tag{43}$$

$$\nu(n,m)=\frac{p_\rho(n)p_\rho(m)}{2\pi\sigma_\rho\sqrt{N}\mu_{N,\rho}(\Theta_{N,\rho})}\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^{N-2}e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}[\bar n+\bar m]} \tag{44}$$

where we use the shortcut notation $\bar n=\xi_\rho(n)=n-\rho$, $\bar m=\xi_\rho(m)=m-\rho$. We can then write

$$Q(m,n)=\frac{\nu(n,m)-\nu(n)\nu(m)}{\nu(n)\nu(m)}=\frac{\mathrm{NUM}}{\mathrm{DEN}} \tag{45}$$

with

$$\begin{aligned}\mathrm{NUM}:=&\int \mathrm{d}t\,\tilde v_\rho(\zeta)^{N-2}e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}[\bar n+\bar m]}\int \mathrm{d}\zeta'\,\tilde v_\rho(\zeta')^N\\ &-\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^{N-1}e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}\bar n}\int \mathrm{d}\zeta'\,\tilde v_\rho(\zeta')^{N-1}e^{i\frac{\zeta'}{\sigma_\rho\sqrt{N}}\bar m}\end{aligned} \tag{46}$$

and

$$\mathrm{DEN}:=\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^{N-1}e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}\bar n}\int \mathrm{d}\zeta'\,\tilde v_\rho(\zeta')^{N-1}e^{i\frac{\zeta'}{\sigma_\rho\sqrt{N}}\bar m}$$

Thanks to the bound of Lemma 3.3 we have $|\tilde v_\rho(\zeta)|^N\leqslant e^{-a\zeta^2}$. Therefore in the integrals above only the region $|\zeta|\leqslant C\log N$ (for some large but fixed C) has to be taken care of. We then observe that there exists $\delta>0$ such that uniformly

$$\tilde v_\rho(\zeta)=1-\frac{\zeta^2}{2N}+O(N^{-1-\delta})\,,\qquad |\zeta|\leqslant C\log N\,. \tag{47}$$

Indeed, by expanding $\tilde v_\rho$ around the origin the third order error term is bounded from above by $C|\zeta|^3(\sigma_\rho\sqrt{N})^{-3}\mu_\rho(|\xi_\rho|^3)$. Observing that $\mu_\rho(|\xi_\rho|^3)\leqslant C\sigma_\rho^2$ and $\sigma_\rho^2\geqslant C^{-1}\rho$ then (47) follows from the assumption $R/2\geqslant\rho\geqslant N^{-3/4}$. Similarly one can write $\tilde v_\rho(\zeta)^N=e^{-\frac{1}{2}\zeta^2}+O(N^{-\delta})$ in the range $|\zeta|\leqslant C\log N$. This gives the uniform estimates

$$\begin{aligned}I_1&:=\int \mathrm{d}\zeta\,\tilde v_\rho(\zeta)^N=\sqrt{2\pi}+O(N^{-\delta})\\ I_2&:=\int \mathrm{d}\zeta\,\zeta^2\,\tilde v_\rho(\zeta)^N=\sqrt{2\pi}+O(N^{-\delta})\\ I_3&:=\int \mathrm{d}\zeta\,\zeta\,\tilde v_\rho(\zeta)^N=O(N^{-\delta})\end{aligned}$$

From (47) we also deduce

$$\tilde{v}_\rho(\zeta)^{N-2} = \tilde{v}_\rho(\zeta)^N \left(1 + \frac{\zeta^2}{N} + O(N^{-1-\delta})\right)$$
$$\tilde{v}_\rho(\zeta)^{N-1} = \tilde{v}_\rho(\zeta)^N \left(1 + \frac{\zeta^2}{2N} + O(N^{-1-\delta})\right)$$

uniformly in the region $|\zeta| \leqslant C \log N$. We then expand

$$e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}\bar{n}} = 1 + i\frac{\zeta\bar{n}}{\sigma_\rho\sqrt{N}} + u_n(\zeta)$$
$$e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}[\bar{n}+\bar{m}]} = \Big(1 + i\frac{\zeta\bar{n}}{\sigma_\rho\sqrt{N}} + u_n(\zeta)\Big)\Big(1 + i\frac{\zeta\bar{m}}{\sigma_\rho\sqrt{N}} + u_m(\zeta)\Big)$$

with error terms u_n satisfying $|u_n(\zeta)| \leqslant C\zeta^2\frac{\bar{n}^2}{N\sigma_\rho^2}$. When we plug all the previous identities into (46), after all the cancellations we arrive at

$$\begin{aligned}\mathrm{NUM} &= -\frac{\bar{n}\bar{m}}{\sigma_\rho^2 N}(I_1I_2 - I_3^2) + R_1(n,m)\\ &= -2\pi\frac{\bar{n}\bar{m}}{\sigma_\rho^2 N} + R_1(n,m) + R_2(n,m)\end{aligned} \tag{48}$$

with remainder terms satisfying

$$|R_1(n,m)| \leqslant C\frac{\bar{n}^2\bar{m}^2}{(\sigma_\rho^2 N)^2} + C\frac{(|\bar{n}|\bar{m}^2 + |\bar{m}|\bar{n}^2)}{(\sigma_\rho^2 N)^{3/2}} + O(N^{-1-\delta})$$

and $|R_2(n,m)| \leqslant CN^{-1-\delta}|\bar{n}|\,|\bar{m}|/\sigma_\rho^2$. Using the bounds $\nu(\xi_\rho^2) \leqslant C\sigma_\rho^2$ and $\nu(\xi_\rho^4) \leqslant C\sigma_\rho^2$ together with $\sigma_\rho^2 \geqslant C^{-1}N^{-\frac{3}{4}}$ we see that

$$\sum_{n,m}\nu(n)\nu(m)|R_i(n,m)|^2 \leqslant O(N^{-2-\delta})\,, \qquad i = 1,2\,. \tag{49}$$

On the other hand similar reasoning implies

$$\mathrm{DEN} = 2\pi + O(N^{-\delta}) \tag{50}$$

In conclusion (37) follows from (48)–(50).

The case $\rho \leqslant N^{-\frac{3}{4}}$. We first check that

$$\sum_{\substack{n,m:\, n+m\,\geqslant\,2,\\ nm\neq 1}} \nu(n)\nu(m)\,Q(n,m)^2 = O(N^{-2-\delta}) \tag{51}$$

To prove (51) we take advantage of the very thin tails of $\nu(n)$ in the range $\rho \leqslant N^{-3/4}$. By a standard argument using Lemma 3.3 (see e.g. the proof of Proposition 3.8 in [5]), from (43)–(44) and (40) one obtains

$$\nu(n) \;\leqslant\; C\,p_\rho(n) \tag{52}$$

and $\nu(n,m) \leqslant Cp_\rho(n)p_\rho(m)$, where C is a uniform constant. Therefore $\nu(n) = O(\rho^n)$ and $\nu(n,m) = O(\rho^{n+m})$. In the same way, writing $\nu(m\,|\,n) := \frac{\nu(m,n)}{\nu(n)}$, we have

$$\frac{\nu(n,m)^2}{\nu(n)\nu(m)} = \nu(m\,|\,n)\nu(n\,|\,m) \leqslant C\,\rho_m^n\,\rho_n^m \leqslant C\rho^{n+m}\,,$$

where $\rho_n = \rho + (\rho - n)/(N-1) \leqslant \rho N/(N-1)$. Therefore

$$\nu(n)\nu(m)Q(n,m)^2 \leqslant C\,\rho^{n+m}$$

In particular, $\sum_{n+m \geqslant 3} \nu(n)\nu(m)Q(n,m)^2 \leqslant C\,\rho^3 \leqslant C\,N^{-9/4}$, since $\rho \leqslant N^{-3/4}$. On the other hand $Q(0,2) = O(\rho)$ since $\nu(0,2) = \nu(2) - \nu(\eta_1 = 2, \eta_2 \geqslant 1) = \nu(0)\nu(2) + O(\rho^3)$. It follows $\nu(0)\nu(2)Q(0,2)^2 = O(\rho^4)$. This completes the proof of (51). In a similar way, using $\sigma_\rho^2 = \rho + O(\rho^2)$ one checks:

$$\sum_{\substack{n,m:\, n+m \geqslant 2,\\ nm \neq 1}} \nu(n)\nu(m)\left|\frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\right|^2 = O(N^{-2-\delta}) \tag{53}$$

It remains to prove that (37) holds when n and m are restricted to $\{0,1\}$:

$$\nu(n)\nu(m)\Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big]^2 = O(N^{-2-\delta})\,, \qquad n,m \in \{0,1\}\,. \tag{54}$$

Recall that $\nu(1) = \rho + O(\rho^2)$ and $\nu(0) = 1 - \rho + O(\rho^2)$. With $\rho_n = \rho - \xi_\rho(n)/(N-1)$ we then have

$$\nu(m\,|\,n) = \begin{cases} (1-\rho_n) + O(\rho^2) & m = 0 \\ \rho_n + O(\rho^2) & m = 1 \end{cases}$$

Therefore

$$Q(m,n) = \frac{\nu(m\,|\,n) - \nu(m)}{\nu(m)} = \begin{cases} \frac{\xi_\rho(n)}{(1-\rho)N} + O(\rho^2) & m = 0 \\ -\frac{\xi_\rho(n)}{\rho N} + O(\rho) & m = 1 \end{cases} \tag{55}$$

Since $\sigma_\rho^2 = \rho + O(\rho^2)$, (55) implies (54). This completes the proof of the proposition.

4. Zero–range processes

The zero range processes fit the general setting of section 2. Here $X = \mathbb{N}$ and the variables η_k are interpreted as occupation numbers. The apriori probability measure μ is of the form

$$p(0) = \frac{1}{Z}\,; \qquad p(n) = \frac{1}{Z}\prod_{i=1}^{n}\frac{1}{c(i)}\,, \quad n \geqslant 1$$

where $p(n) := \mu(\eta_k = n)$, c is a given positive function on $\mathbb{N}_+$ to be interpreted as the rate of escape, see below, and Z is the normalization constant. We shall make assumptions which imply in particular that $c(n) \geqslant \delta n$ for some $\delta > 0$ and all $n \geqslant 1$ so that μ is always well defined (and has all exponential moments).

The conserved quantity is the total number of particles, so $\xi(\eta_k) = \eta_k$. The complete graph dynamics is described as follows. Each site $k \in \{1, \dots, N\}$ is equipped with a Poisson clock which rings at rate 1. When site k rings we choose uniformly another site, say j. If $k \neq j$ and $\eta_k > 0$ we move one particle from k to j with rate $c(\eta_k)$. The rate is independent of the configuration η outside site k, thus justifying the name zero range. The canonical measures $\nu_{N,\rho}$ are reversible since

$$c(n)p(n)p(m) = c(m+1)p(n-1)p(m+1)\,, \qquad n \geqslant 1\,,\ m \geqslant 0.$$

The Dirichlet form is then given by (3) with

$$v_{k,\ell}f(\eta) = \sqrt{c(\eta_k)/2}\,\big[f(T_{k,\ell}\eta) - f(\eta)\big] \tag{56}$$

where

$$\left(T_{k,\ell}\eta\right)_j = \begin{cases} \eta_k - 1 & \text{if } j = k\,,\ \eta_k \geqslant 1 \\ \eta_\ell + 1 & \text{if } j = \ell\,,\ \eta_k \geqslant 1 \\ \eta_j & \text{otherwise}\,. \end{cases}$$

We make two assumptions on the rate $c(\cdot)$:

- c is globally Lipschitz: There exists $a_1 < \infty$ such that

$$\sup_n |c(n+1) - c(n)| \leqslant a_1 \tag{H1}$$

- c grows at infinity: There exists $N_0 < \infty$ and $a_2 > 0$ such that

$$c(n) \geqslant c(m) + a_2\,, \qquad n \geqslant N_0 + m \tag{H2}$$

A very special case is $c(n) = n$, so that the measure μ is Poisson. In this case the process consists of ρN independent random walks on the complete graph and therefore a uniform Poincaré inequality is trivially obtained by tensorization. (H1) and (H2) are the assumptions considered by Landim, Sethuraman and Varadhan [20] and we shall use some key preliminary results of [20] to make our proof. Since the property (MP) discussed in section 2 is immediate for the zero range process (56) one can recover the main results of [20] using (30)–(32) and the theorem below.

Theorem 4.1. *Assume* (H1) *and* (H2). *There exists $C < \infty$ such that*

$$\sup_{N\,\geqslant\,2}\ \sup_{\rho\in\mathbb{N}/N} \gamma(N,\rho) \leqslant C \tag{57}$$

We shall prove the theorem by checking the hypothesis of Proposition 2.1. We follow as closely as possible the analysis of the previous section. However more care is required here in view of the unboundedness of the variables η_k.

Let $\gamma(N)$ be defined as in (10). We first check that $\gamma(N) < \infty$ for all N. From Lemma 3.1 and Lemma 3.2 of [20] we have that for any $f \in L^2(\nu_{N,\rho})$ with $\nu_{N,\rho}(f) = 0$

$$\frac{1}{N}\sum_{k=1}^{N} \nu_{N,\rho}\big(\nu_{N,\rho}(f\,|\,\mathcal{F}_k)^2\big) \leqslant C\Big[\mathcal{E}_{N,\rho}(f) + \frac{1}{N}\sum_{k=1}^{N} \nu\big(\mathrm{Var}_{N-1,\rho_{\eta_k}}(f)\big)\Big]$$

for some uniform constant $C < \infty$. Thus from (13) and (14) we obtain in particular

$$\gamma(N) \leqslant C\,\gamma(N-1) + C\,, \qquad N \geqslant 2\,,$$

which clearly implies that $\gamma(N)$ is finite for every N, since $\gamma(1) = 0$.

Thanks to Lemma 2.2 we reduce the proof of (SGP) to the proof of estimate (SGK). As in (35) we write

$$\langle \varphi, \mathcal{K}\varphi\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \nu(n)\nu(m)Q(n,m)\varphi(n)\varphi(m) \tag{58}$$

with the kernel Q given by (36). In the next subsection we prove (SGK)

4.1. **Proof of (SGK).** As in (38) we define the exponential family $\bar{\mu}_\alpha, \alpha > 0$ and the corresponding measures μ_ρ indexed by the density $\rho > 0$. The latter is given by $\rho = \mu_\rho(\eta_1)$ and a simple computation gives $\alpha(\rho) = \mu_\rho(c(\eta_1))$. The variance σ_ρ^2 is defined as in (39). As shown in [20], Lemma 5.1, the assumptions (H1) and (H2) imply the uniform bounds

$$\delta\rho \leqslant \sigma_\rho^2 \leqslant \delta^{-1}\,\rho\,. \tag{59}$$

for some $\delta \in (0,1)$. We distinguish two regimes according to the value of the density ρ. We speak of *low density* when $\rho < 1$ and of *high density* when $\rho \geqslant 1$. Note that the choice of the critical value 1 is purely conventional. For low densities we use the same strategy as in the previous section with only small modifications. In the case of high density we rely on the uniform local central limit theorem derived in [20], Theorem 6.1.

Low density. When $\rho \leqslant 1$ the system behaves in many respects like the model with cutoff considered in the previous section. In particular when $\rho \to 0$ we have the same estimates as in (40). We are going to prove the following analogon of Proposition 3.2.

There exists $C < \infty$ and $\delta > 0$ such that for any $N \in \mathbb{N}$ and any $\rho \leqslant 1$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \nu(n)\nu(m)\Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big]^2 \leqslant C\,N^{-2-\delta} \tag{60}$$

As seen in the previous section, this bound immediately implies (SGK) in the low density region $\rho \leqslant 1$.

Let v_ρ denote the characteristic function for the random variable ξ_ρ/σ_ρ, see (41). With the observations above, the estimate (59) and the argument of Lemma 3.3 one checks that there exists $a > 0$ independent of $\rho \leqslant 1$ such that

$$|v_\rho(\zeta)| \leqslant e^{-a\zeta^2}\,, \qquad \zeta \in [-\pi\sigma_\rho, \pi\sigma_\rho]\,. \tag{61}$$

When $N^{-\frac{3}{4}} \leqslant \rho \leqslant 1$ the proof of the proposition goes as follows. We write $Q(n,m)$ as in (45). Expanding as in (47) we have the same estimates as in (48)–(50). The only exception is that (50) now holds in the following sense: for every $T > 0$ there exists $\delta > 0$ such that uniformly in $N^{-\frac{3}{4}} \leqslant \rho \leqslant 1$

$$\sup_{\substack{n,m:\\ n+m\,\leqslant\, T\log N}} \big|\,\mathrm{DEN} - 2\pi\,\big| = O(N^{-\delta})\,. \tag{62}$$

In this way we have obtained

$$\sum_{\substack{n,m:\\ n+m\,\leqslant\, T\log N}} \nu(n)\nu(m)\Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big]^2 = O(N^{-2-\delta})$$

On the other hand, since $\nu(n) \leqslant Cp_\rho(n) \leqslant Ce^{-n/C}$ uniformly in N and $\rho \leqslant 1$, we have

$$\sum_{\substack{n,m:\\ n+m>T\log N}} \nu(n)\nu(m)\Big[Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big]^2 = O(N^{-3})$$

provided T is sufficiently large (but independent of ρ and N). This proves the claim in the regime $N^{-\frac{3}{4}} \leqslant \rho \leqslant 1$.

When $\rho \leqslant N^{-\frac{3}{4}}$ we use exactly the same argument as in (51) and (55) which applies without modifications. This ends the proof of (SGK) in the case $\rho \leqslant 1$.

High density. Here the strategy above has to be modified since the Gaussian bound (61) does not hold anymore and one has to control every estimate uniformly as $\rho \to \infty$. The main tool is the uniform Edgeworth expansion derived in [20]. For $M \in \mathbb{N}$, we define

$$W_{M,\rho}(t) = \mu_{M,\rho}\Big(\sum_{k=1}^{M}(\eta_k - \rho) = -t\Big)\,. \tag{63}$$

In (63) and all expressions below when we write $W_{M,\rho}(t)$ we assume that $\rho M - t$ is a non-negative integer. The following lemma is a straightforward consequence of [20], Theorem 6.1, part (b).

Lemma 4.2. *For any $\kappa < 1/6$ there exists $C < \infty$ such that for all $M \geqslant 1$ and all $\rho \geqslant 1$*

$$\sup_{|t|\,\leqslant\,\sigma_\rho M^\kappa}\Big|\,\sigma_\rho\sqrt{M}W_{M,\rho}(t) - \frac{e^{-\frac{t^2}{2\sigma_\rho^2 M}}}{\sqrt{2\pi}}\Big(1 + \frac{A_\rho t}{\sigma_\rho M} + \frac{B_\rho}{M}\Big)\Big| \leqslant C\,M^{-\frac{3}{2}}\,,$$

where A_ρ and B_ρ are real numbers with $\sup_{\rho\,\geqslant\,1}(|A_\rho| + |B_\rho|) < \infty$.

We can express the kernel $Q(n,m)$ in terms of the probabilities (63):

$$Q(n,m) = \frac{W_{N-2,\rho}(\bar{n}+\bar{m})W_{N,\rho}(0) - W_{N-1,\rho}(\bar{n})W_{N-1,\rho}(\bar{m})}{W_{N-1,\rho}(\bar{n})W_{N-1,\rho}(\bar{m})}\,, \tag{64}$$

where $\bar{n} = n - \rho$, $\bar{m} = m - \rho$. We fix $\kappa = 1/10$ and define the sets

$$\mathcal{T}_{N,\rho} = \{(n,m) \in \mathbb{N}^2 \;:\; |\bar{n}| + |\bar{m}| \leqslant \sigma_\rho N^\kappa\}$$

Let us agree to denote by $\varepsilon(N)$ anything which vanishes at least as $O(N^{-\frac{3}{2}})$ uniformly in the sets $\mathcal{T}_{N,\rho}$, $\rho \geqslant 1$. Thus the result of Lemma 4.2, with $M = N-1$ and $t = \bar{n}$, can be written as

$$\sigma_\rho\sqrt{N-1}W_{N-1,\rho}(\bar{n}) = \frac{e^{-\frac{\bar{n}^2}{2\sigma_\rho^2(N-1)}}}{\sqrt{2\pi}}\Big(1 + \frac{A_\rho\bar{n}}{\sigma_\rho(N-1)} + \frac{B_\rho}{N-1}\Big) + \varepsilon(N) \tag{65}$$

We use now (65) to write

$$\begin{aligned}
&2\pi\sigma_\rho^2(N-1)W_{N-1,\rho}(\bar{n})W_{N-1,\rho}(\bar{m}) \\
&\quad = e^{-\frac{\bar{n}^2+\bar{m}^2}{2\sigma_\rho^2(N-1)}}\Big(1 + \frac{A_\rho\bar{n}}{\sigma_\rho(N-1)} + \frac{B_\rho}{N-1}\Big)\Big(1 + \frac{A_\rho\bar{m}}{\sigma_\rho(N-1)} + \frac{B_\rho}{N-1}\Big) + \varepsilon(N) \\
&\quad = e^{-\frac{\bar{n}^2+\bar{m}^2}{2\sigma_\rho^2 N}}\Big(1 + \frac{A_\rho(\bar{n}+\bar{m})}{\sigma_\rho N} + \frac{2B_\rho}{N}\Big) + \varepsilon(N)\,.
\end{aligned}$$

Furthermore, writing $q(N) = (N-1)/\sqrt{N(N-2)} = 1 + O(N^{-2})$, from Lemma 4.2, with $M = N-2$ and $t = \bar{n} + \bar{m}$, one has

$$
\begin{aligned}
&2\pi\sigma_\rho^2(N-1)W_{N-2,\rho}(\bar{n}+\bar{m})W_{N,\rho}(0)\\
&\quad = q(N)\, e^{-\frac{(\bar{n}+\bar{m})^2}{2\sigma_\rho^2(N-2)}}\Big(1 + \frac{A_\rho(\bar{n}+\bar{m})}{\sigma_\rho(N-2)} + \frac{B_\rho}{N-2}\Big)\Big(1 + \frac{B_\rho}{N}\Big) \,+\, \varepsilon(N)\\
&\quad = e^{-\frac{(\bar{n}+\bar{m})^2}{2\sigma_\rho^2 N}}\Big(1 + \frac{A_\rho(\bar{n}+\bar{m})}{\sigma_\rho N} + \frac{2B_\rho}{N}\Big) \,+\, \varepsilon(N)\\
&\quad = e^{-\frac{\bar{n}^2+\bar{m}^2}{2\sigma_\rho^2 N}}\Big(1 - \frac{\bar{n}\bar{m}}{\sigma_\rho^2 N}\Big)\Big(1 + \frac{A_\rho(\bar{n}+\bar{m})}{\sigma_\rho N} + \frac{2B_\rho}{N}\Big) \,+\, \varepsilon(N)\,.
\end{aligned}
$$

Inserting in (64) we have obtained

$$
\sup_{\rho \geqslant 1}\ \sup_{(n,m)\in\mathcal{T}_{N,\rho}} \Big| Q(n,m) + \frac{\xi_\rho(n)\xi_\rho(m)}{\sigma_\rho^2 N}\Big| = O\big(N^{-\frac{3}{2}}\big) \tag{66}
$$

To conclude the proof of (SGK) in the case $\rho \geqslant 1$ it is therefore sufficient to prove

$$
\sum_{(n,m)\notin\mathcal{T}_{N,\rho}} \nu(n,m)|\varphi(n)|\,|\varphi(m)| \leqslant C\, N^{-\frac{3}{2}}\,\langle\varphi,\varphi\rangle\,, \tag{67}
$$

for any $\varphi \in \mathcal{H}$, uniformly over $\rho \geqslant 1$.

We first claim that for any $k \in \mathbb{N}$ there exists $C_k < \infty$ such that

$$
\nu\big(|\eta_1 - \rho_m| \geqslant T\sigma_\rho \mid \eta_2 = m\big) \leqslant C_k\, T^{-2k} \tag{68}
$$

for any $0 \leqslant m \leqslant \rho N/2$ and any $T > 0$, with $\rho_m = \rho + (\rho - m)/(N-1)$.

To prove (68) recall that there exists $C < \infty$ independent of ρ such that for every $n \in \mathbb{N}$ we have $\nu(n) \leqslant C p_\rho(n)$ (this is a consequence of Lemma 4.2 if $\rho \geqslant 1$, otherwise see (52)). Therefore $\nu(n\,|\,m) \leqslant C p_{\rho_m}(n)$ and

$$
\nu\big(|\eta_1 - \rho_m| \geqslant T\sigma_\rho \mid \eta_2 = m\big) \leqslant C\, m_{2k,\rho_m}\,(T\sigma_\rho)^{-2k}\,,
$$

where $m_{2k,\rho} := \mu_\rho\big[(\eta_1 - \rho)^{2k}\big]$. From [20], Lemma 5.2, we know that

$$
M_k := \sup_{\rho \geqslant 1/2} \frac{m_{2k,\rho}}{\sigma_\rho^{2k}} < \infty\,,
$$

for every $k \in \mathbb{N}$. Since $m \leqslant \rho N/2$ implies $\rho_m \geqslant \rho/2 \geqslant 1/2$, the above yields

$$
\nu\big(|\eta_1 - \rho_m| \geqslant T\sigma_\rho\,|\,\eta_2 = m\big) \leqslant C\, M_k\,(T\sigma_\rho/\sigma_{\rho_m})^{-2k}\,.
$$

Now (68) follows since by (59) we have $\sigma_\rho/\sigma_{\rho_m} \geqslant \delta\sqrt{\rho/\rho_m}$ and, using $m \geqslant 0$, $\rho \geqslant \rho_m(N-1)/N$.

Once (68) is established we may prove (67) as follows. Observe that for any $N \geqslant 3$, $(n,m) \notin \mathcal{T}_{N,\rho}$ implies either $|n - \rho_m| \geqslant T\sigma_\rho$ or $|m - \rho_n| \geqslant T\sigma_\rho$, with $T = N^\kappa/4$. By (68) and the Schwarz' inequality we estimate, uniformly in $\rho \geqslant 1$:

$$\begin{aligned}
&\sum_{\substack{(n,m)\notin \mathcal{T}_{N,\rho}\\ n+m\,\leqslant\,\rho N/2}} \nu(n,m)|\varphi(n)|\,|\varphi(m)| \\
&\qquad \leqslant 2 \sum_{m\,\leqslant\,\rho N/2} \nu(m)|\varphi(m)| \sum_{n:|n-\rho_m|\,\geqslant\,T\sigma_\rho} \nu(n\,|\,m)|\varphi(n)| \\
&\qquad \leqslant \sqrt{C_k}\,T^{-k}\sum_m \nu(m)|\varphi(m)|\Big(\sum_n \nu(n\,|\,m)|\varphi(n)|^2\Big)^{\frac12} \leqslant \sqrt{C_k}\,T^{-k}\,\langle\varphi,\varphi\rangle\,.
\end{aligned}$$

Since $T = N^\kappa/4$ we choose k such that $k\kappa > 3/2$ and (67) is proven under the additional requirement $n + m \leqslant \rho N/2$.

It remains to prove

$$\sum_{\substack{n,m:\\ n+m>\rho N/2}} \nu(n,m)|\varphi(n)|\,|\varphi(m)| \;\leqslant\; N^{-\frac32}\langle\varphi,\varphi\rangle \tag{69}$$

This in turn follows from Schwarz' inequality and the uniform bound

$$\sum_{\substack{n,m:\\ n+m>\rho N/2}} \nu(n\,|\,m)\nu(m\,|\,n) \;\leqslant\; N^{-3}\,. \tag{70}$$

To establish (70) we write $\nu(n\,|\,m)\nu(m\,|\,n) \leqslant C\,p_{\rho_m}(n)p_{\rho_n}(m)$ and use the simple bounds $p_\rho(n) \leqslant e^{-n/C}$ valid for $n \geqslant C\rho$, where C is a sufficiently large constant. Recalling that $\rho \geqslant \rho_m(N-1)/N$, $m \geqslant 0$ this immediately implies (70) and therefore (69). This ends the proof of (SGK) in the high density region $\rho \geqslant 1$.

5. Colored exclusion

In this section we consider a model with different kinds of particles, or particles of different colors, with the constraint that each site is occupied at most by a single particle and the number of particles of each kind is conserved. We set $X = \{0, 1, \dots, R\}$ with some positive integer R. If $\eta_k = 0$ we say that site k is empty while if $\eta_k = m$, $m \in \{1, \dots, R\}$ we think of site k as being occupied by a particle with color m. The conservation laws are expressed in terms of the functions

$$\xi^m(\eta_k) = \mathbf{1}_{\{m\}}(\eta_k) = \begin{cases} 1 & \eta_k = m \\ 0 & \eta_k \neq m \end{cases} \tag{71}$$

so that the multicanonical measure $\nu_{N,\bar\rho}$ in (28) is obtained by conditioning on the event

$$\Theta_{N,\bar\rho} = \{\eta \in \Omega_N \;:\; \sum_{k=1}^N \xi^m(\eta_k) = \rho_m N\,,\; m = 1, \dots, R\}\,,$$

with $\bar\rho = (\rho_1, \dots, \rho_R)$ an assigned density vector with $\sum_{m=1}^R \rho_m \leqslant 1$. We say that $\bar\rho$ is trivial if $\rho_m \in \{0, 1\}$ for every $m \in \{1, \dots, R\}$. The dynamics is given by random transpositions so that the Dirichlet form is

$$\mathcal{E}_{N,\bar\rho}(f) = \frac{1}{N}\sum_{k=1}^N\sum_{\ell=1}^N \nu_{N,\bar\rho}\big[\big(f\circ T_{k,\ell} - f\big)^2\big] \tag{72}$$

where $f \in L^2(\nu_{N,\bar\rho})$ and

$$\big(T_{k,\ell}\eta\big)_j = \begin{cases} \eta_k & \text{if } j=\ell \\ \eta_\ell & \text{if } j=k \\ \eta_j & \text{otherwise}. \end{cases}$$

This and related random transposition or card–shuffling models have been studied in great detail by Diaconis and Shashahani [11] with more elaborate techniques. The result we prove below is rather simple but it illustrates well the use of the general arguments outlined in section 2. Note that when $R=1$ we have the usual exclusion process on the complete graph, sometimes called the Bernoulli–Laplace model. When $R=2$ the model was studied by Quastel, [22].

Let $\gamma(N,\bar\rho)$ be the Poincaré constant associated to the couple $(N,\bar\rho)$, as in (4). Note that $\gamma(N,\bar\rho)=0$ when $\bar\rho$ is trivial. Let $\rho^*=\rho^*(N)$ be the density vector corresponding to one particle only: $\rho^*_1=1/N$ and $\rho^*_m=0$, $m=2,\dots,R$. When $\bar\rho=\rho^*$ we have a (rate 2) random walk on the complete graph and a direct computation shows that $\mathcal{E}_{N,\rho^*}(f)=4\mathrm{Var}_{N,\rho^*}(f)$ for every f. Therefore $\gamma(N,\rho^*)=1/4$.

Theorem 5.1. *For any $R\in\mathbb{Z}_+$, $N\geqslant 2$ and any density $\bar\rho$:*

$$\gamma(N,\bar\rho)\leqslant\gamma(N,\rho^*)=\frac{1}{4} \tag{73}$$

Proof. We shall use the notation (34). We write $\rho_0=1-\sum_{m=1}^R\rho_m$ and $\xi^m_{\rho_m}(n)=\mathbf{1}_{\{m\}}(n)-\rho_m$ for every $m=0,1,\dots,R$. Notice that by symmetry we have

$$\nu(m)=\rho_m\,,\qquad m=0,1,\dots,R\,.$$

Therefore

$$\nu(m\,|\,n)=\frac{\nu(m,n)}{\nu(n)}=\frac{\rho_mN-\xi^m(n)}{N-1}=\rho_m-\frac{\xi^m_{\rho_m}(n)}{N-1}\,. \tag{74}$$

Take $\varphi\in\mathcal{H}_0$ and write

$$\begin{aligned}\langle\varphi,\mathcal{K}\varphi\rangle&=\sum_{n=0}^R\sum_{m=0}^R\nu(n)\big[\nu(m\,|\,n)-\nu(m)\big]\varphi(n)\varphi(m)\\ &=-\frac{1}{N-1}\sum_{m=0}^R\varphi(m)\langle\xi^m_{\rho_m},\varphi\rangle \end{aligned}\tag{75}$$

From this we see that whenever $\varphi\in\mathcal{H}_0$ is orthogonal to all $\xi^m_{\rho_m}$ then $\langle\varphi,\mathcal{K}\varphi\rangle=0$. From the analysis in Lemma 2.2 it follows that

$$\nu_{N,\bar\rho}\big(f(1-\mathcal{P})f\big)\geqslant\frac{N-2}{N-1}\nu_{N,\bar\rho}\big(f^2\big) \tag{76}$$

for every $f\in L^2(\nu_{N,\bar\rho})$ with $\nu_{N,\bar\rho}(f)=0$ and any $N\geqslant 3$. Thus if $\gamma(N)$ denotes supremum of $\gamma(N,\bar\rho)$ over all possible values of $\bar\rho$, the argument of Proposition 2.1 gives $\gamma(N)\leqslant\gamma(2)$ for every $N\geqslant 3$. The theorem then follows since $\gamma(2)=\gamma(2,\rho^*)=1/4$. ☐

6. Anisotropic exclusion processes

Here we review recent results obtained in collaboration with F. Martinelli, [5, 6]. The model can be described in the general framework of section 2. We set $X=\{0,1\}^H$ where H

is a positive integer to be interpreted as the height of the system. The measure μ is itself a product of Bernoulli measures

$$\mu = \otimes_{h=1}^{H} \mu_h\,, \qquad \mu_h = \mathrm{Be}(p_h)$$
$$p_h := \frac{q^{2h}}{1+q^{2h}}\,, \qquad q \in (0,1)\,. \tag{77}$$

Then $\Omega_N = X^N$ and a configuration $\eta = \{\eta_i\}_{i=1}^N$ is given in terms of its components $\eta_i = \{\alpha_{(i,h)}\}_{h=1}^H$, with $\alpha_{i,h} \in \{0,1\}$ interpreted as the presence or absence of a particle at site (i,h). The conservation law is given by

$$\xi(\eta_i) = \sum_{h=1}^{H} \alpha_{(i,h)}.$$

The canonical measures $\nu_{N,\rho} = \nu_{H,N,\rho}$ are defined as usual by (2) for every fixed value of H. We may think of identical particles placed at the sites of a two–dimensional cylindrical region $\Lambda = \{1,\dots,N\} \times \{1,\dots,H\}$. Each site can be occupied by at most one particle and the total number of particles is fixed. Since $q<1$ there is anisotropy in the vertical axis and particles prefer to be at the bottom of Λ. The choice of the model (77) is motivated by interesting connections with anisotropic quantum spin chains, see [1, 5, 6, 17, 18] and references therein.

The dynamics can be described as follows. At each site of Λ we have an independent rate 1 Poisson clock. Suppose site (i,h) rings. If $h=H$ we do nothing. If $h<H$ we choose at random one of the sites $(j,h+1)$, $j=1,\dots,N$. The occupation variables $\alpha_{(i,h)}$ and $\alpha_{(j,h+1)}$ are then exchanged with rate

$$c_{(i,h);(j,h+1)}(\alpha) = q^{\alpha_{(i,h)} - \alpha_{(j,h+1)}}\,. \tag{78}$$

That is if a particle is moving upwards the rate is q whereas if it is moving downwards the rate is q^{-1}. We thus obtain a process described by the Dirichlet form (3) with the exchange operators, for $i \neq j$

$$v_{i,j} f(\alpha) = \left(\frac{1}{2} \sum_{h=1}^{H-1} c_{(i,h);(j,h+1)}(\alpha) \left[f(\alpha^{(i,h);(j,h+1)}) - f(\alpha) \right]^2 \right)^{\frac{1}{2}}\,, \tag{79}$$

and $v_{i,i} f = 0$, where we write $\alpha^{(i,h);(j,h+1)}$ for the configuration in which the values of α at (i,h) and $(j,h+1)$ have been exchanged. Notice that the process is local in the vertical direction while it is nonlocal in the horizontal direction. One of the main results of [6] is that for every $q \in (0,1)$ the relaxation time is bounded, uniformly in H, in N and in the number of particles.

Let us recall the definition of the Poincaré constant (4). In order to keep track of the dependence on H we write here $\gamma(H,N,\rho)$ instead of $\gamma(N,\rho)$.

Theorem 6.1. *For every $q \in (0,1)$ there exists $C < \infty$ such that*

$$\sup_{N \geqslant 2} \sup_{H \geqslant 2} \sup_{\rho} \gamma(H,N,\rho) \leqslant C \tag{80}$$

The proof of Theorem 6.1 has been obtained by applying the arguments of Proposition 2.1 and Lemma 2.2. The crucial step in the proof of property (SGK) is a result analogous to Proposition 3.2. We refer to [6] for more details.

Some of the applications of Theorem 6.1, especially those to quantum Heisenberg models, are linked to the restriction of the process to horizontal sums of the basic variables $\alpha_{i,h}$ given by

$$\omega_h = \sum_{i=1}^{N} \alpha_{(i,h)}\,, \qquad h = 1, \dots, H$$

In view of the symmetries of the Dirichlet form $\mathcal{E}_{N,\rho}$ defined by (79) it is not hard to see that the restriction to the variables $\{\omega_h\}$ is again a Markov process. Indeed, the latter can be described as follows. Assign to each row $h = 1, \dots, H-1$ two independent exponentially distributed times (with mean 1), τ_-^h and τ_+^h. When τ_+^h rings the configuration ω is updated with rate $r_{+,h}(\omega) := q^{-1}(N-\omega_h)\omega_{h+1}/N$ to the configuration $\omega^{+,h}$ in which ω_h is increased by 1 and ω_{h+1} is decreased by 1 (while the rest is unchanged). When τ_-^h rings we do the reverse transition ($\omega \to \omega^{-,h}$: ω_h is decreased and ω_{h+1} increased) with rate $r_{-,h}(\omega) := q(N-\omega_{h+1})\omega_h/N$. We can write the Dirichlet form of this process as

$$\frac{1}{2}\sum_{h=1}^{H-1} \widetilde{\nu}\left(r_{+,h}(\omega)\left[f(\omega^{+,h}) - f(\omega)\right]^2 + r_{-,h}(\omega)\left[f(\omega^{-,h}) - f(\omega)\right]^2\right) \tag{81}$$

where $\widetilde{\nu}$ stands for the marginal of $\nu_{H,N,\rho}$ on the variables ω. A simple computation gives the probability $\widetilde{\nu}(\omega)$ of a single ω compatible with the global constraint $\sum_h \omega_h = \rho N$:

$$\widetilde{\nu}(\omega) = \frac{1}{\widetilde{Z}} \prod_{h=1}^{H} \binom{N}{\omega_h} q^{2h\omega_h}\,. \tag{82}$$

The process (81) can be interpreted as describing relaxation of a non-negative profile $\{\omega_h\}_{h=1}^H$ subject to a fixed area constraint. In view of the anisotropy the profile is strongly localized under the measure $\widetilde{\nu}$, i.e. $\omega_h \approx N$ for heights h below ρ and $\omega_h \approx 0$ above ρ with high probability. By Theorem 6.1 relaxation to equilibrium in $L^2(\widetilde{\nu})$ is exponentially fast uniformly in ρ.

In the case $N = 2$ the process (81) admits another interesting interpretation as a model for diffusion limited chemical reactions, see [1] and references therein. Namely describe the state $\omega_h = 2$ as the presence at h of a particle of type A, $\omega_h = 0$ as a particle of type B and $\omega_h = 1$ as the absence of particles. If n_A, n_B denote the size of the two populations we see that the difference $n_A - n_B$ is conserved and we have a model for asymmetric diffusion with creation and annihilation of the two species. Particles of type A have a constant drift towards the bottom while particles of type B have the same drift towards the top. Nearest neighbour pairs can produce the reaction $A + B \to$ *inert* and the reverse reaction *inert* $\to A + B$ with the appropriate rates. While Theorem 6.1 implies immediately a uniform lower bound on the spectral gap for this process, a direct proof of the result for the two-particle model seemed difficult to us.

7. Ginzburg-Landau processes

Here we discuss a recent result ([4]) for the Ginzburg–Landau process. The model is obtained from the general setting in section 2 with $X = \mathbb{R}$ and $\xi(\eta_k) = \eta_k$. The single site probability distribution is of the form

$$\mu(\mathrm{d}\eta) = \frac{e^{-V(\eta)}}{Z}\,\mathrm{d}\eta\,, \tag{83}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a given function with $Z = \int e^{-V(\eta)}\mathrm{d}\eta < \infty$. Precise assumptions on V are specified below. The resulting canonical measure $\nu_{N,\rho}$ on the hyperplane $\sum_{k=1}^N \eta_k = \rho N$

is given by (2), for all $\rho \in \mathbb{R}$. We consider the process defined by the symmetric Dirichlet form $\mathcal{E}_{N,\rho}$ given in (3) with the choice

$$v_{k,\ell} f = \partial_k f - \partial_\ell f\,, \tag{84}$$

where $\partial_k f$ is the partial derivative of f along the k-th coordinate η_k. This yields an ergodic diffusion process on every ρ– hyperplane with reversible invariant measure $\nu_{N,\rho}$ given by (2). In the definition (4) of the Poincaré constant $\gamma(N,\rho)$ the supremum is taken over all smooth functions $f : \mathbb{R}^N \to \mathbb{R}$.

The main result of [4] says that a uniform Poincaré inequality holds whenever V is of the form $V = \varphi + \psi$ with ψ a smooth bounded function and φ a strictly convex function satisfying some mild growth condition at infinity. To describe the latter we define the class Φ of functions $\varphi \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ with second derivative φ'' satisfying

- *Strict convexity:* There exists $\delta > 0$ such that $\varphi'' \geqslant \delta$.
- *Polynomial growth at infinity:* There exist constants $\beta_-, \beta_+ \in [0,\infty)$ and a constant $C \in [1,\infty)$ such that

$$\frac{1}{C} \leqslant \liminf_{x\to\infty} \frac{\varphi''(\pm x)}{x^{\beta_\pm}} \leqslant \limsup_{x\to\infty} \frac{\varphi''(\pm x)}{x^{\beta_\pm}} \leqslant C\,. \tag{85}$$

Clearly, any strictly convex polynomial belongs to Φ. The perturbation will be taken from the class Ψ of functions $\psi \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ such that $|\psi|_\infty < \infty$, $|\psi'|_\infty < \infty$ and $|\psi''|_\infty < \infty$.

Theorem 7.1. *Assume V is of the form $V = \varphi + \psi$ with $\varphi \in \Phi$ and $\psi \in \Psi$. Then*

$$\sup_{N\in\mathbb{N}} \sup_{\rho\in\mathbb{R}} \gamma(N,\rho) \;<\; \infty\,. \tag{86}$$

An immediate corollary of Theorem 7.1 is the uniform diffusive bound for the local dynamics (29). This follows from property (MP) and (30)–(32). Diffusive bounds for the spectral gap of Ginzburg–Landau processes are a key ingredient in the proof of hydrodynamic limits for the nongradient system considered by Varadhan [24]. When there is no perturbation ($\psi = 0$), Theorem 7.1 (without the additional requirement (85)) becomes an immediate consequence of the Brascamp–Lieb inequality [2], see [4]. Since perturbative arguments are very sensitive to the increasing number of dimensions, the case of nonconvex potentials is much more involved. Recently the uniform diffusive estimate has been obtained by Landim, Panizo, Yau [19] in the case $V(x) = ax^2 + \psi(x)$, $a > 0$ and ψ bounded. The results of [19] have been later generalized slightly by Chafai [9] . The proofs of both [19] and [9] are based on the martingale approach ([21]) and the method is sufficiently robust to yield the stronger logarithmic Sobolev inequality. These techniques seem to fail however in the case of non quadratic potentials - thus ruling out natural problems such as quartic potentials.

The proof of Theorem 7.1 is based on the general strategy outlined in Proposition 2.1 and Lemma 2.2. The delicate part of the work is to establish the bound required in condition (SGK). Formally the situation is similar to that encountered in previous sections, but here $\mathcal{K}$ is an integral operator and the technique has to be modified slightly. Moreover, contrary to the case of zero range processes discussed in section 4, here the variance σ_ρ^2 of the grand–canonical measures

$$\mu_\rho(\mathrm{d}x) = \frac{e^{-V(x)-\lambda_\rho x}}{Z_\rho}\,\mathrm{d}x \tag{87}$$

vanishes as $\rho \to \pm\infty$ as soon as φ'' is unbounded. In the above formula λ_ρ is determined as usual by the condition that $(Z_\rho)^{-1}\int x e^{-V(x)-\lambda_\rho x}\mathrm{d}x = \rho$. The technical hypothesis (85) is mainly used to control the speed of decay of σ_ρ^2. Using a uniform local central limit theorem for the measures (87) we prove in [4], Theorem 3.1, that there exists $C < \infty$ independent of ρ and N such that for every $f \in \mathcal{H}_0$ satisfying $\langle f, \xi_\rho \rangle = 0$ one has

$$\left|\langle f, \mathcal{K} f \rangle\right| \leqslant C\, N^{-\frac{3}{2}}\, \langle f, f \rangle\,. \tag{88}$$

8. Exclusion with site–disorder

Here we consider the following non–homogeneous model. The single state space is $X = \{0,1\}$ and the conservation law is $\xi(\eta_k) = \eta_k$, interpreted as the presence or absence of a particle at k. In contrast to previous models here the measure μ is site–dependent. We choose for every $k \in \{1, \dots, N\}$, $N \in \mathbb{N}$, the Bernoulli measures $\mu_k = \mathrm{Be}(\omega_k)$:

$$\mu_k(\eta_k = 1) = \omega_k\,, \qquad \omega_k \in [\delta, 1-\delta]\,, \quad k = 1, \dots, N \tag{89}$$

Here $\delta \in (0, 1/2]$ is fixed and $\omega \in [\delta, 1-\delta]^N$ can be interpreted as a realization of a random field, as in [12, 23]. However, we shall not use any probabilistic structure behind the variables ω and our results will all be uniform in $\omega \in [\delta, 1-\delta]^N$. For every such ω, every $\rho \in [0,1]$, we define the (quenched) canonical measure

$$\nu_{N,\rho} = \bigotimes_{k=1}^{N} \mu_k \Big(\,\cdot\, \mid \sum_{\ell=1}^{N} \eta_\ell = \rho N \Big)\,. \tag{90}$$

The Dirichlet form of the complete graph dynamics is written as in (3) with the choice

$$v_{k,\ell} f = \sqrt{c_{k,\ell}(\eta)}\, \left[f(\eta^{k,\ell}) - f(\eta) \right],$$

where as usual $\eta^{k,\ell}$ denotes the configuration where η_k and η_ℓ have been exchanged, and $c_{k,\ell}$ denotes the associated transition rate. A possible choice of the rates is e.g.

$$c_{k,\ell}(\eta) = \begin{cases} \omega_k(1-\omega_\ell) & (\eta_k, \eta_\ell) = (0,1) \\ \omega_\ell(1-\omega_k) & (\eta_k, \eta_\ell) = (1,0) \end{cases}$$

The result below applies to any choice of rates provided these are uniformly bounded from above and away from zero.

For every fixed ω we call $\gamma^\omega(N, \rho)$ the corresponding Poincaré constant as in (4).

Theorem 8.1. *For every $\delta \in (0, 1/2]$ there exists $C < \infty$ such that*

$$\sup_{N \geqslant 2}\ \sup_{\omega \in [\delta, 1-\delta]^N}\ \sup_{\rho \in (0,1)} \gamma^\omega(N, \rho) \leqslant C\,. \tag{91}$$

Theorem 8.1 is a useful tool in the proof of hydrodynamic limit for the site–disordered simple exclusion process, [12, 23]. One can check that the model described above satisfies the moving particle lemma (MP) of section 2. A little care is required here because of the inhomogeneous medium. We refer to Lemma 3.1 in [23] for details. Thus an immediate corollary of Theorem 8.1 is the diffusive bound on the spectral gap of the local dynamics, see (30)–(32).

8.1. **Proof of Theorem 8.1.** We use the iteration outlined in Proposition 2.1. From a comparison with the homogeneous case $\omega_k \equiv \mathrm{const.}$ we see that $\sup_\omega \sup_\rho \gamma^\omega(N, \rho) \leqslant C^N$ for some $C < \infty$. This guarantees that the first hypothesis of the proposition is satisfied.

If $\mathcal{P}$ denotes the operator introduced in (5) we need to show that (SGP) holds, i.e. that for every $f \in L^2(\nu_{N,\rho})$ with $\nu_{N,\rho}(f) = 0$

$$\nu_{N,\rho}(f(1-\mathcal{P})f) \geqslant \frac{N-2}{N-1}\left[1 - C N^{-1-\epsilon}\right] \nu_{N,\rho}(f^2) \tag{92}$$

with independent constants $\epsilon > 0$, $C < \infty$. As seen in section 2 (see the proof of Lemma 2.2) it is sufficient to prove (92) for functions f of the form $f(\eta) = \sum_{k=1}^N g_k(\eta_k)$ with $g_k : X \to \mathbb{R}$ a mean–zero function. Since here $X = \{0,1\}$, we must have $g_k = \alpha_k(\eta_k - \rho_k)$, $\rho_k := \nu_{N,\rho}(\eta_k)$, for some $\alpha_k \in \mathbb{R}$. That is, we shall prove (92) for functions of the form

$$f(\eta) = \sum_{k=1}^N \alpha_k \bar\eta_k\,, \qquad \alpha \in \mathbb{R}^N \tag{93}$$

with $\bar\eta_k := \eta_k - \rho_k$. We take f as in (93) and compute

$$\nu_{N,\rho}\big(f^2\big) = \sum_{k,\ell} \alpha_k\alpha_\ell \nu_{N,\rho}(\bar\eta_k\bar\eta_\ell) = \langle \tilde\alpha, Q\tilde\alpha\rangle \tag{94}$$

where we use the notation

$$Q_{k,\ell} := \frac{\nu_{N,\rho}(\bar\eta_k\bar\eta_\ell)}{\gamma_k\gamma_\ell}\,, \qquad \tilde\alpha_k := \gamma_k\alpha_k\,, \qquad \gamma_k^2 := \nu_{N,\rho}(\bar\eta_k^2) = \rho_k(1-\rho_k)$$

and $\langle v, w\rangle := \sum_{k=1}^N v_k w_k$ for the scalar product in $\mathbb{R}^N$. Observing that

$$\nu_{N,\rho}(\bar\eta_k \,|\, \eta_j) = \frac{\nu_{N,\rho}(\bar\eta_k\bar\eta_j)}{\gamma_j^2}\,\bar\eta_j$$

one obtains in a similar way

$$\nu_{N,\rho}\big(f\mathcal{P}f\big) = \frac{1}{N}\langle\tilde\alpha, Q^2\tilde\alpha\rangle\,.$$

Q is a non–negative matrix. Setting $\hat\alpha := Q^{\frac12}\tilde\alpha$ we have

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) = \langle\hat\alpha, (1-\frac{Q}{N})\hat\alpha\rangle\,, \qquad \nu_{N,\rho}\big(f^2\big) = \langle\hat\alpha,\hat\alpha\rangle\,. \tag{95}$$

We write now $\Gamma := 1 - Q$, so that

$$\Gamma_{k,\ell} = \begin{cases} -\frac{\nu_{N,\rho}(\bar\eta_k\bar\eta_\ell)}{\gamma_k\gamma_\ell} & k \neq \ell \\ 0 & k = \ell \end{cases}$$

Then (95) reads

$$\nu_{N,\rho}\big(f(1-\mathcal{P})f\big) = \frac{N-1}{N}\,\langle\hat\alpha, (1+\frac{\Gamma}{N-1})\hat\alpha\rangle\,.$$

By (95), the claim (92) follows if we can prove

$$\Gamma \geqslant -CN^{-\epsilon}\,. \tag{96}$$

This in turn will follow from the next lemma.

Lemma 8.2. *There exists $C < \infty$, $\epsilon > 0$ such that for all ω, N, ρ and $k \neq \ell$*

$$\Big|\Gamma_{k,\ell} - \frac{\beta_k\beta_\ell}{N}\Big| \leqslant CN^{-1-\epsilon} \tag{97}$$

with non–negative numbers $\beta_k = \beta_k(\omega, N, \rho)$, $k = 1, \dots, N$ satisfying $\beta_k \leqslant C$ uniformly.

Assuming (97) we conclude

$$\begin{aligned}\langle v, \Gamma v\rangle &= \sum_k\sum_{\ell\neq k} v_k v_\ell\Big(\frac{\beta_k\beta_\ell}{N} + O(N^{-1-\epsilon})\Big) \\ &\geqslant -\frac{1}{N}\sum_k \beta_k^2 v_k^2 - CN^{-\epsilon}\sum_k v_k^2 \geqslant -C'N^{-\epsilon}\langle v, v\rangle\end{aligned}$$

with a constant $C' < \infty$. This gives (96).

8.2. **Proof of Lemma 8.2.** We start with some preliminaries. Let $p_{k,\rho}$ be the grand–canonical single site probabilities

$$p_{k,\rho} := \frac{\omega_k e^{\lambda}}{\omega_k e^{\lambda} + 1 - \omega_k}$$

where $\lambda = \lambda^{\omega}_{N,\rho}$ is a real number such that $\sum_{k=1}^N p_{k,\rho} = \rho N$. We set $\mu_{k,\rho} := \mathrm{Be}(p_{k,\rho})$ and call $\mu_{N,\rho} = \otimes_{k=1}^N \mu_{k,\rho}$ the corresponding grand–canonical measure. We also use the notations

$$\hat\eta_k = \eta_k - p_{k,\rho}\,, \quad \sigma^2_{k,\rho} = p_{k,\rho}(1-p_{k,\rho})\,, \quad \sigma^2_\rho = \frac{1}{N}\sum_{k=1}^N \sigma^2_{k,\rho}$$

Since $\omega_k \in [\delta, 1-\delta]$ it is immediate to check that there exists $C = C(\delta) < \infty$ such that $p_{k,\rho} \leqslant C p_{\ell,\rho}$ for all k, ℓ and ρ. In particular for some $C = C(\delta) < \infty$ one has

$$C^{-1}\rho \leqslant p_{k,\rho} \leqslant C\rho\,, \quad C^{-1}\rho(1-\rho) \leqslant \sigma^2_{k,\rho} \leqslant C\rho(1-\rho) \tag{98}$$

Given $k, \ell \in \{1, \dots, N\}$ consider the events

$$U_1 = \{\eta : \sum_{j\neq k,\ell} \eta_j = \rho N - 1\}\,, \quad U_2 = \{\eta : \sum_{j\neq k,\ell} \eta_j = \rho N - 2\}\,.$$

A simple computation shows that

$$\frac{\rho_k}{\rho_\ell} = \frac{\omega_k\big((1-\omega_\ell)\mu_{N,\rho}(U_1) + \omega_\ell e^{\lambda}\mu_{N,\rho}(U_2)\big)}{\omega_\ell\big((1-\omega_k)\mu_{N,\rho}(U_1) + \omega_k e^{\lambda}\mu_{N,\rho}(U_2)\big)} \tag{99}$$

From the bounds on ω we deduce that there exists $C = C(\delta) < \infty$ such that $\rho_k \leqslant C\rho_\ell$ and similarly

$$C^{-1}\rho \leqslant \rho_k \leqslant C\rho\,, \quad C^{-1}\rho(1-\rho) \leqslant \gamma_k^2 \leqslant C\rho(1-\rho)\,. \tag{100}$$

We turn to the proof of the lemma. By duality we may assume $\rho \leqslant 1/2$. We start with the case $1/2 \geqslant \rho \geqslant N^{-3/4}$. Let $\tilde v_{k,\rho}(\zeta)$ denote the characteristic function

$$\tilde v_{k,\rho}(\zeta) = \mu_{k,\rho}\Big[\exp\big(i\frac{\zeta\hat\eta_k}{\sigma_\rho\sqrt{N}}\big)\Big]\,.$$

Since by (98) $\sigma^2_{k,\rho} \geqslant C^{-1}\sigma^2_\rho$, the argument of Lemma 3.3 implies the Gaussian bound

$$|\tilde v_{k,\rho}(\zeta)| \leqslant e^{-a\zeta^2/N}\,, \quad |\zeta| \leqslant \pi\sigma_\rho\sqrt{N} \tag{101}$$

with some $a > 0$ only depending on δ. Using Fourier transform we see that

$$\nu_{N,\rho}(\bar\eta_k\bar\eta_\ell) = \frac{p_{k,\rho}p_{\ell,\rho}\,A_{k,\ell}}{B^2}$$

with

$$B := \int \mathrm{d}\zeta \prod_{k=1}^N \tilde v_{k,\rho}(\zeta)$$

and

$$A_{k,\ell} := \int \mathrm{d}\zeta\, e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}(2-p_{k,\rho}-p_{\ell,\rho})} \prod_{j\neq k,\ell} \tilde{v}_{j,\rho}(\zeta) \int \mathrm{d}\zeta' \prod_j \tilde{v}_{j,\rho}(\zeta')$$
$$- \int \mathrm{d}\zeta\, e^{i\frac{\zeta}{\sigma_\rho\sqrt{N}}(1-p_{k,\rho})} \prod_{j\neq k} \tilde{v}_{j,\rho}(\zeta) \int \mathrm{d}\zeta'\, e^{i\frac{\zeta'}{\sigma_\rho\sqrt{N}}(1-p_{\ell,\rho})} \prod_{j\neq \ell} \tilde{v}_{j,\rho}(\zeta')\,. \tag{102}$$

Here all integrals are in the range $[-\pi\sigma_\rho\sqrt{N}, \pi\sigma_\rho\sqrt{N}]$. Using the hypothesis $\rho \geqslant N^{-3/4}$, the bounds (98) and the computation of section 3, see (47), we have

$$\tilde{v}_{j,\rho}(\zeta) = 1 - \frac{\zeta^2\sigma_{j,\rho}^2}{2\sigma_\rho^2 N} + O\big(N^{-1-\epsilon}\big)\,, \qquad |\zeta| \leqslant C\log N\,.$$

Since $\sigma_\rho^2 = (\sum_k \sigma_{k,\rho}^2)/N$ we have

$$\prod_j \tilde{v}_{j,\rho}(\zeta) = 1 - \frac{\zeta^2}{2} + O\big(N^{-\epsilon}\big)\,, \quad |\zeta| \leqslant C\log N\,.$$

As in (50) we deduce

$$B = \sqrt{2\pi} + O(N^{-\epsilon})$$

Moreover

$$\prod_{j\neq k,\ell} \tilde{v}_{j,\rho}(\zeta) = \Big(1 + \frac{\zeta^2(\sigma_{k,\rho}^2 + \sigma_{\ell,\rho}^2)}{2\sigma_\rho^2 N} + O\big(N^{-1-\epsilon}\big)\Big) \prod_j \tilde{v}_{j,\rho}(\zeta)\,,$$
$$\prod_{j\neq k} \tilde{v}_{j,\rho}(\zeta) = \Big(1 + \frac{\zeta^2\sigma_{k,\rho}^2}{2\sigma_\rho^2 N} + O\big(N^{-1-\epsilon}\big)\Big) \prod_j \tilde{v}_{j,\rho}(\zeta)\,, \qquad |\zeta| \leqslant C\log N\,.$$

If we plug these expansions in (102) and open all the brackets as in the derivation of (48) we obtain the estimate

$$p_{k,\rho}p_{\ell,\rho}A_{k,\ell} = -2\pi\frac{\sigma_{k,\rho}^2\sigma_{\ell,\rho}^2}{\sigma_\rho^2 N} + O(\sigma_\rho^2 N^{-1-\epsilon})$$

uniformly in the case $\rho \geqslant N^{-3/4}$ ($\sigma_\rho^2 \geqslant C^{-1}N^{-3/4}$). Using $\gamma_k = O(\sigma_{k,\rho}) = O(\sigma_\rho)$ the Lemma follows with $\beta_k = \sigma_{k,\rho}^2/(\gamma_k\sigma_\rho) = O(1)$ by (98) and (100). This proves (8.2) in the case $\rho \geqslant N^{-3/4}$.

We now prove the lemma for densities $\rho \leqslant N^{-3/4}$. We set $\hat{\omega}_k := \omega_k/(1-\omega_k)$ and rewrite (99) as

$$\frac{\rho_k}{\rho_\ell} = \frac{\hat{\omega}_k}{\hat{\omega}_\ell}\,\frac{1+e^\lambda\hat{\omega}_\ell W}{1+e^\lambda\hat{\omega}_k W}\,, \qquad W := \frac{\mu_{N,\rho}(U_2)}{\mu_{N,\rho}(U_1)}\,. \tag{103}$$

When $\rho N = 1$ we have $W = 0$ and $\rho_k/\rho_\ell = \hat{\omega}_k/\hat{\omega}_\ell$. Suppose $\rho N \geqslant 2$. Define the event

$$V^m = \{\eta : \sum_{j\neq k,\ell,m} \eta_j = \rho N - 2\}\,.$$

Then

$$\mu_{N,\rho}(U_1) = \frac{1}{\rho N - 1} \sum_{m \neq k,\ell} p_{m,\rho} \mu_{N,\rho}(V^m)\,,$$
$$\mu_{N,\rho}(U_2) = \frac{1}{N(1-\rho)} \sum_{m \neq k,\ell} (1 - p_{m,\rho}) \mu_{N,\rho}(V^m)\,.$$

Since $p_{m,\rho} \geqslant C^{-1}\rho$ we see that $W = \mu_{N,\rho}(U_2)/\mu_{N,\rho}(U_1) \leqslant C$ uniformly. Using $e^\lambda \leqslant C p_{k,\rho} \leqslant C'\rho$, from (103) we have

$$\frac{\rho_k}{\rho_\ell} = \frac{\hat\omega_k}{\hat\omega_\ell}\left[1 + O(\rho)\right]. \tag{104}$$

Summing over k in (104) we arrive at the estimate

$$\rho_\ell = \frac{\rho N \hat\omega_\ell}{\sum_k \hat\omega_k} + O(\rho^2)\,. \tag{105}$$

Set now $\rho_\ell^{(j)} := \nu_{N,\rho}(\eta_\ell \,|\, \eta_j = 1)$. From (105) applied to $N-1$ sites with $\rho N - 1$ particles:

$$\rho_\ell^{(j)} = \frac{(\rho N - 1)\hat\omega_\ell}{\sum_{k \neq j} \hat\omega_k} + O(\rho^2) = \rho_\ell - \frac{\hat\omega_\ell}{\sum_k \hat\omega_k} + O(\rho^2)\,.$$

Since $\nu_{N,\rho}(\bar\eta_k \bar\eta_\ell) = -\rho_k(\rho_\ell - \rho_\ell^{(k)})$, $\Gamma_{k,\ell}$ can be written as

$$\Gamma_{k,\ell} = \frac{\rho_k \hat\omega_\ell}{\gamma_k \gamma_\ell \sum_j \hat\omega_j} + O(\rho^2) = \frac{\rho N \hat\omega_k \hat\omega_\ell}{\gamma_k \gamma_\ell (\sum_j \hat\omega_j)^2} + O(\rho^2)\,.$$

Since $\rho^2 \leqslant N^{-3/2}$, (97) follows with $\beta_k := N\hat\omega_k \sqrt{\rho}/\gamma_k(\sum_j \hat\omega_j)$. Then $\beta_k = O(1)$ by (100). This completes the proof of Lemma 8.2.

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