Harmonic Crystals: Statistical Mechanics and Large Deviations

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To Valeria

Abstract

This work is concerned with the probabilistic analysis of large fluctuations of an harmonic interface on the integer lattice. We study the problem in the frame of the Gibbsian formalism of classical statistical mechanics. The model is explicitly expressed in terms of massless Gaussian random fields. The quadratic interaction is chosen in such a way that the corresponding covariances are given by the Green function of a lattice random walk. We put special emphasis on the non local model obtained by considering long-range random walks, since traditional techniques often fail in this case in view of the loss of quasilocality.

In the first part of the thesis we consider the general model obtained from an arbitrary symmetric, homogeneous and transient random walk. We first reconstruct the classical thermodynamical picture, proving the standard identity connecting the relative entropy (or information-gain) functional to the free energy functional. We also establish a version of Gibbs' variational principle, showing that translation invariant Gibbs measures are characterized as minimizers of free energy. We then study the large fluctuations of the interface in the frame of the Donsker-Varadhan theory of large deviations. We show that the empirical field of a Gibbs measure satisfies a weak large deviation principle at the volume order, with rate function given by the relative entropy functional.

In the second part we investigate second order fluctuations, socalled critical large deviations. In view of Gibbs' variational principle these are the macroscopic fluctuations of the interface within the class of translation invariant Gibbs measures. We consider here the long-range model obtained from random walks which are in the domain of attraction of a symmetric stable law. The aim is to generalize the work of Bolthausen and Deuschel [5], who gave a detailed account of the large deviations of the local model obtained from short-range random walks. We find that the empirical field satisfies a strong large deviation principle at the capacity order, and we obtain an expression for the rate function in terms of the Dirichlet form of the limiting symmetric stable process. This is the analogue of the results of [5], where the rate function was given in terms of the Dirichlet form of the Brownian motion. As in [5], the result is achieved through the analysis of large deviations of the empirical profile of the interface. We also discuss fluctuations of the field under the constraint of zero boundary conditions (that is, the interface is tied down at the boundary of a box). In this case we prove a large deviation principle at capacity order, with rate function given by the Dirichlet form of the killed symmetric stable process.

Zusammenfassung

In dieser Arbeit werden große Fluktuationen einer harmonischen Interaktionsfläche auf dem Standardgitter mit Methoden der Wahrscheinlichkeitstheorie analysiert. Wir studieren das Problem im Rahmen des Gibbsschen Formalismus der klassischen Statistischen Mechanik. Das Modell wird explizit repräsentiert durch stark korrelierte Gaußsche Felder. Die quadratische Wechselwirkung wird so gewählt, daß die Kovarianzen durch die Greensche Funktion einer Irrfahrt auf dem Gitter gegeben sind. Besonderes Gewicht legen wir auf langreichweitige Irrfahrten. In diesem Fall nämlich sind die Gibbsschen Spezifikationen unseres Modells nicht quasilokal und daher traditionelle Techniken nicht mehr anwendbar.

Im ersten Teil dieser Dissertation betrachten wir das allgemeine Modell, dem eine beliebige symmetrische, homogene, transiente Irrfahrt zugrunde gelegt wird. Zunächst bestätigen wir das klassische thermodynamische Bild, indem wir die übliche Beziehung zwischen der relativen Entropie und der freien Energie herleiten. Außerdem beweisen wir eine Version des Gibbsschen Variationsprinzips, indem wir die translationsinvarianten Gibbsmaße charakterisieren als die Minimierer der freien Energie. Sodann studieren wir die großen Fluktuationen der Interaktionsfläche im Rahmen der Donsker-Varadhan-Theorie der Großen Abweichungen. Wir zeigen, daß das empirische Feld eines Gibbsmaßes ein schwaches Prinzip Großer Abweichungen mit Volumenrate und der relativen Entropie als Ratenfunktion erfüllt.

Im zweiten Teil untersuchen wir Fluktuationen zweiter Ordnung, sogenannte kritische Große Abweichungen. Im Hinblick auf das Gibbssche Variationsprinzip sind dies die makroskopischen Fluktuationen der Interaktionsfläche innerhalb der Klasse der translationsinvarianten Gibbsmaße. Wir betrachten hier das nicht-lokale Modell, das durch eine Irrfahrt definiert wird, die im Anziehungsbereich einer symmetrischen stabilen Verteilung liegt. Wir erweitern Resultate von Bolthausen und Deuschel [5], die die Großen Abweichungen im lokalen Fall detailliert untersuchten. Wir zeigen, daß das empirische Feld einem starken Prinzip Großer Abweichungen mit Kapazitätsrate genügt, und beschreiben die Ratenfunktion mit Hilfe der Dirichletform des zugehörigen symmetrischen stabilen Prozesses. Dies ist analog zu Ergebnissen von [5], wo die Ratenfunktion in Termen der Dirichletform der Brownschen Bewegung angegeben wurde. Wie in [5] erreichen wir das Ergebnis durch die Untersuchung der Großen Abweichungen des empirischen Profils der Fläche. Wir diskutieren auch die Fluktuationen des Feldes unter Null-Randbedingungen (d. h., die Fläche wird am Rand der Box zu Null gedrückt). In diesem Fall beweisen wir ein Prinzip Großer Abweichungen mit Kapazitätsrate, wobei die Ratenfunktion mit Hilfe der Dirichletform des beim Verlassen der Box gestoppten symmetrischen stabilen Prozesses beschrieben wird.

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It is hardly necessary to mention how much I am in debt with my parents and the rest of my family. Their encouragement and support have always been there.

Finally, this thesis is dedicated to Valeria, who enlightened the past few years with her kindness and love.

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Contents

Chapter 0

Introduction

0.1 Harmonic Crystal: a Gaussian model

Harmonic crystals are Gibbsian fields associated to a quadratic interaction. The Gaussian nature of the Boltzmann factor makes it possible to give a rather explicit description of the statistical mechanics of the model (physicists call it an "exactly solvable model" since the free energy can be explicitly computed). In view of its simplicity, the harmonic crystal is well suited for investigation of fine properties, even in strongly correlated systems, i.e. when covariances are not summable. Since the early stages of modern statistical mechanics the Gaussian model has played a fundamental role as a first approximation for systems with a richer structure. A quick look at the literature might help to clarify these observations. We shall not discuss the point of view of quantum field theory here, since this would lead us too far. We just recall that, in the field-theoretic context, the Gaussian field - also called "free" field - is the building stone on top of which a truly interacting theory is constructed. Moreover, along with the development of Euclidean (quantum) field theory in the 70's, the statistical mechanics of Gaussian fields also entered as an important ingredient in the construction of quantum fields, see [43] and references therein.

An early reference is the paper of Kac and Berlin [46] in 1952. The Gaussian model was introduced as a useful comparison tool for the Ising model and the N-vector model. In 1966 Dobrushin proposed to study Gaussian fields in the frame of the just arising Gibbsian formalism. In this direction the main results were obtained by Rozanov [67], Dobrushin [25] and Künsch [49]. We shall discuss the matter in some detail in chapter 2. An important step was taken by Spitzer [70] in 1973, who studied nearest neighbour Gaussian fields from the point of view of the underlying lattice random walk. He showed, in particular, how the (discrete) potential theory for random walks could be used to derive properties of the random field. A theme which is going to be recurrent in the following chapters. Indeed, this was also the starting observation in the work of Bolthausen and Deuschel [5] in 1991, perhaps the main source of inspiration for the present work.

Thermodynamical properties of Gaussian fields have been discussed by Künsch [47]. With special emphasis on the *local* (or Markov) case, i.e. the case of finite range interaction, he also showed interesting applications of the thermodynamical formalism to statistics. Apart from the basic results on the structure of Gibbs phases, most of the literature is concerned with the local case. We shall see that some basic questions concerning thermodynamical functionals remained without an answer in the *non local* setting. The reason being that the interaction is not only non local in this case, but also non *quasi-local*, and standard arguments do not apply.

Properties of the Gaussian model are also of great importance in the context of lattice spin systems with a continuous symmetry, such as the XY model or the Heisenberg model. In the late '70s and early '80s the model was shown to be an approximation (so-called "spin-wave" picture) of the low temperature phase for such systems, yielding the fundamental results of Fröhlich, Simon and Spencer [35] on the spontaneous breaking of the continuous symmetry in dimension $d \geqslant 3$ and of Fröhlich and Spencer [36] on the existence of so-called Kosterlitz-Thouless transitions and roughening transitions.

Harmonic crystals are also the simplest examples of gradient fields. In particular they can be seen as explicit models for fluctuating interfaces. In recent years much work has been devoted to the rigorous understanding of relevant physical phenomena related to random interfaces, such as entropic repulsion, localization/delocalization and wetting transitions, see [10] for an introduction. Harmonic crystals are often the only interfaces for which a detailed mathematical treatment is available and even a slight perturbation of the quadratic interaction can increase enormously the difficulty of the problems. We refer to the recent notes of Bolthausen [4] for a review of the above mentioned problems and for reference to the large literature devoted to them. There are in general good reasons to believe that the Gaussian model should capture the correct physics of the problem, in the sense that qualitative behaviour should not be altered by certain perturbations of the quadratic interaction. For the case of strictly convex interactions a recent result of Naddaf and Spencer [63] indeed shows that the field is essentially Gaussian at large scales. In the case of the SOS model, i.e. when the interaction of neighbouring heights is given by the absolute value (instead of the square) of their distance, the situation seems to be different. In fact, the recent works [15] and [7] on the wetting transition show that the two models can give rise to completely different physics.

0.2 About this work

Throughout this work we are concerned with the class of harmonic crystals obtained by representing the covariance of the corresponding Gaussian fields as the Green function of a lattice random walk. In particular, the interaction is non local if the random walk can take jumps of arbitrary magnitude. We shall not address the interesting question of perturbations of the quadratic interaction. The thesis is divided in two parts.

Part I

In part I we first introduce the model and recall the results about the explicit representation of the set of Gibbs measures, to which we add a theorem characterizing ergodic Gibbs states. We then discuss properties of the relative entropy functional and the associated variational principle. Here we are able to settle the open problems for the non local case. The rest of part I is devoted to the analysis of volume-order large deviations, in the spirit of classical Donsker-Varadhan theory [26]. The results of part I have been published in [13]. They are valid for the general model obtained from an arbitrary symmetric, homogeneous and transient random walk [1].

Part II

Here we consider the class of harmonic crystals obtained from random walks which are in the domain of attraction of the symmetric α-stable law. We mainly follow ideas introduced in [5] for the local case. We study the critical large deviations of the empirical field and of the empirical profile measures for both the infinite-volume field and the zero boundary condition field. We show that these are naturally linked to capacity-order asymptotics and identify the rate functions with suitable Dirichlet forms of the symmetric α-stable process. When $\alpha = 2$, in particular, our analysis yields the results of [5]. We plan to publish the results of part II in [14].

0.3 Part I

Connections between equilibrium statistical mechanics and the theory of large deviations have been known at least since the pioneering work of Lanford [51] in the early '70s[2]. In the context of Gibbsian fields associated to bounded interactions Donsker-Varadhan-type estimates for the empirical field have been established by several authors [19, 65, 32, 41]. These works show that probabilities of large fluctuations of the system in the thermodynamical limit are exponentially small in the volume, with rate function given by the relative entropy functional. Relative entropy, on the other hand, can be expressed as a difference of free energies and the Gibbs' variational principle yields a complete characterization of equilibrium measures of the system, i.e. translation invariant Gibbs measures, as minima of the rate function. The presence or absence of a phase transition in the system is then explicitly reflected in the large deviation picture, in the sense that if there exists more than one translation invariant Gibbs state, then large fluctuations can occur with probabilities decaying slower than exponentially in the volume of the system.

[1] These results can actually be extended to some non-random walk harmonic crystals with non-ferromagnetic interactions, see [13]. This extension is not discussed here.

[2] We refer to [57] for a nice overview.

In this context, moreover, the interplay between the theory of large deviations and the variational principle in statistical mechanics has found interesting applications in testing the Gibbsianity of certain random fields [29].

On the other hand, the case of unbounded spin systems is far less investigated, the reason being that in the non-compact setting, even existence of thermodynamics becomes nontrivial. In the past, existence problems have been solved by assuming superstability of the interaction, [53]. Assuming superstability and other regularity properties, a version of the variational principle was discussed in [48]. On the other hand it is quite discouraging that no results are available for non-superstable unbounded spin systems, superstability being a very strong assumption which typically does not hold in strongly correlated systems. For instance the massless free field and in general any strongly correlated Gaussian field (see below) is not superstable. In this sense we may motivate the circle of problems treated in part I of this work as a first attempt to extend the classical theory to a class of strongly correlated unbounded systems.

0.3.1 The model

The harmonic crystal is an unbounded spin system on $\mathbb{Z}^d$, $d \geqslant 1$, with formal Hamiltonian given by

$$\mathcal{H}(\phi) = \frac{1}{2} \sum_{x,y} J(x,y) \phi_x \phi_y \,, \tag{0.1}$$

where $\phi_x \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and the couplings $J(x,y)$ are symmetric and translation invariant. In chapter 2 we will review the basic result which allows to write any Gibbs measure for (0.1) as a mixture of Gaussian fields with covariance $G = J^{-1}$. In particular, existence of infinite volume Gibbs measure is related to the existence of the Green function G.

We restrict our attention to couplings J of the following form. Let $p(x,y)$ be the transition function of a symmetric, homogeneous random walk on $\mathbb{Z}^d$. We consider

$$J(x,y) = \beta \begin{cases} 1 - p(0,0) + m^2 & \text{if } x = y \\ -p(x,y) & \text{if } x \neq y \end{cases} \tag{0.2}$$

where $\beta > 0$ is called the inverse temperature and $m \geqslant 0$ is called the mass. The parameter β plays no role (there is no "temperature") and it will be set equal to one[3].

When $m > 0$ we shall see that the Gibbs measure associated to (0.1) is essentially unique and is given by a weakly dependent centered Gaussian field P^m, with

[3] All results discussed in this work are easily seen to hold for any choice of $\beta > 0$.

summable covariance[4]

$$\sum_{y\in\mathbb{Z}^d} P^m(\phi_x\phi_y) = \frac{1}{m^2}\,, \quad x\in\mathbb{Z}^d\,.$$

We shall see that the massive field P^m satisfies hypercontractive estimates which allow to use standard tools to study large deviations, [18, 23]. Moreover, one can show that the interaction (0.2) is superstable if and only if $m>0$, [48].

We are primarily interested in the strongly correlated or *massless* regime $m=0$. In this case, due to the conservation law

$$\sum_{y\in\mathbb{Z}^d} p(x,y) = 1\,, \quad x\in\mathbb{Z}^d\,,$$

the Hamiltonian (0.1) takes the form

$$\mathcal{H}(\phi) = \frac{1}{4}\sum_{x,y} p(x,y)(\phi_x-\phi_y)^2\,. \tag{0.3}$$

The system (0.3) can be then interpreted as a model for an harmonic interface, with ϕ_x, $x\in\mathbb{Z}^d$, representing the heights of a d-dimensional surface in a $d{+}1$-dimensional medium[5]. The global shift symmetry $\phi_x\to\phi_x+t$, $t\in\mathbb{R}$, is spontaneously broken as soon as Gibbs measures exist, yielding a phase transition. We call $\mathcal{G}$ the set of Gibbs measures. As a consequence of the general Mermin-Wagner theory, $\mathcal{G}$ is empty for short-range models in dimension $d=1,2$, see e.g. [33]. We ensure existence by requiring the random walk p to be *transient*. In particular, low-dimensional models can be treated if the random walk takes sufficiently long jumps. In this setting (see Theorem 2.1) any Gibbs measure is a mixture of Gaussian fields with harmonic mean vector ξ, i.e.

$$\xi_x = \sum_{y\in\mathbb{Z}^d} p(x,y)\xi_y\,, \quad x\in\mathbb{Z}^d\,,$$

and covariance given by the Green function of the random walk, i.e.

$$G(x,y) = \sum_{n=0}^{\infty} p^n(x,y)\,,$$

where p^n denotes the n-th iterate of p. We shall often take as a reference measure the Gibbs field P, defined as the Gaussian field with covariance G and mean zero

$$P(\phi_x)=0\,,\;\; P(\phi_x\phi_y) = G(x,y)\,. \tag{0.4}$$

[4]Note, however, that since we allow long range transitions in p, the covariances $P^m(\phi_x\phi_y)$ do not necessarily decay exponentially in the distance between x and y.

[5]The interaction is of *ferromagnetic* type, in the sense that the heights ϕ_x tend to be close to one another in order to lower the energy. Moreover, it is known that so called FKG inequalities hold for the model (0.3), [44].

We then see that among the extremal elements of the convex set $\mathcal{G}$, the translation invariant ones are

$$\{\gamma_t\,,\quad t \in \mathbb{R}\} \tag{0.5}$$

where γ_t is the Gaussian field with constant mean t and covariance G,

$$\gamma_t(\phi_x) = t\,, \quad \gamma_t(\phi_x\phi_y) - \gamma_t(\phi_x)\gamma_t(\phi_y) = G(x,y)\,. \tag{0.6}$$

With the above notation, $P = \gamma_0$. Theorem 2.2 will show that if the random walk is irreducible the set (0.5) coincides with the so called *pure phases* of the system, i.e. the elements of $\mathcal{G}$ which are extremal in the convex set of translation invariant Gibbs measures[6].

A classical example is the *massless free field*. It is obtained when p is the simple random walk

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.7}$$

i.e. $2dJ = -\Delta$, if Δ is the usual discrete Laplace operator on $\mathbb{Z}^d$. The Green function

$$G = 2d\,(-\Delta)^{-1}$$

and the corresponding measure P are well defined for $d \geqslant 3$ only.

As a typical example of a non local interaction we consider the following model. For $\alpha \in (0,2)$, let $p_\alpha(x,y)$ be an isotropic random walk, and suppose that

$$\lim_{|x|\to\infty} |x|^{d+\alpha} p_\alpha(0,x) = v_{\alpha,d}\,, \tag{0.8}$$

for some $v_{\alpha,d} \in (0,\infty)$. The random walk will then be in the domain of attraction of the symmetric α-stable law[7]. Moreover, if $\alpha < d$ the random walk is transient and Gibbs measures exist in any dimension, cf. chapter 5. The reference field $P = P_\alpha$ obtained from this model will be the central object of part II. Harmonic crystals with a random walk satisfying (0.8) have been considered in [6], where the authors derive precise asymptotics of entropic repulsion events.

0.3.2 Results of part I

We now turn to a brief description of the main results of part I. We refer to the relevant chapters for precise statements, more background ideas and proofs.

[6] We refer to [29], Sect. 2.4, for the standard interpretation of these notions in statistical mechanics.

[7] The paths of such random walks exhibit interesting cluster-like structures which have been used to model non-standard transport phenomena, [62].

Statistical mechanics

A first result of chapter 2 is a theorem concerning the structure of translation invariant Gibbs measures. It states that for models generated by an *irreducible* random walk, translation invariant Gibbs measures cannot be realized as mixtures of non-translation invariant extremal Gibbs measures. Or, in other words, the set of translation invariant Gibbs measures is a face of the convex set of Gibbs measures. On the other hand it is easily seen that this need not be the case if the random walk is not irreducible. It is also not difficult to show that for Harmonic crystals with non-ferromagnetic couplings the result cannot hold, see [13].

We introduce thermodynamics in section 2.2. The internal energy $e(\mu)$ and the entropy $s(\mu)$ are viewed as functionals on the space of translation invariant probability measures μ. The free energy is defined as

$$f(\mu) = e(\mu) - s(\mu)\,.$$

We then introduce the relative entropy of a translation invariant measure μ with respect to the reference measure P,

$$h(\mu|P) = \lim_{N\to\infty} N^{-d} H_N(\mu|P)\,, \tag{0.9}$$

where H_N is the relative entropy on a cubic box Λ_N of side N

$$H_N(\mu|P) = \begin{cases} \mu\Big(\log \dfrac{\mathrm{d}\mu_N}{\mathrm{d}P_N}\Big) & \text{if } \mu_N \ll P_N \\ +\infty & \text{otherwise} \end{cases}$$

where μ_N and P_N denote marginals of μ and P on Λ_N. We prove existence of the limit (0.9) together with an identity expressing the latter as the difference of free energies

$$h(\mu|P) = f(\mu) - f(P)\,.$$

In the case of Gibbsian fields with bounded interactions, this is a classical result which goes back at least to Föllmer's paper [31]. For Gaussian random fields the above identity was proven by Künsch under the assumption that either covariances are summable or $J(x,y)$ decays at least as $|x-y|^{-3d/2}$, as $|x-y| \to \infty$, cf. [47], Theorem 2.11. In our setting the result is true with no further assumption on the decay of J.

The last result in chapter 2 establishes a version of Gibbs' variational principle. Namely, translation invariant Gibbs states with finite variance are characterized as minima of the free energy f or, equivalently, as zeros of relative entropy $h(\cdot|P)$, i.e.

$$h(\mu|P) = 0 \quad \Longleftrightarrow \quad \mu \in \mathcal{G}_2^S\,, \tag{0.10}$$

where $\mathcal{G}_2^S$ denotes the class of shift invariant $\mu \in \mathcal{G}$ with $\mu(\phi_x^2) < \infty$. The "if" part in (0.10), i.e. that any $\mu \in \mathcal{G}_2^S$ minimizes the free energy, is not hard. In the finite range case, the proof that any minimizer must be Gibbsian follows from the standard argument based on quasilocality, [66, 40]. The non-local case is much more difficult in the unbounded-spin setting, since specifications are no longer quasilocal. Under assumptions of superstability and rapid decay of the interaction the issue was settled in [48], but the general case remained open. Quasilocality essentially means that we can approximate Gibbs specifications by local specifications, in the uniform norm, i.e. uniformly in the value of far away boundary conditions. In the absence of quasilocality, one therefore seeks for a weaker type of approximation but, of course, the convergence of the local specification must still be strong enough so that the DLR equation remains closed. The same problem appears for compact spins in the realm of weakly Gibbsian measures [59], where the issue of proving variational principle seems to be rather challenging. Partial results in this direction have been recently obtained in the context of so called renormalization group pathologies, [54, 61]. Different types of continuity for non quasilocal Gibbsian specifications were also discussed in the paper [60]. Our proof in the harmonic crystal setting relies on the explicit nature of Gauss-Gibbs specifications, but it may give a flavour of the main ideas involved in the more general setting.

Volume-order large deviations

Volume-order large deviations are discussed in chapter 3. The *empirical field* on the cubic box Λ_N is the random probability measure defined by

$$R_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \delta_{\theta_x \phi}\,, \tag{0.11}$$

where δ_ϕ stands for the Dirac measure at ϕ and $\theta_x \phi$ denotes the shifted configuration $\phi_{\cdot+x}$. Since our reference measure P is ergodic with respect to lattice shifts,

$$R_N \;\Rightarrow\; P\,, \quad P-\text{a.s.}\,, \quad N \to \infty\,,$$

in the sense of weak convergence. In particular, for any open set G of probability measures such that $P \in G$, we have $P(R_N \notin G) \to 0$, as $N \to \infty$. We show that these probabilities are exponentially decaying in the volume N^d, with rate given by the relative entropy $h(\cdot|P)$. Since we are interested in the massless regime, classical large deviations results for stationary Gaussian processes based on the continuity of the spectral density (cf. e.g. [27, 11]) cannot be used. Following an idea introduced in [5], the exponential bounds are first established for the massive regime and then lifted to the general case. In particular, for any $m > 0$, we prove the bounds

$$\begin{aligned} -\inf_{\mu \in \Gamma^\circ} h(\mu|P^m) &\leqslant \liminf_{N\to\infty} N^{-d} \log P^m\big(R_N \in \Gamma\big) \\ &\leqslant \limsup_{N\to\infty} N^{-d} \log P^m\big(R_N \in \Gamma\big) \leqslant -\inf_{\mu \in \overline{\Gamma}} h(\mu|P^m) \end{aligned} \tag{0.12}$$

where Γ is an arbitrary measurable set of shift invariant probability measures, Γ° and $\overline{\Gamma}$ denote the interior and the closure of Γ respectively. The lower bound above follows from a standard change of measure argument, and a lemma on ergodic approximations. Once the hypercontractivity properties of the massive field are established, one can prove the convergence of the asymptotic logarithmic moment generating function (see appendix B). Then the upper bound in (0.12) follows from standard ideas. To remove the mass m, i.e. to take the limit $m \to 0$ in (0.12), we shall use a refinement of the argument of [5]. In fact, in the local case one can get away with rather crude estimates, while we need to exploit here the full strength of the random walk representation for the covariances (see appendix A). When removing the mass the lower bound is stable and we have

$$\liminf_{N\to\infty} N^{-d} \log P\big(R_N \in \Gamma\big) \geqslant -\inf_{\mu\in\Gamma^\circ} h(\mu|P)\,. \tag{0.13}$$

On the other hand, in order to have a stable upper bound we need to restrict to sets Γ such that the closure $\overline{\Gamma}$ satisfies

$$\sup_{\mu\in\overline{\Gamma}} \mu(\phi_x^2) < \infty\,.$$

In particular, $\overline{\Gamma}$ must be compact in the weak topology. With the above restriction, we may remove the mass and obtain

$$\limsup_{N\to\infty} N^{-d} \log P\big(R_N \in \Gamma\big) \leqslant -\inf_{\mu\in\overline{\Gamma}} h(\mu|P) \tag{0.14}$$

In this sense we speak of a *weak* large deviations principle for the empirical field of P. As discussed in chapter 3, the compactness constraint cannot be removed by standard arguments since the massless field P is not exponentially tight at volume-order. Moreover, the rate function is easily seen to violate lower semi-continuity outside of sets with uniformly finite second moments. Clearly, non-tightness is due to the broken shift symmetry $\phi_x \to \phi_x + t$. Consider for instance the massless free field (0.7). If instead of the field ϕ one considers its gradient field

$$\eta_x^i = (\nabla_i\phi)_x = \phi_{x+e_i} - \phi_x\,, \quad i = 1,\dots,d\,,\ x\in\mathbb{Z}^d\,,$$

where e_i denotes the unit lattice vector in the i-th direction, then it is not hard to prove exponential tightness at volume-order. However, apart from the easy one-dimensional case (where the variables η_x become independent) the proof of a large deviations principle for the field η seems to be a interesting open question[8]. For the moment we are not able to make substantial progress in this direction and we do not include discussion of the problem in this work.

[8]Note that a solution of this problem would enable to treat perturbations of the Gaussian which are not necessarily strictly convex. In the strictly convex regime, properties of the gradient field η have been studied recently in a series of remarkable papers, see [63, 38, 42, 22].

It is natural to ask about the influence of *boundary conditions* on the large deviations estimates. Namely, what happens to the bounds (0.13) and (0.14) when we replace the infinite volume field P by finite volume Gibbs measures P_N^τ with a given boundary condition τ outside of Λ_N. We find that if the boundary configuration τ is not growing too fast at infinity the large deviations estimates remain unaffected. This condition can be expressed in terms of the surface tension induced by the boundary. The growth restriction of course depends on the couplings $p(x, y)$. We find for instance that if p is irreducible and has finite range, then the effect of any τ with strictly sublinear growth is negligeable on the thermodynamical level. For the long range case (0.8) we find that $|\tau_x| \leqslant |x|^\beta$ for large $|x|$, $\beta < \alpha/2$, is sufficient.

0.4 Part II

The large deviation lower bound (0.13) together with the variational principle shows that deviations within the set of Gibbs measures are of order $o(N^d)$ on the exponential scale. We then speak of *critical* (or second order) large deviations. This observation shows that the study of critical large deviations has good chances to give insight in the physics of coexisting phases. In particular this program has been carried out with great success in the past decade for the Ising model in the phase-coexistence region, providing a description of the rich structure of phase separation. We refer the reader to the recent review [3] for an exhaustive account on this beautiful subject. In the Ising model, coexistence of phases is characterized by the creation of an interface between the '+' phase and the '−' phase. In a box of side N the typical cost in energy of such configurations is of the surface order N^{d-1}. In particular, critical large deviations occur at the surface order N^{d-1}. On the other hand, for short-range systems with a continuous symmetry such as the XY or Heisenberg model, the situation seems to be completely different. Indeed, the energy cost of typical configurations in the phase-transition regime is much less than surface order, since the continuous symmetry makes it possible to interpolate smoothly from one phase to another. We refer to [34] for a proof of the vanishing of surface tension at all temperatures for the XY model. Unfortunately, there are no results about precise asymptotics and the problem seems to be out of reach with present techniques (see, however, [16] for a preliminary attempt in this direction). In this sense we can view the harmonic crystal as a prototype model providing insight in the difficult matter of low-temperature phases for systems with a continuous symmetry. We shall see that the decay of the interaction plays a prominent role in determining the order of critical large deviations.

Capacity-order large deviations

Critical large deviations have been studied for the massless free field, and more generally for any P arising from a finite range irreducible random walk, by Bolthausen and Deuschel in the paper [5]. For such random walks a classical invariance principle applies and the order of critical large deviations is the capacity-order N^{d-2}, for $d \geqslant 3$. Moreover, the authors prove a strong large deviations principle for R_N at the order N^{d-2}, with rate function given by a suitable Dirichlet form associated to brownian motion. Part II of this work is mainly concerned with the extension of the analysis of [5] to the case of long range random walks. In order to have a treatable limiting process we assume that the random walk is in the domain of attraction of the symmetric α-stable law. In particular, we consider massless fields P_α, $\alpha \in (0, 2\wedge d)$ and $d \geqslant 1$, defined by (0.4) with $p = p_\alpha$ an homogeneous, isotropic random walk satisfying (0.8). The stable random walk is discussed in chapter 5, where we also introduce a technical assumption in addition to (0.8). After rescaling, the lattice process converges to the symmetric α-stable process on $\mathbb{R}^d$, and its capacities scale like $N^{d-\alpha}$.

A first idea why the correct order of critical large deviations should be the capacity-order $N^{d-\alpha}$, can be given by the following simple observations about the empirical mean

$$M_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \phi_x \,. \tag{0.15}$$

According to P_α, M_N is a Gaussian variable with mean zero and variance

$$\sigma_N^2 = P_\alpha(M_N^2) = N^{-2d} \sum_{x,y\in\Lambda_N} G(x,y)\,.$$

In particular,

$$P\left(M_N \geqslant m\right) = \frac{1}{\sqrt{2\pi}} \int_{m/\sigma_N}^{\infty} \mathrm{e}^{-y^2/2}\, dy\,.$$

Using the asymptotics (see chapter 5)

$$G(x,y) \sim |x-y|^{\alpha-d}\,,$$

we have

$$\lim_{N\to\infty} N^{d-\alpha}\sigma_N^2 = c > 0\,.$$

From the elementary Gaussian bounds

$$\frac{x}{x^2+1}\,\mathrm{e}^{-x^2/2} \leqslant \int_x^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y \leqslant \frac{1}{x}\,\mathrm{e}^{-x^2/2}\,, \quad x>0\,,$$

one then obtains

$$\lim_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(M_N \geqslant m\big) = -\frac{m^2}{2c}\,, \quad m>0\,. \tag{0.16}$$

0.4.1 Results of part II

We give here a rough description of our main results in part II. For statements and proofs we refer to the relevant chapters, where main ideas of proof are also discussed in some detail.

Symmetric stable process

In the first chapter of part II we introduce the symmetric α-stable process and discuss its potential theory in terms of the associated Dirichlet forms. This potential-theoretic intermezzo is intended to be a reference chapter of independent interest. Results collected here are of a fairly standard nature. However, we include some of the proofs since we could not always find explicit references. The most interesting statement here is a theorem relating the Dirichlet form of the embedded process to the Dirichlet form of the free process through a variational formula. The latter is essential for the proof of the limit theorems of chapter 5.

The infinite volume field

In analogy with [5], our first main result establishes a strong large deviations principle for the empirical field of the infinite volume measure P_α at the capacity-order $N^{d-\alpha}$. The rate function is shown to have a representation in terms of the Dirichlet form of the process *embedded* in the rescaled unit box $V \sim \Lambda_N/N$. We prove that for any measurable set Γ of shift invariant probability measures, the empirical field R_N satisfies the bounds

$$\begin{aligned} -\inf_{\mu\in\Gamma^\circ} \mathcal{C}_\alpha(\mu) &\leqslant \liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in \Gamma\big) \\ &\leqslant \limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in \Gamma\big) \leqslant -\inf_{\mu\in\overline{\Gamma}} \mathcal{C}_\alpha(\mu) \end{aligned} \tag{0.17}$$

In particular, there is no compactness restriction for the upper bound, since it is not difficult to prove exponential tightness of the field at capacity-order. Also, the rate function $\mathcal{C}_\alpha$ is lower semi-continuous and has compact level sets. The proof of the lower bound is based on the important observation, first made in [5], that for certain Gaussian measures γ^ψ, indexed by profiles ψ, the relative entropy $H_N(\gamma^\psi|P)$ converges at capacity-order to the Dirichlet form of the embedded process computed on ψ. We shall discuss this convergence and its relation to capacity in chapter 5. In particular, when ψ is a constant profile the measure γ^ψ coincides with γ_t for some $t \in \mathbb{R}$, where γ_t is the global shift of P defined in (0.6). This is shown to imply the convergence

$$N^{\alpha-d} H_N(\gamma_t|P) \to \frac{t^2}{2}\,\mathrm{cap}_\alpha(V)\,, \tag{0.18}$$

where $\mathrm{cap}_\alpha(V)$ is the capacity of the rescaled box V for the symmetric α-stable process. This indicates that the typical energy cost of global shift (or 0-mode) fluctuations is of order $N^{d-\alpha}$. Note that this exceeds surface order if $\alpha \in (0,1)$. Once the above mentioned convergence of relative entropies has been established, the lower bound in (0.17) is fairly standard. One may think at this point that $\mathcal{C}_\alpha(\mu)$ is simply given by the specific capacity entropy $N^{\alpha-d}H_N(\mu|P)$ in the limit $N \to \infty$. However, as already noted in [5], this is only true when μ is one of the extremal measures γ_t, see (0.6).

The proof of the upper bound is more delicate. Following the approach of [5] we first analyze the ergodic behaviour of random *profile measures*. We refer to chapter 6 for a discussion and precise results.

The 0-boundary condition field

Our next result is a strong large deviations principle for the empirical field of $P^0_{\alpha,N}$, the harmonic crystal on Λ_N with zero boundary conditions outside. This is the centered Gaussian field with covariance Γ_N, the Green function of the stable random walk killed upon exiting Λ_N. The rate function is now expressed in terms of the Dirichlet form of the symmetric α-stable process *killed* upon exiting the rescaled unit box $V \sim \Lambda_N/N$. In particular we prove the bounds

$$\begin{aligned} -\inf_{\mu\in\Gamma^\circ} \mathcal{C}^0_\alpha(\mu) &\leqslant \liminf_{N\to\infty} N^{\alpha-d}\log P^0_{\alpha,N}\big(R_N\in\Gamma\big) \\ &\leqslant \limsup_{N\to\infty} N^{\alpha-d}\log P^0_{\alpha,N}\big(R_N\in\Gamma\big) \leqslant -\inf_{\mu\in\overline{\Gamma}} \mathcal{C}^0_\alpha(\mu) \end{aligned} \tag{0.19}$$

Again there is no difficulty in proving exponential tightness. Also we show that the rate function $\mathcal{C}^0_\alpha(\mu)$ is lower semi-continuous and has compact level sets. We shall see that there are interesting relations between the killed rate function $\mathcal{C}^0_\alpha$ and the free rate function $\mathcal{C}_\alpha$, and that, in contrast with the volume-order case, boundary conditions may have a dramatic influence on capacity order asymptotics. In particular, as should be expected on intuitive grounds, $\mathcal{C}^0_\alpha(\mu) \geqslant \mathcal{C}_\alpha(\mu)$ for any measure μ. It is also not difficult to see that $\mathcal{C}^0_\alpha(\gamma_t) > \mathcal{C}_\alpha(\gamma_t)$ for any $t \neq 0$. Moreover, we shall give an argument to see that $\mathcal{C}^0_\alpha(\gamma_t) = +\infty$, for $t \neq 0$ and $\alpha \in (1,2)$, $d \geqslant 2$, while $\mathcal{C}_\alpha(\gamma_t)$ is always finite. This is clearly due to the fact that a global shift deviation $P \to \gamma_t$ in the zero boundary case has always at least a surface-order price, due to the abrupt raising of the field at the boundary.

The proof of the lower bound relies on the convergence of the relative entropy $H_N(\gamma^\psi|P^0_{\alpha,N})$ to the Dirichlet form of the killed process. The upper bound is much more delicate. We refer to chapter 6 for a thorough presentation.

The question of zero b.c. was not addressed in [5]. We observe that our method is easily seen to apply to their case, when the stable random walk is replaced by any irreducible, isotropic random walk with finite range and the stable process is replaced by brownian motion, i.e. $\alpha = 2$. It also covers the case of infinite range but finite variance.

Part I

Part one

Chapter 1

Preliminaries

We start with a presentation of the model. Definitions and notations given here will be used throughout the whole work.

1.1 The Model

Lattice

We consider $\mathbb{Z}^d$, the d-dimensional integer lattice, with *sites*

$$x = (x_1, x_2, \ldots, x_d) \quad x_i \in \mathbb{Z},\ i = 1, \ldots, d$$

where d is an arbitrary integer with $d \geqslant 1$. For $x \in \mathbb{Z}^d$, $|x|$ denotes the Euclidean norm

$$|x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} .$$

If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality (volume) of Λ is denoted $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. The complement of a set is denoted $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. Given an even integer $N \in 2\mathbb{Z}_+$ we define Λ_N to be the cube of side N centered at the origin, i.e.

$$\Lambda_N = [-N/2, N/2)^d \cap \mathbb{Z}^d\,, \quad |\Lambda_N| = N^d\,.$$

Configurations

We consider a field of real variables attached to the sites of the lattice. The space of configurations is $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, and a single configuration $\phi \in \Omega$ is denoted

$$\phi = \left\{ \phi_x\,,\ x \in \mathbb{Z}^d \right\}\,.$$

We equip Ω with the product topology and call $\mathcal{F}$ the corresponding Borel σ-algebra. Given $\Lambda \subset \mathbb{Z}^d$ we write $\Omega_\Lambda = \mathbb{R}^\Lambda$, and $\pi_\Lambda : \Omega \to \Omega_\Lambda$ denotes the canonical projection of ϕ onto $\phi_\Lambda \equiv \{\phi_x,\, x \in \Lambda\}$, the restriction of ϕ to the region Λ. If $\Lambda = \{x\}$ we simply write π_x. $\mathcal{F}_\Lambda$ denotes the σ-algebra generated by the canonical projections $\{\pi_x,\, x \in \Lambda\}$. Lattice *shifts* (or translations) $\{\theta_y,\, y \in \mathbb{Z}^d\}$, the natural action of the group $\mathbb{Z}^d$ on Ω, are defined by $(\theta_y\phi)_x = \phi_{x+y}$, $x, y \in \mathbb{Z}^d$. Given $\xi, \eta \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$, the scalar product $\xi_\Lambda \cdot \eta_\Lambda$ is written

$$\langle \xi, \eta \rangle_\Lambda = \sum_{x \in \Lambda} \xi_x \eta_x \,.$$

A function $f : \Omega \to \mathbb{R}$ is called *local* if there exists a $\Lambda \in \mathbb{F}$ such that f is $\mathcal{F}_\Lambda$-measurable. The smallest such Λ is called the support of f and is denoted S_f.

Probability measures

The distribution of the random field ϕ is given by probability measures on $(\Omega, \mathcal{F})$. These are denoted by $\mathcal{M}_1(\Omega)$. A measure $\mu \in \mathcal{M}_1(\Omega)$ is sometimes interpreted as a *state* of the system. Given a measurable function $f : \Omega \to \mathbb{R}$ we write

$$\mu(f) = \int_\Omega f \,\mathrm{d}\mu \,.$$

Given $\Lambda \in \mathbb{F}$ we write μ_Λ or $\mu \circ \pi_\Lambda^{-1}$ for the marginal of μ on Ω_Λ.

The set $\mathcal{M}_1(\Omega)$ is equipped with the *weak topology*, i.e. the topology induced by convergence against bounded continuous functions on Ω. A measure $\mu \in \mathcal{M}_1(\Omega)$ is said to be *tail-trivial* if $\mu(A) = 0$ or 1 for any event $A \in \mathcal{F}$ such that A is $\mathcal{T}$-measurable, where $\mathcal{T}$ is the tail σ-algebra

$$\mathcal{T} = \bigcap_{\Lambda \in \mathbb{F}} \mathcal{F}_{\Lambda^c} \,.$$

It is known that μ is tail-trivial if and only if μ is *extremal* in the convex set $\mathcal{M}_1(\Omega)$ (i.e. it has no non-trivial convex decomposition).

Elements of $\mathcal{M}_1(\Omega)$ which are invariant under the transformations induced by lattice shifts are called *shift invariant* (or translation invariant) states, and are denoted $\mathcal{M}_1^S(\Omega)$. $\mathcal{M}_1^S(\Omega)$ is a closed convex subset of $\mathcal{M}_1(\Omega)$. A measure $\mu \in \mathcal{M}_1^S(\Omega)$ is called *ergodic* if $\mu(A) = 0$ or 1 for any shift invariant event $A \in \mathcal{F}$, i.e. such that $\theta_x A = A$ for all $x \in \mathbb{Z}^d$. Since shift invariant events are tail-measurable, all shift invariant states which are tail-trivial are also ergodic.

1.2 Random walks

We define our random walk as the discrete time jump process on $\mathbb{Z}^d$ with transitions given by a function $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ satisfying the following basic assumptions:

- stochasticity

$$\sum_{y\in\mathbb{Z}^d} p(x,y) = 1\,, \quad x\in\mathbb{Z}^d\,. \tag{1.1.1}$$

- symmetry and translation invariance

$$p(x,y) = p(0,y-x) = p(y,x)\,, \quad x,y\in\mathbb{Z}^d\,.$$

- transience

$$\sigma^2 = \sum_{n=0}^{\infty} p^n(0,0) < \infty\,, \tag{1.1.2}$$

where p^n denotes the $n-$th iterate of p.

In particular, $p(x,y)$ is interpreted as the probability to jump from x to y. Let $x\in\mathbb{Z}^d$ and denote by $\xi_0 = x, \xi_1, \xi_2, \ldots$ the path of the random walk started in $\xi_0 = x$. We write the corresponding probability and expectation $\mathbb{P}_x$ and $\mathbb{E}_x$ respectively, so that

$$p^n(x,y) = \mathbb{P}_x(\xi_n = y) = \mathbb{E}_x\left[1_{\{y\}}(\xi_n)\right]\,, \quad n\in\mathbb{Z}_+.$$

It is useful to introduce the Fourier transform of p,

$$\hat{p}(\theta) = \sum_{x\in\mathbb{Z}^d} p(0,x)\,\mathrm{e}^{i\,x\cdot\theta}\,, \quad \theta\in\mathbb{T}^d\,, \tag{1.1.3}$$

where $\mathbb{T}^d$ stands for the torus $(-\pi,\pi]^d$ (the first Brillouin zone) and $x\cdot\theta = \sum_{i=1}^d x_i\theta_i$. We also consider the generator of the continuous time random walk

$$J(x,y) = \delta(x,y) - p(x,y)\,, \tag{1.1.4}$$

where $\delta(x,y) = 1$ if $x = y$, and $\delta(x,y) = 0$ otherwise. We write $J = 1\!\!1 - p$ and $\hat{J} = 1 - \hat{p}$. The random walk defined by p is called *irreducible* if for every $x,y\in\mathbb{Z}^d$, there exists an integer $n\in\mathbb{Z}_+$ such that $p^n(x,y) > 0$. It is said to be *aperiodic* if $\hat{J}(\theta) = 0$ implies $\theta = 0$ (note that $\hat{J}(0) = 0$ always). Irreducible random walks are always aperiodic (cf. chapter 2 of [71]).

1.3 Green functions

Thanks to the transience assumption (1.1.2), the Green function

$$G(x,y) = \sum_{n=0}^{\infty} p^n(x,y) = \mathbb{E}_x\left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n)\right] \tag{1.1.5}$$

is well defined. Indeed, $G(x,y) \leqslant \sigma^2$, for any $x,y \in \mathbb{Z}^d$. In terms of Fourier transforms, letting $f = (1-\hat{p})^{-1}$, we have

$$G(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta)\, \mathrm{e}^{i\,(y-x)\cdot\theta}\, \mathrm{d}\theta\,. \tag{1.1.6}$$

Let $\Lambda \in \mathbb{F}$. We use the notations

$$p_\Lambda = \{p(x,y)\}_{x,y\in\Lambda}\,, \quad J_\Lambda = \{J(x,y)\}_{x,y\in\Lambda}\,, \quad \Gamma_\Lambda = J_\Lambda^{-1}\,.$$

Then Γ_Λ has the Neumann series

$$\Gamma_\Lambda(x,y) = \sum_{n=0}^{\infty} p_\Lambda^n(x,y)\,, \quad x,y \in \Lambda\,. \tag{1.1.7}$$

To have a probabilistic interpretation of Γ_Λ we use the metaphor of "killing". If σ_Λ is the first exit time

$$\sigma_\Lambda = \inf\{n \geqslant 0\,,\ \ \xi_n \notin \Lambda\}\ , \tag{1.1.8}$$

then

$$p_\Lambda^n(x,y) = \mathbb{P}_x\left(\xi_n = y\,;\ \sigma_\Lambda > n\right)\ .$$

Also,

$$\Gamma_\Lambda(x,y) = \mathbb{E}_x\left[\sum_{n=0}^{\sigma_\Lambda - 1} 1_{\{\xi_n = y\}}\right]\,. \tag{1.1.9}$$

We interpret p_Λ as the transition function of the random walk which is *killed* upon exiting the set Λ. Let G_Λ be the restriction of G to Λ

$$G_\Lambda = \{G(x,y)\}_{x,y\in\Lambda}\,.$$

In the appendix, cf. Lemma A.1, we show that G_Λ has an expansion analogous to (1.1.7) where p_Λ must be replaced by an effective transition function $\tilde{p}_\Lambda$. This gives

$$G_\Lambda^{-1} = \mathbb{I}_\Lambda\ -\ \tilde{p}_\Lambda \tag{1.1.10}$$

Moreover $\tilde{p}_\Lambda$ has the following simple probabilistic interpretation. Let $\tilde{\sigma}_\Lambda$ be the first hitting time for Λ after time zero

$$\tilde{\sigma}_\Lambda = \inf\{n \geqslant 1\,,\ \ \xi_n \in \Lambda\}\ . \tag{1.1.11}$$

Then

$$\tilde{p}_\Lambda(x,y) = \mathbb{P}_x(\xi_{\tilde{\sigma}_\Lambda} = y)\,. \tag{1.1.12}$$

The renormalized transition $\tilde{p}_\Lambda$ defines the so called *embedded* random walk in Λ. The right hand side of (1.1.12) can be split into two terms depending on whether $\tilde{\sigma}_\Lambda = 1$ (i.e. $\xi_1 \in \Lambda$) or $\tilde{\sigma}_\Lambda > 1$ (i.e. $\xi_1 \notin \Lambda$). The first term just yields $p(x,y)$ and from (1.1.10) and (1.1.7) we see that for any $x, y \in \Lambda$

$$\begin{aligned} \Gamma_\Lambda^{-1}(x,y) - G_\Lambda^{-1}(x,y) &= \tilde{p}_\Lambda(x,y) - p_\Lambda(x,y) \\ &= \mathbb{P}_x(\xi_{\tilde{\sigma}_\Lambda} = y\,;\ \tilde{\sigma}_\Lambda > 1)\,. \end{aligned} \tag{1.1.13}$$

The matrix $G_\Lambda = (J^{-1})_\Lambda$ is also interpreted as the Green function of J on Λ with *free* boundary conditions, while $\Gamma_\Lambda = (J_\Lambda)^{-1}$ is associated to *Dirichlet* boundary conditions.

1.4 The reference measure

Given $\Lambda \in \mathbb{F}$, we call P_Λ the centered Gaussian measure on Ω_Λ with covariance G_Λ, i.e. with density

$$\frac{\mathrm{d}P_\Lambda}{\mathrm{d}\phi_\Lambda}(\phi) = (2\pi)^{-|\Lambda|/2} (\det G_\Lambda)^{-1/2} \exp\left(-\frac{1}{2}\,\langle \phi, G_\Lambda^{-1}\phi\rangle_\Lambda\right).$$

Here $\mathrm{d}\phi_\Lambda = \prod_{x\in\Lambda} d\phi_x$ denotes Lebesgue measure on Ω_Λ. The measures P_Λ, $\Lambda \in \mathbb{F}$ form a consistent family and there exists a unique measure $P \in \mathcal{M}_1(\Omega)$ such that its marginal on Λ is given by P_Λ. We also refer to P as the centered Gaussian field on $\mathbb{Z}^d$, with covariance G, or the infinite volume harmonic crystal measure. In view of (1.1.6), P is also referred to as the Gaussian field with *spectral density* f.

Note that $P \in \mathcal{M}_1^S(\Omega)$, since G is translation invariant. From (1.1.6) and Riemann-Lebesgue lemma, we have

$$G(x,y) \to 0\,, \quad |x-y| \to \infty\,.$$

This implies that P is tail-trivial (cf. e.g. [40], chapter 7). In particular, P is ergodic w.r.t. lattice shifts.

The following random walk representation for the conditional field plays an important role in many proofs. Given $\Lambda \in \mathbb{F}$, the conditional field $P(\cdot|\phi_{\Lambda^c})$ is the Gaussian field with mean

$$P(\phi_x|\phi_{\Lambda^c}) = \sum_{y\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = y\right)\phi_y \tag{1.1.14}$$

and covariance Γ_Λ (independent of ϕ_{Λ^c}). This also holds for an arbitrary (possibly infinite) subset Λ of $\mathbb{Z}^d$, see appendix A.

We shall use the following notations. Given $\eta \in \Omega$, we denote by γ^η the translate of P by the configuration η

$$\gamma^\eta = P * \delta_{-\eta} \tag{1.1.15}$$

where δ_ξ is the Dirac mass at $\xi \in \Omega$ and $*$ means convolution. Thus γ^η is simply the Gaussian field with mean and covariance given by

$$\gamma^\eta(\phi_x) = \eta_x \, , \quad \gamma^\eta(\phi_x \phi_y) - \gamma^\eta(\phi_x)\gamma^\eta(\phi_y) = G(x, y) \, .$$

When η is a constant configuration $\eta_x = t \in \mathbb{R}$, $x \in \mathbb{Z}^d$, we also write $\gamma_t = \gamma^\eta$. Note that γ_t is ergodic, for all $t \in \mathbb{R}$.

Chapter 2

Statistical Mechanics

In this chapter we discuss the model in the framework of equilibrium statistical mechanics. We recall a basic result characterizing Gibbs measures for the harmonic crystal in terms of Gaussian fields. We then prove a result concerning the extremality of ergodic Gibbs measures. In section 2 we introduce thermodynamic functionals and the relative entropy density. The classical identity is established, expressing the latter as a difference of free energies. Finally in section 3 we prove a version of Gibbs' variational principle.

2.1 Gibbs Measures

2.1.1 Gaussian specifications

Let $J = \mathbb{1} - p$ be defined as in (1.1.4) and let

$$\Omega_J = \left\{ \phi \in \Omega : \sum_{y \in \mathbb{Z}^d} |J(x,y)\phi_y| < \infty \,, \text{ for all } x \in \mathbb{Z}^d \right\} . \tag{2.2.1}$$

Note that Ω_J is a tail event. It is clear that if the random walk has finite range then $\Omega_J = \Omega$. Fix $\Lambda \in \mathbb{F}$ and a boundary condition $\tau \in \Omega_J$. The Hamiltonian[1] is defined by

$$H_\Lambda^\tau(\phi) = \frac{1}{2} \sum_{x,y \in \Lambda} J(x,y)\phi_x\phi_y + \sum_{x \in \Lambda, y \notin \Lambda} J(x,y)\phi_x\tau_y \,. \tag{2.2.2}$$

We define *specifications* $\gamma_\Lambda^\tau \in \mathcal{M}_1(\Omega)$, $\tau \in \Omega_J$, by

$$\gamma_\Lambda^\tau(\,\mathrm{d}\phi) = \begin{cases} (Z_\Lambda^\tau)^{-1} \exp\left(-H_\Lambda^\tau(\phi)\right) \mathrm{d}\phi_\Lambda & \text{if } \phi_{\Lambda^c} = \tau_{\Lambda^c} \\ 0 & \text{otherwise} \end{cases} \tag{2.2.3}$$

[1] Note that we are in the massless regime $m = 0$. The case of a positive mass can be treated in the same way, setting $J = (1+m)\mathbb{1} - p$. The massive model will be mentioned in the next chapter.

with

$$Z_\Lambda^\tau = \int e^{-H_\Lambda^\tau(\phi)} \, d\phi_\Lambda \tag{2.2.4}$$

A simple computation (cf. [40], chapter 13) shows that γ_Λ^τ is the Gaussian measure on $(\Omega, \mathcal{F})$ with mean

$$m_x^{\tau,\Lambda} = \gamma_\Lambda^\tau(\phi_x) = \begin{cases} \sum_{y\in\Lambda} \sum_{z\notin\Lambda} \Gamma_\Lambda(x,y) p(y,z) \tau_z & \text{if } x \in \Lambda \\ \tau_x & \text{if } x \notin \Lambda \end{cases} \tag{2.2.5}$$

and covariance (independent of $\tau \in \Omega_J$)

$$\gamma_\Lambda^\tau(\phi_x\phi_y) - \gamma_\Lambda^\tau(\phi_x)\gamma_\Lambda^\tau(\phi_y) = \begin{cases} \Gamma_\Lambda(x,y) & \text{if } x, y \in \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{2.2.6}$$

The partition function (2.2.4) is explicitly given by

$$Z_\Lambda^\tau = (2\pi)^{|\Lambda|/2} \, (\det \Gamma_\Lambda)^{1/2} \, \exp\left(\frac{1}{2} \left\langle m^{\tau,\Lambda}, \Gamma_\Lambda^{-1} m^{\tau,\Lambda} \right\rangle_\Lambda \right). \tag{2.2.7}$$

Recalling the probabilistic interpretation of Γ_Λ, see (1.1.9), we can derive the following representation for the mean values (2.2.5). If σ_Λ denotes the first exit time for Λ (1.1.8), we have

$$m_x^{\tau,\Lambda} = \sum_{z\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = z\right) \, \tau_z = \mathbb{E}_x \, \tau_{\xi_{\sigma_\Lambda}} \, , \quad \tau \in \Omega_J \, . \tag{2.2.8}$$

2.1.2 Basic theorem

Given a local, bounded continuous function g, in symbols $g \in \mathcal{C}_{b,\text{loc}}(\Omega)$, we denote by $\gamma_\Lambda(g)$ the $\mathcal{F}_{\Lambda^c}$-measurable function

$$\Omega_J \ni \tau \to \gamma_\Lambda^\tau(g) = \int \gamma_\Lambda^\tau(\,d\phi) g(\phi) \, .$$

A measure $\mu \in \mathcal{M}_1(\Omega)$ is said to be a *Gibbs measure* relative to J if

$$\mu(\Omega_J) = 1 \tag{2.2.9}$$

and

$$\mu(\gamma_\Lambda(g)) = \mu(g) \, , \quad \text{for all } g \in \mathcal{C}_{b,\text{loc}}(\Omega), \, \Lambda \in \mathbb{F}. \tag{2.2.10}$$

Note that the requirement (2.2.9) becomes trivial in the finite range case. Conditions (2.2.10) are called DLR equations. We call

$$\mathcal{G} = \mathcal{G}_J$$

the set of Gibbs measures relative to J. Let $\mathcal{G}_{\text{ext}}$ denote the set of extremal points of the convex set $\mathcal{G}$. An important property of $\mathcal{G}$ is that any $\mu \in \mathcal{G}$ has a unique decomposition in terms of its extremal components, i.e. $\mathcal{G}$ is a simplex, cf. [40]. The following result, due to Rozanov and Dobrushin gives explicitly the set $\mathcal{G}_{\text{ext}}$ in terms of the Gaussian fields γ^η defined in (1.1.15). Let $\mathcal{H}_J$ denote the set of *J-harmonic* configurations

$$\mathcal{H}_J = \{\phi \in \Omega : \sum_{y \in \mathbb{Z}^d} J(x,y)\phi_y = 0, \text{ for all } x \in \mathbb{Z}^d\}.$$

Theorem 2.1. *(Rozanov [67], Dobrushin [25], Künsch [49])*

$$\mathcal{G}_{\text{ext}} = \{\gamma^\eta, \ \eta \in \mathcal{H}_J\}. \tag{2.2.11}$$

For a proof of Theorem 2.1 we refer to the original works. See also Theorems (13.24) and (13.36) of [40]. Let us briefly describe the basic ideas behind this result. One first shows that P itself is a Gibbs measure. To see this one observes that γ_Λ^τ is a version of the conditional probability $P(\cdot|\mathcal{F}_{\Lambda^c})$, cf. (1.1.14), and the DLR equation follows. Since $P(\phi_x^2) = \sigma^2 < \infty$, condition (2.2.9) follows from the absolute summability of J. Now assume $\eta \in \mathcal{H}_J$. From (2.2.2) we see that

$$H_\Lambda^{\tau-\eta}(\cdot - \eta) = H_\Lambda^\tau(\cdot) + \text{const}(\eta, \tau),$$

which implies $\gamma_\Lambda^{\tau-\eta}(\cdot - \eta) = \gamma_\Lambda^\tau(\cdot)$, and therefore $\gamma^\eta = P(\cdot + \eta)$ satisfies the DLR equation, since P does. Moreover (2.2.9) is satisfied since $\mathcal{H}_J \subset \Omega_J$. This shows that any γ^η belongs to $\mathcal{G}$. Extremality follows from tail-triviality of P. To show that $\mu \in \mathcal{G}_{\text{ext}}$ implies $\mu = \gamma^\eta$, for some $\eta \in \mathcal{H}_J$ one roughly proceeds as follows. As a corollary of general results (see e.g. Theorem (7.12) of [40]), $\mu \in \mathcal{G}_{\text{ext}}$ can be written as the limit of γ_Λ^τ, for some $\tau \in \Omega_J$, as $\Lambda \nearrow \mathbb{Z}^d$. Any such limit of Gaussian measures must be itself Gaussian, with mean and covariance given by the limiting values of mean and covariance of γ_Λ^τ. The harmonicity of the mean is a consequence of (2.2.5). The limiting covariance is given by $G(x,y)$, the limiting value of $\Gamma_\Lambda(x,y)$.

Remark 2.1. The theorem above shows that there are uncountably many Gibbs measures. We see that the continuous symmetry

$$\phi_x \to \phi_x + t, \quad t \in \mathbb{R},$$

is spontaneously broken. Indeed, by construction, any constant configuration is J-harmonic. For random walks with finite variance, i.e.

$$\sum_{x \in \mathbb{Z}^d} |x|^2 |J(0,x)| < \infty, \tag{2.2.12}$$

the transience requirement forces $d \geqslant 3$, cf. e.g. [71]. This is in accord with the general Mermin-Wagner theory, which asserts that spontaneous breaking of a continuous symmetry for an interaction satisfying (2.2.12) can only occur for $d \geqslant 3$, see e.g. [33]. We also have a breaking of translation invariance, since there may well be non constant harmonic configurations (for example for the massless free field (0.7) any hyperplane $\phi_x = u \cdot x$, $u \in \mathbb{R}^d$, is harmonic).

Remark 2.2. Theorem 2.1 shows that all extremal elements of $\mathcal{G}$ are Gaussian. On the other hand it is clear that a mixture of Gaussians need not be Gaussian. This is the case if and only if the measure specifying its extremal decomposition is itself Gaussian. Here we want to show that by taking suitable mixtures we can exhibit elements of $\mathcal{G}$ which are shift invariant Gaussians and have arbitrarily large second moments. This non-compact character of the set of Gibbs measures will be seen to have important consequences in the next chapter. Let $\theta_0 \in \mathbb{T}^d$ be a zero of $\hat{J}$ (e.g. $\theta_0 = 0$). For any $T > 0$ let ρ_T be the centered Gaussian field with spectral density

$$\hat{\rho}_T = \frac{T}{2}\,(2\pi)^d\,(\delta_{\theta_0} + \delta_{-\theta_0})\ ,$$

or, in other words, ρ_T is the centered Gaussian field with covariance

$$\rho_T(\sigma_x\sigma_y) = T\,\cos\theta_0 \cdot (x-y)\,, \quad x, y \in \mathbb{Z}^d\,.$$

(cf. e.g. [40], Proposition (13.A7) to check that ρ_T is well defined). Then, for any $T > 0$,

$$\mu_T = \int \gamma^\sigma \rho_T(d\sigma) \tag{2.2.13}$$

is a centered Gaussian field, since it is a convolution of two centered Gaussian measures. Moreover it is translation invariant, since its covariance

$$G_T(x,y) = G(x,y) + T\,\cos\theta_0 \cdot (x-y) \quad x, y \in \mathbb{Z}^d\,,$$

is homogeneous on $\mathbb{Z}^d$. In view of Theorem 2.1, to show that $\mu_T \in \mathcal{G}$ we have to check that $\rho_T(\mathcal{H}_J) = 1$. But this follows from

$$\rho_T\big(|\sum_{y\in\mathbb{Z}^d} J(x,y)\sigma_y|^2\big) = T \sum_{y,z\in\mathbb{Z}^d} J(x,y)J(x,z)\cos\theta_0\cdot(y-z) = T\hat{J}(\theta_0)^2 = 0\,.$$

Remark 2.3. In finite range models condition (2.2.9) is automatically satisfied. Moreover, there can be configurations in $\mathcal{H}_J$ which have exponential growth at infinity ([25]). It is then useful to restrict oneself to measures on smaller subsets of Ω. One can introduce sets of configurations $\overline{\Omega} \subset \Omega$, and define *tempered Gibbs* *measures* as the set of $\mu \in \mathcal{G}$ such that $\mu(\overline{\Omega}) = 1$. A natural choice in finite range models is $\overline{\Omega} = \mathcal{S}'(\mathbb{Z}^d)$, where $\mathcal{S}'(\mathbb{Z}^d)$ denotes the space of configurations which increase at most polynomially at infinity, [25]. On the other hand, in long-range models such as e.g. the α-stable model (0.8), temperedness is automatically implied by (2.2.9).

2.1.3 Extremal vs. ergodic Gibbs states

A general result in the theory of Gibbs measures (see [40], Theorem (14.17)) states that a shift invariant Gibbs measure has a unique decomposition in terms of ergodic Gibbs measures. But an ergodic Gibbs measure need not be extremal in $\mathcal{G}$ in general (an ergodic measure is not necessarily tail-trivial). Indeed, there are many well known models in statistical mechanics in which a shift invariant Gibbs measures can be written as mixture of non shift invariant elements of $\mathcal{G}_{\text{ext}}$. Typically, these are non-ferromagnetic models such as e.g. the Ising anti-ferromagnet, cf. [29], section 2. However, irreducibility also plays a role. Namely the above phenomenon can also happen in ferromagnetic systems with non-irreducible couplings[2].

In this subsection we prove that for harmonic crystals with an irreducible random walk this cannot happen, at least if we restrict to tempered measures. In other words all ergodic Gibbs measures are extremal in $\mathcal{G}$ and "extremal shift invariant Gibbs = shift invariant extremal Gibbs". It is also not difficult to give an example of a model with non-irreducible random walk where this is violated. Recall that shift invariant elements of $\mathcal{G}_{\text{ext}}$ are given by the measures γ_t, $t \in \mathbb{R}$, where $\gamma_t = \gamma^\eta$ with $\eta_x = t \in \mathbb{R}$ for all x.

Theorem 2.2. *Let μ be a shift invariant Gibbs measure for $J = \mathbb{1} - p$ with p irreducible, and assume μ has finite first moments, i.e. $\mu(|\phi_0|) < \infty$. Then μ is a mixture of shift invariant elements of $\mathcal{G}_{\text{ext}}$. Namely, there exists a probability measure ρ_0 on $\mathbb{R}$ with finite first moment, such that*

$$\mu = \int_{-\infty}^{\infty} \gamma_t \, \rho_0(\,\mathrm{d}t)\,. \tag{2.2.14}$$

Proof. Let μ be given as in the hypothesis, and let $\rho \in \mathcal{M}_1(\Omega)$ be its extremal decomposition in $\mathcal{G}$, i.e.

$$\mu = \int_{\Omega} \gamma^\sigma \, \rho(\,\mathrm{d}\sigma)\,. \tag{2.2.15}$$

---

[2] The following example is due to A. van Enter (private communication). Consider the 2D Ising model at low temperature, with couplings $J_{xy} = 1$ if $x - y$ is in the even sub-lattice $2\mathbb{Z}^2$, with $|x-y| = 2$, and $J_{xy} = 0$ otherwise. Then the even and the odd sub-lattices are decoupled. Let μ_1 (μ_2) be the product measure obtained by choosing the '+' state on the even (odd) sub-lattice and the '−' state on the odd (even) sub-lattice. Then μ_1 and μ_2 are non-translation invariant extremal Gibbs measures, but the mixture $\frac{1}{2}(\mu_1 + \mu_2)$ is translation invariant.

In particular, cf. (2.2.11), $\rho(\mathcal{H}_J) = 1$. Moreover, ρ is shift invariant and has finite first moments. We want to show that ρ is actually concentrated on the constant elements of $\mathcal{H}_J$. This would reduce the integral on Ω in (2.2.15) to the integral on $\mathbb{R}$ in (2.2.14), thereby proving the theorem. It is sufficient to show that for any $x, y \in \mathbb{Z}^d$, for any $\beta > 0$,

$$\rho\,\{|\phi_x - \phi_y| > \beta\} = 0\,. \tag{2.2.16}$$

In order to prove (2.2.16) we may proceed as follows. Observe that $\phi \in \mathcal{H}_J$ is equivalent to $p_t\phi = \phi$ for all $t \geqslant 0$, where

$$p_t(x, y) = \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!}\, p^n(x, y)$$

is the continuous time kernel on $\mathbb{Z}^d$ with Fourier transform $\exp(-t\hat{J})$. Then we can replace ϕ in (2.2.16) by $p_t\phi$, and by Chebyshev's inequality and the shift invariance of ρ,

$$\rho\,\{|\phi_x - \phi_y| > \beta\} \leqslant \beta^{-1}\,\rho(|\phi_0|) \sum_{z\in\mathbb{Z}^d} |p_t(x, z) - p_t(y, z)|\,.$$

Since t is arbitrary, (2.2.16) follows from

$$\lim_{t\to\infty} \sum_{z\in\mathbb{Z}^d} |p_t(x, z) - p_t(y, z)| = 0\,. \tag{2.2.17}$$

If p is the simple random walk, (2.2.17) appears in [52], Corollary 1.2.3. Using the standard local limit theorem estimates (cf. [71], chapter 7) we may extend (2.2.17) to any irreducible random walk with finite range. For a general irreducible random walk we can use the following argument, cf. [20]. Let $K > 0$ be a large number such that $\delta \in (0, 1)$ if

$$\delta = \sum_{x:\,|x|\,\leqslant\,K} p(0, x)\,.$$

Define two homogeneous transitions $r(x, y)$ and $s(x, y)$, by

$$\begin{aligned} r(x, y) &= \delta^{-1} p(x, y) 1_{\{|x-y|\,\leqslant\,K\}}\,,\\ s(x, y) &= (1-\delta)^{-1} p(x, y) 1_{\{|x-y|>K\}}\,. \end{aligned}$$

Notice that since p is irreducible, K can be chosen in such a way that r is irreducible. If r_t and s_t are the corresponding continuous time kernels, then it is easy, using e.g. the Fourier transform, to see that

$$p_t(x, z) = \sum_{z'\in\mathbb{Z}^d} r_{\delta t}(x, z')\, s_{(1-\delta)t}(z', z)\,.$$

We have

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} |p_t(x,z)-p_t(y,z)| &\leqslant \sum_{z,z'\in\mathbb{Z}^d} |r_{\delta t}(x,z')-r_{\delta t}(y,z')|\, s_{(1-\delta)t}(z',z)\\ &= \sum_{z'\in\mathbb{Z}^d} |r_{\delta t}(x,z')-r_{\delta t}(y,z')| \end{aligned} \tag{2.2.18}$$

Now (2.2.17) follows from (2.2.18) since r has finite range. □

A non-irreducible random walk which violates Theorem 2.2 is e.g. the simple random walk on the even sub-lattice,

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } x-y\in 2\mathbb{Z}^d \text{ and } |x-y|=2\\ 0 & \text{otherwise}\end{cases}$$

Then, choosing $\theta_0 = (\pi,\dots,\pi)$, we have $\hat{p}(\theta_0)=1$, i.e. $\hat{J}(\theta_0)=0$. Then the measure μ_T, $T>0$, given in (2.2.13) is a shift invariant element of $\mathcal{G}$, but the oscillating covariance G_T shows that μ_T cannot have the form (2.2.14).

2.2 Thermodynamics

We define the sets of translation invariant measures

$$\mathcal{K}_L = \left\{\mu\in\mathcal{M}_1^S(\Omega)\,,\quad \mu(\phi_0^2)\leqslant L\right\}\,,\qquad \mathcal{K}_\infty = \bigcup_{L\,\geqslant\,0}\mathcal{K}_L \tag{2.2.19}$$

A tightness argument shows that $\mathcal{K}_L$ is compact in the weak topology for any $L\geqslant 0$.

Given two measures μ,ν, let μ_N,ν_N denote their marginals on Λ_N. The *relative entropy* of μ with respect to ν on Λ_N is defined by

$$H_N(\mu|\nu) = \begin{cases} \int d\mu_N \log\dfrac{d\mu_N}{d\nu_N} & \text{if } \mu_N\ll\nu_N\\ +\infty & \text{otherwise}\end{cases} \tag{2.2.20}$$

Let also $H_N(\mu)$ denote the relative entropy of μ with respect to Lebesgue measure on Λ_N. The *entropy density* of $\mu\in\mathcal{M}_1^S(\Omega)$ is defined as

$$s(\mu) = -\lim_{N\to\infty} N^{-d} H_N(\mu) \tag{2.2.21}$$

A general subadditivity argument shows that this limit always exists, although it may be equal to $-\infty$. Moreover, s is *affine* (see e.g. chapter 15 of [40]). The *internal energy density* of $\mu\in\mathcal{K}_\infty$ is defined as

$$e(\mu) = \frac{1}{2}\lim_{N\to\infty} N^{-d}\sum_{x,y\in\Lambda_N} J(x,y)\mu(\phi_x\phi_y)\,. \tag{2.2.22}$$

As a consequence of the absolute summability of J and the shift invariance of μ, we obtain that, for $\mu \in \mathcal{K}_\infty$, the limit exists and equals

$$e(\mu) = \frac{1}{2} \sum_{y \in \mathbb{Z}^d} J(0,y)\mu(\phi_0\phi_y)\,. \tag{2.2.23}$$

The *pressure* is defined by

$$p = \lim_{N\to\infty} N^{-d} \log Z^0_{\Lambda_N}\,, \tag{2.2.24}$$

where $Z^0_{\Lambda_N}$ is given by (2.2.4) with $\tau = 0$. By Theorem 2.5 in [47] the above limit exists and equals

$$p = \frac{1}{2}\log(2\pi) + \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \log f(\theta)d\theta \tag{2.2.25}$$

where $f = \hat{J}^{-1}$ is the spectral density of P (cf. (1.1.6)). We define the *relative entropy density* of μ w.r.t. ν as

$$h(\mu|\nu) = \lim_{N\to\infty} N^{-d} H_N(\mu|\nu)\,. \tag{2.2.26}$$

Proposition 2.3. *The function $h(\,\cdot\,|P)$ is well defined on $\mathcal{M}_1^S(\Omega)$, and*

$$h(\mu|P) = \begin{cases} +\infty & \text{if } \mu \in \mathcal{M}_1^S(\Omega)\setminus\mathcal{K}_\infty \\ e(\mu) \;-\; s(\mu) \;+\; p & \text{if } \mu \in \mathcal{K}_\infty \end{cases} \tag{2.2.27}$$

Moreover, $h(\,\cdot\,|P)$ is affine on $\mathcal{K}_\infty$.

Proof. Let $\mu \in \mathcal{M}_1(\Omega)$. We check the first line of (2.2.27). An alternative expression for H_N is given by (cf. e.g. Lemma (3.2.13) of [23])

$$H_N(\mu|P) = \sup_\psi \left\{\mu(\psi) \;-\; \log P(\mathrm{e}^\psi)\right\} \tag{2.2.28}$$

where the supremum is taken over bounded $\mathcal{F}_{\Lambda_N}-$measurable functions. Let $\alpha \in (0, \sigma^2/2)$, so that $\log P(\mathrm{e}^{\alpha\phi_0^2}) \equiv c(\alpha) < \infty$. Then choosing $\psi_T(\phi) = \alpha(\phi_0^2 \wedge T)$, $T > 0$ in (2.2.28) gives

$$H_N(\mu|P) \geqslant \alpha\mu(\phi_0^2 \wedge T) \;-\; c(\alpha)\,.$$

Letting $T \to \infty$, this implies $H_N(\mu|P) \geqslant \alpha\mu(\phi_0^2) - c(\alpha)$. Hence,

$$h(\mu|P) = \infty\,, \quad \mu \notin \mathcal{K}_\infty\,.$$

We turn to the proof of the identity in the second line of (2.2.27). We may assume that μ_N has a density, otherwise the identity holds trivially, since $s(\mu) = -\infty$ in this case. Let ψ_N be the density of P_N, and call ψ_N^0 the density of $\gamma^0_{\Lambda_N}$, cf. (2.2.3). We have

$$H_N(\mu|P) = H_N(\mu) - \mu(\log\psi_N^0) + \mu(\log\psi_N^0\psi_N^{-1})$$

Since

$$\lim_{N\to\infty} N^{-d}\mu(\log\psi_N^0) = -p - e(\mu)\,,$$

we need to prove

$$\lim_{N\to\infty} N^{-d}\mu(\log\psi_N^0\psi_N^{-1}) = 0\,. \tag{2.2.29}$$

Recall that

$$\psi_N^0\psi_N^{-1}(\phi) = \det\left(G_N\Gamma_N^{-1}\right)^{1/2}\exp\left(-\frac{1}{2}\,\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle_{\Lambda_N}\right),$$

where $G_N \equiv G_{\Lambda_N}$ and $\Gamma_N \equiv \Gamma_{\Lambda_N}$. The argument of Theorem 2.5 in [47] shows that

$$\lim_{N\to\infty} N^{-d}\log\det\left(G_N\Gamma_N^{-1}\right) = 0\,.$$

We are left with

$$\lim_{N\to\infty} N^{-d}\mu\left[\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\right] = 0\,. \tag{2.2.30}$$

Let $x, y \in \Lambda_N$. From (1.1.13) we have

$$\begin{aligned}\sum_{y\in\Lambda_N}\left(\Gamma_N^{-1}(x,y) - G_N^{-1}(x,y)\right) &= \mathbb{P}_x\left(1 < \tilde\sigma_{\Lambda_N} < \infty\right) \\ &\leqslant \mathbb{P}_x\left(1 < \tilde\sigma_{\Lambda_N}\right) = \mathbb{P}_x\left(\xi_1 \notin \Lambda_N\right)\,.\end{aligned} \tag{2.2.31}$$

Since $\mu \in \mathcal{K}_\infty$, using Schwarz' inequality we write

$$\left|\mu\left[\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\right]\right| \leqslant \mu(\phi_0^2)\sum_{x\in\Lambda_N}\mathbb{P}_x\left(\xi_1\notin\Lambda_N\right) \tag{2.2.32}$$

Then (2.2.30) follows from

$$\lim_{N\to\infty} N^{-d}\sum_{x\in\Lambda_N}\mathbb{P}_x\left(\xi_1\notin\Lambda_N\right) = 0\,. \tag{2.2.33}$$

To prove (2.2.33) we may use the following straightforward argument. Let $\delta > 0$ and define the cubes $\Lambda_N^\delta = N[-1/2+\delta, 1/2-\delta]^d \cap \mathbb{Z}^d$, so that $\Lambda_N^\delta \subset \Lambda_N$ and a site $x \in \Lambda_N^\delta$ is at a distance at least δN from any $y \notin \Lambda_N$. Then, for any $\delta > 0$ it is obvious that

$$\sup_{x\in\Lambda_N^\delta}\mathbb{P}_x\left(\xi_1\notin\Lambda_N\right)\to 0\,,\quad N\to\infty\,. \tag{2.2.34}$$

On the other hand $N^{-d}|\Lambda_N \setminus \Lambda_N^\delta| \leqslant c\delta$ for some $c < \infty$, uniformly on N. Therefore (2.2.33) follows from (2.2.34) and the arbitrariness of δ. ☐

2.3 Variational principle

In classical statistical mechanics Gibbs' variational principle can be stated as follows. A measure $\mu \in \mathcal{M}_1^S(\Omega)$ is a minimum for the *free energy*

$$f(\mu) = e(\mu) \; - \; s(\mu)$$

if and only if μ is a translation invariant Gibbs state. In view of Proposition 2.3 this can be turned into a statement about relative entropy density. Define

$$\mathcal{G}_2^S = \mathcal{G} \cap \mathcal{K}_\infty \, ,$$

the set of translation invariant Gibbs states with finite second moments.

Theorem 2.4. *Let $\mu \in \mathcal{M}_1^S(\Omega)$. Then,*

$$\mu \in \mathcal{G}_2^S \quad \Longleftrightarrow \quad h(\mu|P) = 0 \tag{2.2.35}$$

Proof. Let $\mu \in \mathcal{G}_2^S$, and denote by $\rho \in \mathcal{M}_1(\Omega)$ its extremal decomposition,

$$\mu = \int \gamma^\sigma \, \rho(\,\mathrm{d}\sigma) \, , \quad \rho(\mathcal{H}_J) = 1 \, .$$

Note that $\mu \in \mathcal{K}_\infty$ implies $\rho \in \mathcal{K}_\infty$. By convexity

$$H_N(\mu|P) \leqslant \int H_N(\gamma^\sigma|P) \, \rho(\,\mathrm{d}\sigma) \, . \tag{2.2.36}$$

Moreover, by explicit computation,

$$H_N(\gamma^\sigma|P) = \frac{1}{2} \, \left\langle \sigma, G_N^{-1} \sigma \right\rangle \, .$$

For $\sigma \in \mathcal{H}_J$ we can write

$$\left\langle \sigma, G_N^{-1} \sigma \right\rangle = \left\langle \sigma, (G_N^{-1} - J_N) \sigma \right\rangle \; - \sum_{x \in \Lambda_N} \sum_{y \notin \Lambda_N} J(x,y) \sigma_x \sigma_y \, .$$

Here $J_N = \Gamma_N^{-1}$ is the restriction of J to Λ_N. We can now use the argument of Proposition 2.3, cf. (2.2.30)-(2.2.33), and the fact that ρ is concentrated on $\mathcal{H}_J$ to obtain

$$\lim_{N \to \infty} N^{-d} \rho \left[\left\langle \sigma, G_N^{-1} \sigma \right\rangle \right] = 0 \, .$$

From (2.2.36) we see that

$$h(\mu|P) = \lim_{N \to \infty} N^{-d} H_N(\mu|P) = 0 \, .$$

We now turn to the more delicate part of (2.2.35). Namely, suppose $\mu \in \mathcal{M}_1^S(\Omega)$, and

$$h(\mu|P) = 0\,. \tag{2.2.37}$$

We are going to prove that μ satisfies (2.2.9) and the DLR equations (2.2.10). Note that Proposition 2.3 implies that μ has finite second moments, and therefore (2.2.9) follows from

$$\mu\Big(\sum_{y\in\mathbb{Z}^d} |J(x,y)|\,|\phi_y|\Big) = \mu\left(|\phi_0|\right) \sum_{y\in\mathbb{Z}^d} |J(x,y)| < \infty\,, \quad x \in \mathbb{Z}^d\,.$$

Let g be a local, bounded continuous function, and let $\Lambda \in \mathbb{F}$. We will show

$$\mu(\gamma_\Lambda(g)) = \mu(g)\,. \tag{2.2.38}$$

Let $V \in \mathbb{F}$ be a large set containing both Λ and the support of g. We write

$$f_V = \frac{d\mu_V}{dP_V}$$

for the density of μ w.r.t. P on V. Then, since $\mu(g) = P(f_V g)$, we have

$$\begin{aligned} &|\mu(\gamma_\Lambda(g)) - \mu(g)| \\ &\quad\leqslant |\mu(\gamma_\Lambda(g)) \,-\, P(f_V\gamma_\Lambda(g))| + |P(f_V\gamma_\Lambda(g)) \,-\, P(f_V g)|\,. \end{aligned} \tag{2.2.39}$$

To prove (2.2.38) we show that both terms in the r.h.s. of (2.2.39) can be made arbitrarily small by choosing V large enough. To be more precise, (2.2.38) is a consequence of the next two lemmas.

Lemma 2.5. *For any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{Z}_+$ such that for any $V \supset \Lambda_N$*

$$|\mu(\gamma_\Lambda(g)) \,-\, P(f_V\gamma_\Lambda(g))| < \varepsilon \tag{2.2.40}$$

Lemma 2.6. *For any $\varepsilon > 0$ and for all $N \in \mathbb{Z}_+$, there exists a finite set $V = V(\varepsilon, N)$, such that $V \supset \Lambda_N$, and*

$$|P(f_V\gamma_\Lambda(g)) \,-\, P(f_V g)| < \varepsilon \tag{2.2.41}$$

Note that Lemma 2.5 becomes trivial if p has finite range. In the non-local case the continuity expressed by (2.2.40) replaces the classical argument based on quasilocality.

Proof of Lemma 2.5. Notice that

$$P(f_V\gamma_\Lambda(g)) = P(f_V P(\gamma_\Lambda(g)|\mathcal{F}_V)) = \mu(P(\gamma_\Lambda(g)|\mathcal{F}_V))\,. \tag{2.2.42}$$

Define

$$\gamma^{\tau}_{\Lambda,V}(g) = \gamma_{\Lambda}^{\tau_V 0_{V^c}}(g)\,,$$

i.e. we put 0-boundary conditions outside of V. Note that $\gamma_{\Lambda,V}(g)$ is now a $\mathcal{F}_V$-measurable function. Let us fix a sequence $\{V_N\}$, with $V_N \supset \Lambda_N$, $N \in \mathbb{Z}_+$. We define

$$\varphi_N = \gamma_\Lambda(g) - \gamma_{\Lambda,V_N}(g)\,.$$

Using (2.2.42) we see that for any N

$$|\mu(\gamma_\Lambda(g)) - P(f_{V_N}\gamma_\Lambda(g))| \leqslant \mu(|\varphi_N|) + \mu(P(|\varphi_N|\,|\mathcal{F}_{V_N}))\,.$$

To prove (2.2.40) it is then sufficient to show

$$\lim_{N\to\infty} \mu(|\varphi_N|) = 0 \tag{2.2.43}$$

and

$$\lim_{N\to\infty} \mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) = 0\,. \tag{2.2.44}$$

We may assume g to be a bounded Lipshitz-continuous local function. For such a function, define

$$\|g\|_{x,\mathrm{L}} = \sup\left\{ \frac{|g(\sigma)-g(\eta)|}{|\sigma_x - \eta_x|}\,,\quad \sigma,\eta\in\Omega:\ \sigma_y = \eta_y,\ \forall\, y\neq x \right\}\,.$$

From (2.2.5) and (2.2.6) we know that the measure $\gamma_\Lambda^{\tau_{V_N} 0_{V_N^c}}$ can be seen as the measure γ^τ_Λ shifted by the mean values vector

$$m_x^N(\tau) \equiv \gamma_\Lambda^{\tau_{V_N} 0_{V_N^c}}(\phi_x) - \gamma_\Lambda^\tau(\phi_x) = \sum_{y\in\Lambda}\sum_{z\notin V_N} \Gamma_\Lambda(x,y)J(y,z)\tau_z$$

We then have

$$\begin{aligned} |\varphi_N(\tau)| &\leqslant \int \gamma^\tau_\Lambda(\,\mathrm{d}\phi)\,|g(\phi + m^N(\tau)) - g(\phi)| \\ &\leqslant \sum_{x\in\Lambda} \|g\|_{x,\mathrm{L}}|m_x^N(\tau)| \leqslant C(g,\Lambda)\sum_{y\in\Lambda}\sum_{z\notin V_N}|J(y,z)|\,|\tau_z|\,, \end{aligned} \tag{2.2.45}$$

having defined

$$C(g,\Lambda) = |\Lambda|\,\left(\sup_{x\in\Lambda}\|g\|_{x,\mathrm{L}}\right)\,\left(\sup_{x,y\in\Lambda}|\Gamma_\Lambda(x,y)|\right)\,.$$

Using the shift invariance of μ and the fact that $\mu \in \mathcal{K}_\infty$, we have

$$\begin{aligned}\mu(|\varphi_N|) &\leqslant C(g,\Lambda) \sum_{y\in\Lambda}\sum_{z\notin V_N} |J(y,z)|\, \mu(|\phi_z|)\\ &= C(g,\Lambda)\, \mu(|\phi_0|) \sum_{y\in\Lambda}\sum_{z\notin V_N} |J(y,z)|\,. \end{aligned} \tag{2.2.46}$$

Now, the r.h.s. of (2.2.46) vanishes as $N\to\infty$, and (2.2.43) follows.

We turn to the proof of (2.2.44). From (2.2.45) we have

$$\mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) \leqslant C(g,\Lambda) \sum_{y\in\Lambda}\sum_{z\notin V_N} |J(y,z)|\, \mu(P(|\phi_z|\,|\mathcal{F}_{V_N}))\,. \tag{2.2.47}$$

Let us fix $\tau\in\Omega_J$ and denote by $P(\cdot|\tau_{V_N})$ the conditional expectation $P(\cdot|\mathcal{F}_{V_N})$ evaluated at τ_{V_N}. Then by the random walk representation (appendix A), $P(\cdot|\tau_{V_N})$ is the Gaussian field on $\Omega_{V_N^c}$ with mean

$$P(\phi_z|\tau_{V_N}) = \sum_{y\in V_N} \mathbb{P}_z\left(\xi_{s_{V_N}} = y\right)\tau_y\,, \quad z\in V_N^c\,, \tag{2.2.48}$$

and covariance

$$\begin{aligned}&P(\phi_z\phi_{z'}|\tau_{V_N}) - P(\phi_z|\tau_{V_N})P(\phi_{z'}|\tau_{V_N})\\ &\qquad = \Gamma_{V_N^c}(z,z') = \mathbb{E}_z\left[\sum_{n=0}^{s_{V_N}-1} 1_{\xi_n=z'}\right]\,, \quad z,z'\in V_N^c\,,\end{aligned} \tag{2.2.49}$$

where s_{V_N} is the time of first entrance in V_N. We then have

$$\begin{aligned}P(\phi_z^2\,|\tau_{V_N}) &= \Gamma_{V_N^c}(z,z) + (P(\phi_z|\tau_{V_N}))^2\\ &\leqslant \sigma^2 + \sum_{x,x'\in V_N} |\tau_x\tau_{x'}|\mathbb{P}_z(\xi_{s_{V_N}}=x)\mathbb{P}_z(\xi_{s_{V_N}}=x')\,,\end{aligned}$$

where we have used (1.1.2) to bound $\Gamma_{V_N^c}(z,z)$. We now use Schwarz' inequality and the shift invariance of μ to write

$$\mu\left(P(\phi_z^2\,|\mathcal{F}_{V_N})\right) \;\leqslant\; \sigma^2 \;+\; \mu(\tau_0^2)\,.$$

Therefore,

$$\mu\left(P(|\phi_z|\,|\mathcal{F}_{V_N})\right) \leqslant \mu\left(P(\phi_z^2\,|\mathcal{F}_{V_N})\right)^{1/2} \;\leqslant\; \left(\sigma^2+\mu(\tau_0^2)\right)^{1/2}\,. \tag{2.2.50}$$

Plugging (2.2.50) into (2.2.47) shows that

$$\mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) \leqslant C(g,\Lambda)\,\left(\sigma^2+\mu(\tau_0^2)\right)^{1/2} \sum_{y\in\Lambda}\sum_{z\notin V_N} |J(y,z)| \to 0\,,$$

as $N \to \infty$. This completes the proof of Lemma 2.5. $\square$

Proof of Lemma 2.6. Since $f_{V\setminus\Lambda} = d\mu_{V\setminus\Lambda}/dP_{V\setminus\Lambda}$ is $\mathcal{F}_{\Lambda^c}-$measurable, the DLR property of P implies

$$P(f_{V\setminus\Lambda}\gamma_\Lambda(g)) = P(f_{V\setminus\Lambda}g)\,.$$

By adding and subtracting $P(f_{V\setminus\Lambda}g)$ we then write

$$|P(f_V\gamma_\Lambda(g)) - P(f_V g)| \leqslant 2\|g\|_\infty P(|f_{V\setminus\Lambda} - f_V|)\,.$$

The Lemma now follows from a classical argument (cf. [66, 40]), which we reproduce here for the sake of completeness. We shall adapt to our setting the proof given in [40], Step 2 in the proof of Theorem (15.37).

Consider the function

$$\psi(x) = 1 - x + x\log x\,, \quad x \geqslant 0.$$

It is strictly convex and it equals zero only at $x = 1$. Moreover, a simple graphical consideration shows that for any $\epsilon > 0$, there exists a number $r = r(\epsilon) \in (0,\infty)$, such that

$$|x - 1| \leqslant r\psi(x) + \frac{\epsilon}{2} \tag{2.2.51}$$

Now recall that the relative entropy on $V \in \mathbb{F}$ is an increasing function of V and is given by

$$H_V(\mu|P) = P(f_V \log f_V),$$

and therefore

$$H_V(\mu|P) - H_{V\setminus\Lambda}(\mu|P) = P(f_{V\setminus\Lambda}\psi(f_V/f_{V\setminus\Lambda})). \tag{2.2.52}$$

Using (2.2.51) and (2.2.52) we see that for any $\epsilon > 0$, we can find $r = r(\epsilon) \in (0,\infty)$, such that

$$P(|f_V - f_{V\setminus\Lambda}|) \leqslant r\,\left[H_V(\mu|P) - H_{V\setminus\Lambda}(\mu|P)\right] + \frac{\epsilon}{2}. \tag{2.2.53}$$

We are going to show that, for fixed Λ, the r.h.s. of (2.2.53) can be made arbitrarily small by choosing V large enough. More precisely, we claim that, for any $\delta > 0$, for any $N \in \mathbb{Z}_+$ with $\Lambda \subset \Lambda_N$, we can find $V = V(\delta) \in \mathbb{F}$, such that $\Lambda \subset \Lambda_N \subset V$ and

$$|H_V(\mu|P) - H_{V\setminus\Lambda}(\mu|P)| \leqslant \delta \tag{2.2.54}$$

By choosing $\delta = \epsilon/2r$, the result (2.2.41) follows from (2.2.53) and (2.2.54). We are left with the proof of (2.2.54). We fix $\delta > 0$ and $N \in \mathbb{Z}_+$ with $\Lambda \subset \Lambda_N$. From (2.2.37) we see that we can choose an integer $M \geqslant N$, with $\Lambda_M \supset \Lambda_N$, and

$$H_M(\mu|P) \leqslant \frac{\delta}{2^d N^d} M^d. \tag{2.2.55}$$

Let $m = [M/N]$, the integer part. We choose m^d sites $x(k)$, such that $\Lambda_N(k) \equiv \Lambda_N + x(k)$, $k = 1, \ldots, m^d$, are pairwise disjoint subsets of Λ_M, and we set

$$W(k) = \bigcup_{i \leqslant k} \Lambda_N(i)\,, \qquad \Lambda(k) = \Lambda + x(k)\,.$$

From the monotonicity of relative entropies we then have

$$\begin{aligned}
&\sum_{k=1}^{m^d} \big(H_{W(k)}(\mu|P) - H_{W(k)\setminus\Lambda(k)}(\mu|P)\big) \qquad (2.2.56)\\
&\qquad \leqslant \sum_{k=1}^{m^d} \big(H_{W(k)}(\mu|P) - H_{W(k)\setminus\Lambda_N(k)}(\mu|P)\big) = H_{W(m^d)}(\mu|P) \leqslant H_M(\mu|P).
\end{aligned}$$

Dividing by m^d and using $m \geqslant M/2N$, from (2.2.55) and (2.2.56) we have

$$m^{-d} \sum_{k=1}^{m^d} \big(H_{W(k)}(\mu|P) - H_{W(k)\setminus\Lambda(k)}(\mu|P)\big) \leqslant \delta$$

which shows that at least for an index k we have

$$H_{W(k)}(\mu|P) - H_{W(k)\setminus\Lambda(k)}(\mu|P) \leqslant \delta.$$

The claim now follows by setting $V = W(k) - x(k)$, since the shift invariance of μ and P implies

$$H_{W(k)}(\mu|P) - H_{W(k)\setminus\Lambda(k)}(\mu|P) = H_V(\mu|P) - H_{V\setminus\Lambda}(\mu|P).$$

$\square$

Chapter 3

Volume-order large deviations

3.1 Main result

In this chapter we discuss volume order large deviations of the empirical field with respect to the measure P. From now on, in addition to the usual assumptions (1.1.1) and (1.1.2), we assume that the coupling coefficients satisfy the following mild decay assumption.

Assumption H

- there exists $\beta > 0$, such that

$$\lim_{|x|\to\infty} |x|^{d+\beta} p(0,x) = 0\,. \tag{3.3.1}$$

We define the function

$$\mathcal{I}(\mu) = \begin{cases} +\infty & \mu \in \mathcal{M}_1(\Omega) \setminus \mathcal{M}_1^S(\Omega) \\ h(\mu|P) & \mu \in \mathcal{M}_1^S(\Omega) \end{cases} \tag{3.3.2}$$

The next theorem states that the empirical field

$$R_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \delta_{\theta_x\,\phi}\,, \tag{3.3.3}$$

satisfies a weak-type large deviations principle with rate function $\mathcal{I}$. Recall the definition (2.2.19) of the compact sets $\mathcal{K}_L \subset \mathcal{M}_1^S(\Omega)$.

Theorem 3.1. *For any $L > 0$, the restriction of $\mathcal{I}$ to $\mathcal{K}_L$ is lower semi-continuous. Moreover, the following bounds hold:*

- *For any closed set $F \subset \mathcal{M}_1(\Omega)$, for any $L > 0$,*

$$\limsup_{N\to\infty} N^{-d} \log P\left(R_N \in F \cap \mathcal{K}_L\right) \leqslant -\inf_{\mu\in F\cap\mathcal{K}_L} \mathcal{I}(\mu)\,. \tag{3.3.4}$$

- *For any open set* $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \in G\right) \geqslant - \inf_{\mu \in G} \mathcal{I}(\mu)\,. \tag{3.3.5}$$

Remark 3.1. Since the upper bound (3.3.4) is restricted to compact sets we speak of a *weak* large deviations principle. The standard way to remove this restriction would be to prove exponential tightness in the form

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{-d} \log P\left(R_N \notin \mathcal{K}_L\right) = -\infty\,.$$

But this is false for the massless field P. Indeed, for fixed $L > 0$, the measure μ_T defined by (2.2.13) (choosing $\theta_0 = 0$ there) is not in $\mathcal{K}_L$, for any $T > L - \sigma^2$. But $\mu_T \in \mathcal{G}_2^S$, and therefore the lower bound (3.3.5) together with Theorem 2.4 implies

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \notin \mathcal{K}_L\right) \geqslant - h(\mu_T|P) = 0\,.$$

Note, moreover, that the upper bound (3.3.4) does not extend to *any* compact subset of $\mathcal{M}_1^S(\Omega)$ (it suffices to consider compact sets of measures which have finite first moment but infinite second moment). It is also not difficult to see that, in contrast to the hypercontractive case discussed in Theorem 3.2 below, $\mathcal{I}$ cannot be lower semi-continuous on the whole of $\mathcal{M}_1(\Omega)$. Indeed, consider e.g. sequences $\mu_n \in \mathcal{G}$ given by (see also (2.2.14))

$$\mu_n = \int_{-\infty}^{\infty} \gamma_t\, \psi_n(t)\ \mathrm{d}t\,, \tag{3.3.6}$$

where $\psi_n \in L^2(\mathbb{R})$ is a probability density such that ψ_n converges in $L^1(\mathbb{R})$ to some $\psi \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$. Then $\mu_n \to \mu$ weakly, as $n \to \infty$, where μ is the mixture given by $\psi(t)$. From Theorem 2.4 we have $\mathcal{I}(\mu_n) = 0$ for all n. On the other hand, since $\mu \notin \mathcal{K}_\infty$, Proposition 2.3 shows that $\mathcal{I}(\mu) = \infty$. This violates lower semi-continuity.

Idea of proof

The proof of Theorem 3.1 will be given in the following sections. We follow the argument introduced in [5] for the case of finite range, irreducible random walks. The main line of the proof of the upper bound can be roughly described as follows. We perturb the original measure P by removing the singularity of its spectral density, cf. (3.3.7). This defines a massive Gaussian field P^ε, $\varepsilon > 0$, with bounded spectral density. The field P^ε is weakly correlated and hypercontractive. Moreover, P^ε satisfies a strong large deviations principle, cf. Theorem 3.2. This is a multidimensional version of the sort of results proven in section 5.4 of [23], see also [18]. The hypercontractivity properties are obtained via Nelson's celebrated theorem [64], see Lemma B.2 in the appendix. We then show that the bounds satisfied by P^ε are stable (provided we restrict to uniformly finite second moments) when the mass is removed (or, equivalently, the singularity in the spectral density is restored), cf. Theorem 3.5. This is the most delicate part of the proof. In particular, the arguments of [5] have to be refined, see Lemma 3.8.

The proof of the lower bound in Theorem 3.1 is rather standard, see Proposition 3.4. We use the idea of ergodic approximations, as e.g. in [32], see Proposition 3.4 below. Our proof of Theorem 3.1 also shows, however, how the lower bound for P can be in principle derived from a lower bound on P^ε.

3.2 The massive field

For each $\varepsilon > 0$, we define the measure $P^\varepsilon \in \mathcal{M}_1(\Omega)$, as the centered Gaussian field on $\mathbb{Z}^d$ with spectral density (cf. (1.1.6))

$$f^\varepsilon = (\hat{J} + \varepsilon)^{-1} = \frac{f}{1 + \varepsilon f}\,. \tag{3.3.7}$$

In other words, P^ε is the centered Gaussian field with covariance

$$G^\varepsilon(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f^\varepsilon(\theta)\, \mathrm{e}^{i\,(y-x)\cdot\theta}\, \mathrm{d}\theta\,. \tag{3.3.8}$$

3.2.1 Random walk with uniform killing

We shall use the following random walk representation for G^ε. Let us define the transition probabilities

$$p_\varepsilon(x,y) = \frac{p(x,y)}{1+\varepsilon}\,. \tag{3.3.9}$$

We may consider the random walk on $\mathbb{Z}^d$ which moves from x to y according to $p_\varepsilon(x,y)$, which is killed at each step with a probability $\varepsilon/(1+\varepsilon)$. The *killing time* τ_ε is then an independent, exponentially distributed random variable, and assuming that killing starts at time zero we set

$$\mathbb{P}(\tau_\varepsilon \geqslant n) = \mathrm{e}^{-m(\varepsilon)(n+1)}\,, \quad m(\varepsilon) = \log(1+\varepsilon)\,. \tag{3.3.10}$$

We refer to this process as the random walk with uniform killing rate $m(\varepsilon)$. Since f^ε is the Fourier transform of $(1+\varepsilon)^{-1}(\mathbb{1} - p_\varepsilon)^{-1}$, we have

$$\begin{aligned} G^\varepsilon(x,y) &= (1+\varepsilon)^{-1} \sum_{n=0}^{\infty} p_\varepsilon^n(x,y) \\ &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n)\,;\ \tau_\varepsilon \geqslant n+1\right] \end{aligned} \tag{3.3.11}$$

where, for simplicity, we use the symbols $\mathbb{P}_x, \mathbb{E}_x$ also for the coupling of the free random walk ξ with the random variable τ_ε. We then interpret G^ε as the Green function of the random walk with uniform killing (with killing starting at time zero). Let $\Lambda \in \mathbb{F}$ and define

$$G^\varepsilon_\Lambda = \{G^\varepsilon(x,y)\}_{x,y\in\Lambda}\,. \tag{3.3.12}$$

Then an application of Lemma A.1 yields

$$\left(G^\varepsilon_\Lambda\right)^{-1} = (1+\varepsilon)\left(\mathbb{1}_N - \tilde{p}^\varepsilon_\Lambda\right)\,, \tag{3.3.13}$$

with renormalized transition

$$\tilde{p}^\varepsilon_\Lambda(x,y) = \mathbb{P}_x\left(\xi_{\tilde{\sigma}_\Lambda} = y\,;\ \tau_\varepsilon > \tilde{\sigma}_\Lambda\right) \tag{3.3.14}$$

where $\tilde{\sigma}_\Lambda$ is the hitting time (1.1.11) and $x, y \in \Lambda$.

Remark 3.2. From (3.3.10) and (3.3.11) we see that

$$\sum_{y\in\mathbb{Z}^d} G^\varepsilon(x,y) = \mathbb{E}(\tau_\varepsilon) = \frac{1}{\varepsilon}\,. \tag{3.3.15}$$

In particular, G^ε is absolutely summable, i.e. the Gaussian field P^ε is weakly correlated. In the case of a finite range random walk p, it is easy to see that the transformation $P \to P^\varepsilon$ results in the creation of a *mass gap*, i.e. the massless field P is transformed into a massive field P^ε with exponentially decaying correlations. On the other hand, when p has long range it is clear that G^ε cannot decay exponentially, since $G^\varepsilon(x,y) \geqslant (1+\varepsilon)^{-1}p(x,y)$. With some abuse we shall still call P^ε a massive field. As we show in the appendix, the assumption (3.3.1) is sufficient to prove hypercontractivity of P^ε for any $\varepsilon > 0$. For a thorough discussion of properties of inverses of matrices with decaying off-diagonal coefficients we refer to [39]. For instance, it follows from the results of [39] that if $p(x,y) \sim |x-y|^{-d-\alpha}$, then $G^\varepsilon(x,y) \sim |x-y|^{-d-\alpha}$ as well, for any $\varepsilon, \alpha > 0$.

3.2.2 Large deviations principle

Let $h(\mu|P^\varepsilon)$ be the relative entropy density associated to the measure P^ε, cf. (2.2.26). We define

$$\mathcal{I}_\varepsilon(\mu) = \begin{cases} +\infty & \mu \in \mathcal{M}_1(\Omega)\setminus\mathcal{M}_1^S(\Omega) \\ h(\mu|P^\varepsilon) & \mu \in \mathcal{M}_1^S(\Omega) \end{cases} \tag{3.3.16}$$

Theorem 3.2. *The function $\mathcal{I}_\varepsilon : \mathcal{M}_1(\Omega) \to [0,\infty]$ is lower semi-continuous and its level sets*

$$\{\mu \in \mathcal{M}_1(\Omega) : \ \mathcal{I}(\mu) \leqslant L\}, \quad L \geqslant 0$$

are compact. For each $\mu \in \mathcal{K}_\infty$ it satisfies

$$\mathcal{I}_\varepsilon(\mu) = \mathcal{I}(\mu) + \frac{\varepsilon}{2}\mu(\phi_0^2) + \phi(\varepsilon), \quad \mu \in \mathcal{K}_\infty, \tag{3.3.17}$$

where $\phi(\varepsilon)$ is given by

$$\phi(\varepsilon) = -\frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \log(1 + \varepsilon f(\theta))\, \mathrm{d}\theta. \tag{3.3.18}$$

Moreover,

- *For any closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\lim_{N\to\infty} N^{-d} \log P^\varepsilon(R_N \in F) \leqslant - \inf_{\mu \in F} \mathcal{I}_\varepsilon(\mu), \tag{3.3.19}$$

- *For any open set $G \subset \mathcal{M}_1(\Omega)$,*

$$\lim_{N\to\infty} N^{-d} \log P^\varepsilon(R_N \in G) \geqslant - \inf_{\mu \in G} \mathcal{I}_\varepsilon(\mu). \tag{3.3.20}$$

We first check the following properties of the rate function $\mathcal{I}_\varepsilon$.

Lemma 3.3. *For each $\varepsilon > 0$, $\mathcal{I}_\varepsilon$ satisfies the variational principle*

$$\mathcal{I}_\varepsilon(\mu) = 0 \iff \mu = P^\varepsilon, \quad \mu \in \mathcal{M}_1(\Omega) \tag{3.3.21}$$

and the identity

$$\mathcal{I}_\varepsilon(\mu) = \mathcal{I}(\mu) + \frac{\varepsilon}{2}\mu(\phi_0^2) + \phi(\varepsilon), \quad \mu \in \mathcal{K}_\infty. \tag{3.3.22}$$

Proof. From Proposition 2.3 we have

$$h(\mu|P^\varepsilon) = e^\varepsilon(\mu) - s(\mu) + p^\varepsilon \tag{3.3.23}$$

where the entropy s is given by (2.2.21) (independent of ε), and e^ε, p^ε are the internal energy and the pressure associated to the potential $J + \varepsilon 1\!\!1$. According to (2.2.25) and (3.3.7) we see that

$$p^\varepsilon - p = \phi(\varepsilon).$$

Then (3.3.22) follows from the obvious identity (cf. (2.2.23))

$$e^\varepsilon(\mu) - e(\mu) = \frac{\varepsilon}{2}\mu(\phi_0^2).$$

To prove the variational principle we may use Theorem 2.4 (which of course applies to the massive field) and the fact that P^ε is the only shift invariant Gibbs measure with finite second moment for the potential $J + \varepsilon 1\!\!1$, cf. e.g. Corollary (13.40) in [40]. □

Lower bound

Since our proof of the estimate (3.3.20) does not rely on hypercontractivity, we state and prove the following proposition directly in the massless case $\varepsilon = 0$.

Proposition 3.4. *For any $\mu \in \mathcal{M}_1^S(\Omega)$ and for any open neighbourhood G of μ,*

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \in G\right) \geqslant \, -\mathcal{I}(\mu)\,. \tag{3.3.24}$$

Proof. A classical change of measure argument, see e.g. Lemma 5.4.21 in [23], implies that (3.3.24) is always satisfied if μ is ergodic with respect to lattice shifts. The general case follows from ergodic approximations. Namely, let $\mu \in \mathcal{M}_1^S(\Omega)$, and assume $\mu \in \mathcal{K}_\infty$ (otherwise (3.3.24) is trivial). Let ν_N^x be the marginal of μ on the shifted box $\Lambda_N + Nx$ for $x \in \mathbb{Z}^d$ and define the product measure

$$\nu_N = \prod_{x\in\mathbb{Z}^d} \nu_N^x\,.$$

Then the measure

$$\mu_N = N^{-d} \sum_{x\in\Lambda_N} \theta_x \nu_N$$

is shift invariant ($\theta_x\nu$ stands for the shifted measure $\nu \circ \theta_x^{-1}$), and $\mu_N \in \mathcal{K}_\infty$. Also, it is not difficult to show that μ_N converges weakly to μ as $N \to \infty$, and that μ_N is ergodic for each fixed N, see e.g. Theorem (14.12) of [40]. Let us check that

$$\lim_{N\to\infty} h(\mu_N|P) = h(\mu|P)\,. \tag{3.3.25}$$

Thanks to Proposition 2.3 we need to prove

$$s(\mu_N) \to s(\mu) \quad \text{and} \quad e(\mu_N) \to e(\mu)\,.$$

The convergence of entropy is proven in Proposition (16.34) of [40]. On the other hand, the convergence of the internal energy is explicitly checked by using the expression (2.2.23). Once we have the convergence (3.3.25) it is clear that (3.3.24) follows from the lower bound for ergodic measures. □

Upper bound

We establish the hypercontractivity of P^ε in the appendix B, see Lemma B.2. Once this is done, the proof of the upper bound in Theorem 3.2 follows the standard line, cf. section 5.4 of [23]. We point out the main steps needed.

Step 1. Let $Q \subset\subset \mathbb{Z}^d$ be a fixed cube containing the origin. Using Lemma B.2 one can show that for all bounded measurable functions f such that $S_f \subset Q$, the limit

$$\Lambda_Q(f) = \lim_{N\to\infty} N^{-d} \log P^\varepsilon\Big(\exp \sum_{x\in\Lambda_N} f \circ \theta_x\Big) \tag{3.3.26}$$

exists, cf. (B.B.9). An application of Hölder's inequality shows that Λ_Q is a convex functional. Its *Legendre transform* is the convex function on $\mathcal{M}_1(\Omega_Q)$ given by

$$\Lambda_Q^*(\mu) = \sup_f \; \{\mu(f) \; - \; \Lambda_Q(f)\} \; , \tag{3.3.27}$$

where the supremum is taken over continuous bounded f with $S_f \subset Q$.

Step 2. Following Lemma 5.4.13 and Corollary 5.1.11 in [23], one can check that Λ^* has compact level sets on $\mathcal{M}_1(\Omega_Q)$ and for each closed set $F_Q \subset \mathcal{M}_1(\Omega_Q)$,

$$\limsup_{N\to\infty} N^{-d} \log P^\varepsilon \left(R_N \circ \pi_Q^{-1} \in F_Q\right) \leqslant \; - \inf_{\mu \in F_Q} \Lambda_Q^*(\mu) \, , \tag{3.3.28}$$

with $R_N \circ \pi_Q^{-1}$ denoting the marginal of R_N on Q.

Step 3. Define

$$\Lambda^*(\mu) = \sup_Q \Lambda_Q^*(\mu \circ \pi_Q^{-1}) \, , \quad \mu \in \mathcal{M}_1(\Omega) \, ,$$

where the supremum is taken over all finite cubes containing the origin. By definition, the function Λ^* is non-negative, convex and lower semicontinuous. Moreover, the standard projective limit argument (cf. e.g. [21], Theorem 4.6.1) shows that Λ^* has compact level sets, and (3.3.28) can be lifted to

$$\limsup_{N\to\infty} N^{-d} \log P^\varepsilon \left(R_N \in F\right) \leqslant \; - \inf_{\mu \in F} \Lambda^*(\mu) \, , \tag{3.3.29}$$

for all closed sets $F \subset \mathcal{M}_1(\Omega)$.

Step 4. With the same arguments as in [23], Lemma 5.4.17, one then proves the identity

$$\Lambda^*(\mu) = \mathcal{I}_\varepsilon(\mu) \, , \quad \mu \in \mathcal{M}_1(\Omega) \, .$$

From step 3, this implies that $\mathcal{I}_\varepsilon$ is lower semi-continuous, has compact level sets and satisfies the upper bound (3.3.19) of Theorem 3.2.

3.3 Removing the mass

We are now in a position to start the proof of Theorem 3.1. The semi-continuity of $\mathcal{I}$ on each $\mathcal{K}_L$ is a consequence of (3.3.17) and the semi-continuity of $\mathcal{I}_\varepsilon$. We turn to the proof of the bounds (3.3.4) and (3.3.5). Let $\varepsilon > 0$ be fixed, and choose an integer N. We define a new measure $P^{N,\varepsilon} \in \mathcal{M}_1(\Omega)$ by

$$\frac{\mathrm{d}P^{N,\varepsilon}}{\mathrm{d}P}(\phi) = F_{N,\varepsilon}(\phi) \equiv \left(Z_{N,\varepsilon}\right)^{-1} \exp\left(-\frac{\varepsilon}{2} \sum_{x \in \Lambda_N} \phi_x^2\right) . \tag{3.3.30}$$

The normalizing factor $Z_{N,\varepsilon}$ can be computed

$$Z_{N,\varepsilon} = P\left[\exp\left(-\frac{\varepsilon}{2}\sum_{x\in\Lambda_N}\phi_x^2\right)\right] = \det\left(1\!\!\mathrm{I}_N + \varepsilon G_N\right)^{-1/2}.$$

In particular, the argument of Theorem 2.5 in [47] shows that

$$\lim_{N\to\infty} N^{-d}\log Z_{N,\varepsilon} = \phi(\varepsilon)\,. \tag{3.3.31}$$

where $\phi(\varepsilon)$ is given in (3.3.18).

The key intermediate step in deriving the bounds of Theorem 3.1 is given by the following result.

Theorem 3.5. *For each closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\limsup_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(R_N \in F\right) \leqslant -\inf_{\mu\in F}\mathcal{I}_\varepsilon(\mu)\,, \tag{3.3.32}$$

and for each open set $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(R_N \in G\right) \geqslant -\inf_{\mu\in G}\mathcal{I}_\varepsilon(\mu)\,. \tag{3.3.33}$$

Let us assume, in fact, the validity of Theorem 3.5. Let $L > 0$ be fixed. For each closed set $F \subset \mathcal{M}_1(\Omega)$, setting $F_L \equiv F \cap \mathcal{K}_L$, we have

$$\begin{aligned} P\left(R_N \in F_L\right) &= P^{N,\varepsilon}\left(F_{N,\varepsilon}^{-1}\,;\, R_N \in F_L\right) \\ &\leqslant Z_{N,\varepsilon}\,\exp\left(\varepsilon\, LN^d/2\right) P^{N,\varepsilon}\left(R_N \in F_L\right)\,, \end{aligned} \tag{3.3.34}$$

where we have used the fact that if $R_N(\phi) \in \mathcal{K}_L$, then

$$\sum_{x\in\Lambda_N}\phi_x^2 \leqslant L\,N^d\,.$$

On the other hand, for any open set $G \subset \mathcal{M}_1(\Omega)$, we have

$$\begin{aligned} P\left(R_N \in G\right) &= P^{N,\varepsilon}\left(F_{N,\varepsilon}^{-1}\,;\, R_N \in G\right) \\ &\geqslant Z_{N,\varepsilon}\,P^{N,\varepsilon}\left(R_N \in G\right)\,. \end{aligned} \tag{3.3.35}$$

Let us prove the upper bound (3.3.4). From (3.3.34) and (3.3.32) we have

$$\limsup_{N\to\infty} N^{-d}\log P\left(R_N \in F_L\right) \leqslant -\inf_{\mu\in F_L}\mathcal{I}_\varepsilon(\mu) + \phi(\varepsilon) + \frac{1}{2}\,\varepsilon\, L\,.$$

Using (3.3.17) we have

$$\limsup_{N\to\infty} N^{-d}\log P\left(R_N \in F_L\right) \leqslant -\inf_{\mu\in F_L}\mathcal{I}(\mu) + \frac{1}{2}\,\varepsilon\, L\,. \tag{3.3.36}$$

Then (3.3.4) follows from (3.3.36) letting $\varepsilon \to 0$. To prove the lower bound (3.3.5), we show that for each $\mu \in G$,

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \in G\right) \geqslant -\mathcal{I}(\mu)\,. \tag{3.3.37}$$

In view of Proposition 2.3, we may assume $\mu \in \mathcal{K}_\infty$. From (3.3.35) and (3.3.33), we have

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \in G\right) \geqslant -\mathcal{I}_\varepsilon(\mu) + \phi(\varepsilon)\,.$$

Using (3.3.17) this implies

$$\liminf_{N\to\infty} N^{-d} \log P\left(R_N \in G\right) \geqslant -\mathcal{I}(\mu) - \frac{1}{2}\,\varepsilon\mu(\phi_0^2)\,.$$

Letting $\varepsilon \to 0$ we obtain (3.3.37). This completes the proof of Theorem 3.1.

Before proving the bounds (3.3.32) and (3.3.33) of Theorem 3.5, we shall discuss the random walk representation for the field $P^{N,\varepsilon}$.

3.3.1 Random walk with killing inside Λ_N

For $x, y \in \mathbb{Z}^d$, we define the transition probabilities

$$p_{\varepsilon,N}(x,y) = \begin{cases} p_\varepsilon(x,y) & \text{if } x \in \Lambda_N \\ p(x,y) & \text{if } x \notin \Lambda_N \end{cases} \quad, y \in \mathbb{Z}^d \tag{3.3.38}$$

where $p_\varepsilon(x,y) = p(x,y)/(1+\varepsilon)$, cf. (3.3.9). We consider the random walk on $\mathbb{Z}^d$ generated by $p_{\varepsilon,N}(x,y)$, i.e. the process with killing occurring *only inside* Λ_N. Then its Green function is given by

$$\begin{aligned} G^{\varepsilon,N}(x,y) &= (1+\varepsilon)^{-1} \sum_{n=0}^{\infty} p^n_{\varepsilon,N}(x,y) \\ &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n)\, \mathrm{e}^{-m(\varepsilon)\sum_{k=0}^n 1_{\Lambda_N}(\xi_n)}\right]. \end{aligned} \tag{3.3.39}$$

Remark 3.3. We may introduce the killing time $\tau_{\varepsilon,N}$ for the random walk with transition $p_{\varepsilon,N}$. We observe that there is an obvious *coupling* $\tilde{\mathbb{P}}_x$ between the walk ξ^ε with uniform killing and the walk $\xi^{\varepsilon,N}$ with killing inside Λ_N when both start at $x \in \Lambda_N$. Namely, we let the two processes coincide with the free random walk ξ until the time $\tau_\varepsilon - 1$, see (3.3.10). Then, if $\xi_{\tau_\varepsilon - 1} \in \Lambda_N$, we set $\tau_{\varepsilon,N} = \tau_\varepsilon$ so that the two walkers die together. On the other hand, if $\xi_{\tau_\varepsilon-1} \notin \Lambda_N$, the second walker survives and keeps jumping until the first come-back to Λ_N. Here we start again the exponential clock associated to τ_ε until the next exit and so on. In particular, until the first exit time

$$\sigma_N = \inf\{n \geqslant 0\,,\ \xi_n \notin \Lambda_N\} \tag{3.3.40}$$

the two processes $\xi^{\varepsilon,N}$ and ξ^ε coincide. Let

$$\mathcal{T} = \inf\{n \geqslant 0\,,\ \xi^\varepsilon_n \neq \xi^{\varepsilon,N}_n\}\,.$$

We then have, for any $x \in \Lambda_N$,

$$\tilde{\mathbb{P}}_x\left(\mathcal{T} < \infty\right) = \tilde{\mathbb{P}}_x\left(\tau_{\varepsilon,N} > \tau_\varepsilon\right) \leqslant \mathbb{P}_x\left(\tau_\varepsilon > \sigma_N\right)\,. \tag{3.3.41}$$

Lemma 3.6. *Let $\Lambda \subset \Lambda_N$. Then the marginal of $P^{N,\varepsilon}$ on Λ is the centered Gaussian measure with covariance*

$$G^{\varepsilon,N}_\Lambda = \{G^{\varepsilon,N}(x,y)\}_{x,y\in\Lambda}\,. \tag{3.3.42}$$

Moreover, letting

$$\tilde{p}^{\varepsilon,N}_\Lambda(x,y) = \mathbb{P}_x\left(\xi_{\tilde{\sigma}_\Lambda} = y\,;\ \tau_{\varepsilon,N} > \tilde{\sigma}_\Lambda\right) \tag{3.3.43}$$

where $\tilde{\sigma}_\Lambda$ is the hitting time (1.1.11) and $x, y \in \Lambda$, we have

$$\left(G^{\varepsilon,N}_\Lambda\right)^{-1} = (1+\varepsilon)\left(\mathbb{1}_N - \tilde{p}^{\varepsilon,N}_\Lambda\right)\,. \tag{3.3.44}$$

Proof. From the definition (3.3.30) we see that the marginal of $P^{N,\varepsilon}$ on Λ_N is the centered Gaussian measure with covariance

$$C^{\varepsilon,N} = \left(G_N^{-1} + \varepsilon\mathbb{1}_N\right)^{-1}\,. \tag{3.3.45}$$

From Lemma A.1 we have $G_N^{-1} = \mathbb{1}_N - \tilde{p}_N$ and we can write

$$\begin{aligned} C^{\varepsilon,N} &= (1+\varepsilon)^{-1}\left(\mathbb{1}_N - (1+\varepsilon)^{-1}\tilde{p}_N\right)^{-1} \\ &= (1+\varepsilon)^{-1}\sum_{n=0}^{\infty}(1+\varepsilon)^{-n}\tilde{p}_N^n\,. \end{aligned} \tag{3.3.46}$$

From (3.3.46) and (3.3.39) we see that

$$C^{\varepsilon,N} = G^{\varepsilon,N}\,.$$

Then the marginal of $P^{N,\varepsilon}$ on $\Lambda \subset \Lambda_N$ is the centered Gaussian measure with covariance $G^{\varepsilon,N}_\Lambda$ given in (3.3.42). Finally, by applying Lemma A.1 to (3.3.46), we check that the effective couplings in (3.3.44) are given precisely by (3.3.43). □

Lemma 3.7. *For any $\Lambda \subset \Lambda_N$ and $x \in \Lambda$, we have (cf. (3.3.43) and (3.3.14))*

$$\sum_{y\in\Lambda}\left[\tilde{p}_\Lambda^{\varepsilon,N}(x,y)-\tilde{p}_\Lambda^{\varepsilon}(x,y)\right] \leqslant 2\,\mathbb{P}_x\left(\tau_\varepsilon > \sigma_N\right)\,. \tag{3.3.47}$$

Proof. We use the coupling argument introduced in the above remark, cf. (3.3.41). Indeed, by (3.3.43) and (3.3.14) we have

$$\begin{aligned}\sum_{y\in\Lambda}\left[\tilde{p}_\Lambda^{\varepsilon,N}(x,y)-\tilde{p}_\Lambda^{\varepsilon}(x,y)\right] &\leqslant 2\,\tilde{\mathbb{P}}_x\left(\mathcal{T}<\infty\right)\\ &\leqslant 2\,\mathbb{P}_x\left(\tau_\varepsilon > \sigma_N\right)\,.\end{aligned}$$

$\square$

3.3.2 Proof of Theorem 3.5

We start with the proof of the upper bound (3.3.32). We use here the projective limit approach. Let $M \in \mathbb{Z}_+$ and let $\pi_M : \Omega \to \Omega_M$ be the canonical projection $\phi \to \phi_{\Lambda_M}$. We will also use the notation $\mu \circ \pi_M^{-1}$ to denote the marginal μ_M on Λ_M of a measure $\mu \in \mathcal{M}_1(\Omega)$. The Dawson-Gärtner theorem (see e.g. Theorem 4.6.1 in [21]) shows that in order to establish the upper bound (3.3.32) it is enough to prove, for each $M \in \mathbb{Z}_+$, for each closed set $F_M \subset \mathcal{M}_1(\Omega_M)$,

$$\begin{aligned}&\limsup_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(R_N\circ\pi_M^{-1}\in F_M\right)\\ &\qquad\leqslant\ -\inf\left\{\mathcal{I}_\varepsilon(\mu)\,,\ \mu\in\mathcal{M}_1(\Omega):\ \ \mu_M\in F_M\right\}\,.\end{aligned} \tag{3.3.48}$$

We need to introduce the so called periodized empirical field $\tilde{R}_N$, defined by

$$\tilde{R}_N(\phi) = N^{-d}\sum_{x\in\Lambda_N}\delta_{\theta_x\phi_N}\,, \tag{3.3.49}$$

where ϕ_N denotes the configuration on Λ_N periodized to $\mathbb{Z}^d$, i.e.

$$(\phi_N)_x = \begin{cases}\phi_x & \text{if } x\in\Lambda_N\\ \phi_{\bar{x}} & \text{if } x_i=\bar{x}_i \bmod N,\ i=1,\ldots,d,\ \bar{x}\in\Lambda_N\end{cases} \tag{3.3.50}$$

With this definition $\tilde{R}_N(\phi)$ is translation invariant, i.e. $\tilde{R}_N(\phi) \in \mathcal{M}_1^S(\Omega)$ for all $\phi \in \Omega$. Moreover, the map $\phi \to R_N(\phi)$ is $\mathcal{F}_{\Lambda_N}$-measurable. It is well known, and easy to see that as far as the large deviations behaviour is concerned (3.3.49) is equivalent to the empirical field (0.11). Indeed, for any bounded local function f we have

$$\lim_{N\to\infty}\sup_{\phi\in\Omega}\left|\tilde{R}_N(\phi)(f)-R_N(\phi)(f)\right| = 0\,, \tag{3.3.51}$$

where $R_N(\phi)(f)$ denotes the mean of f w.r.t. $R_N(\phi)$. In particular, (3.3.48) is equivalent to the statement obtained by replacing R_N with $\tilde{R}_N$. The same holds for the lower bound of Theorem 3.5.

Let us fix $M \in \mathbb{Z}^+$ and denote by ρ_M a metric on $\mathcal{M}_1(\Omega_M)$, compatible with the weak topology and such that ρ_M is dominated by the total variation norm (cf. e.g. [28]). For a given $\delta \in (0,1)$, define

$$\tilde{R}_{N,\delta}(\phi) = \frac{1}{|\Lambda_{N,\delta}|} \sum_{x \in \Lambda_{N,\delta}} \delta_{\theta_x \phi_N} ,$$

where

$$\Lambda_{N,\delta} \equiv [-(1-\delta)N/2, (1-\delta)N/2]^d \cap \mathbb{Z}^d .$$

It is easy to check that there exist constants $\delta_0(d) > 0$ and $c(d) < \infty$, only depending on d, such that for any bounded measurable function f, for any $N \in \mathbb{Z}^+$,

$$\sup_{\phi \in \Omega} \left| \tilde{R}_N(\phi)(f) - \tilde{R}_{N,\delta}(\phi)(f) \right| < \delta\, c(d) \|f\|_\infty$$

provided $\delta < \delta_0(d)$. This implies

$$\sup_{\phi \in \Omega} \rho_M \left(\tilde{R}_N(\phi), \tilde{R}_{N,\delta}(\phi) \right) < \delta\, c(d) \qquad (3.3.52)$$

for $\delta < \delta_0(d)$, and integers $N > M$. Now observe that, if e.g. $N > 2M/\delta$, the map

$$\phi \to \tilde{R}_{N,\delta}(\phi) \circ \pi_M^{-1}$$

is $\mathcal{F}_{\Lambda_{N,\delta/2}}$-measurable. Let us define the density

$$\phi_{N,\delta} = \frac{\mathrm{d}P^{N,\varepsilon}_{N,\delta/2}}{\mathrm{d}P^{\varepsilon}_{N,\delta/2}} , \qquad (3.3.53)$$

where $P^{N,\varepsilon}_{N,\delta/2}$ and $P^{\varepsilon}_{N,\delta/2}$ are the marginals of $P^{N,\varepsilon}$ and P^ε respectively, on the box $\Lambda_{N,\delta/2}$. We define the set

$$F^{(\delta)} = \{\nu \in \mathcal{M}_1(\Omega_M) \, : \, \rho_M(\nu, F_M) \leqslant \delta\, c(d)\} .$$

Then, by (3.3.52),

$$\begin{aligned} P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in F_M\right) &\leqslant P^{N,\varepsilon}\left(\tilde{R}_{N,\delta} \circ \pi_M^{-1} \in F^{(\delta)}\right) \\ &= P^{\varepsilon}\left(\phi_{N,\delta} \, ; \, \tilde{R}_{N,\delta} \circ \pi_M^{-1} \in F^{(\delta)}\right) . \end{aligned}$$

For each $1 < p < \infty$, $q = p/(p-1)$, Hölder's inequality yields

$$\log P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in F_M\right) \leqslant \frac{1}{p}\log P^{\varepsilon}\left(\phi^p_{N,\delta}\right) + \frac{1}{q}\log P^{\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in F^{(2\delta)}\right), \tag{3.3.54}$$

where we have used once more (3.3.52). We shall need the following key approximation result.

Lemma 3.8. *Let $\phi_{N,\delta}$ be the density defined in (3.3.53). For all $p \geqslant 1$, $\varepsilon > 0$ and $\delta > 0$, we have*

$$\lim_{N\to\infty} N^{-d}\log P^{\varepsilon}\left(\phi^p_{N,\delta}\right) = \lim_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(\phi^{-p}_{N,\delta}\right) = 0. \tag{3.3.55}$$

Let us show that Lemma 3.8 allows to prove the desired estimate. Suppose (3.3.55) holds. Then (3.3.54) together with the upper bound of Theorem 3.2 imply

$$\limsup_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in F_M\right) \leqslant -\frac{1}{q}\inf\left\{\mathcal{I}_\varepsilon(\mu)\,,\ \mu \in \mathcal{M}_1(\Omega)\ :\ \mu\circ\pi_M^{-1} \in F^{(2\delta)}\right\}.$$

This implies (3.3.48) since q and δ are arbitrarily close to 1 and 0 respectively. The proof of (3.3.32) is complete.

For the lower bound (3.3.33) we may use similar arguments. Namely, let us fix an open set $G_M \subset \mathcal{M}_1(\Omega_M)$, for a given $M \in \mathbb{Z}^+$. Given $\delta > 0$ define

$$G^{(\delta)} = \{\mu \in \mathcal{M}_1(\Omega_M)\ :\ \rho_M(\mu, G_M^c) \geqslant \delta\, c(d)\}\,.$$

Then, by (3.3.52)

$$P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in G_M\right) \geqslant P^{N,\varepsilon}\left(\tilde{R}_{N,\delta} \circ \pi_M^{-1} \in G^{(\delta)}\right) = P^{\varepsilon}\left(\phi_{N,\delta}\,;\ \tilde{R}_{N,\delta} \circ \pi_M^{-1} \in G^{(\delta)}\right).$$

For any $1 < p < \infty$, $p/q = p-1$, Hölder's inequality yields

$$\log P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in G_M\right) \geqslant -\frac{q}{p}\log P^{N,\varepsilon}\left(\phi^{-p}_{N,\delta}\right) + q\log P^{\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in G^{(2\delta)}\right). \tag{3.3.56}$$

From Lemma 3.8, the arbitrariness of q and δ, and from the lower bound of Theorem 3.2, we have

$$\liminf_{N\to\infty} N^{-d}\log P^{N,\varepsilon}\left(\tilde{R}_N \circ \pi_M^{-1} \in G_M\right) \geqslant -\mathcal{I}_\varepsilon(\mu)\,,$$

for any $\mu \in \mathcal{M}_1(\Omega)$ such that $\mu_M \in G_M$. This implies (3.3.33). The proof of Theorem 3.5 is completed. $\square$

Proof of Lemma 3.8. We fix $\varepsilon > 0$ and $\delta > 0$. Let $G^\varepsilon, G^{\varepsilon,N}$ be the covariance functions of the fields P^ε and $P^{N,\varepsilon}$ respectively, see (3.3.11) and Lemma 3.6. Let us define the matrices

$$C_N = \left\{G^{\varepsilon,N}(x,y)\right\}_{x,y\in\Lambda_{N,\delta/2}}\,, \quad D_N = \left\{G^{\varepsilon}(x,y)\right\}_{x,y\in\Lambda_{N,\delta/2}}\,.$$

Then (3.3.53) can be written

$$\phi_{N,\delta}(\sigma) = \det\left(D_N C_N^{-1}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\left\langle\sigma,(C_N^{-1} - D_N^{-1})\sigma\right\rangle_{\Lambda_{N,\delta/2}}\right). \tag{3.3.57}$$

An explicit computation shows that

$$P^\varepsilon\left(\phi^p_{N,\delta}\right) = \det\left(\mathbb{I}_N + B_N\right)^{p/2}\det\left(\mathbb{I}_N + p\,B_N\right)^{-\frac{1}{2}}\,, \tag{3.3.58}$$

where we have defined

$$B_N = D_N C_N^{-1} - \mathbb{I}_N\,,$$

$\mathbb{I}_N$ denoting the unit diagonal matrix on $\Lambda_{N,\delta/2}$. Moreover, setting $q = p/(p-1)$,

$$P^{N,\varepsilon}\left(\phi^{-p}_{N,\delta}\right) = \det\left(\mathbb{I}_N + B_N\right)^{-p/2q}\det\left(\mathbb{I}_N - \frac{p}{q}\,B_N\right)^{-\frac{1}{2}}. \tag{3.3.59}$$

Let us denote by

$$\{\lambda_j(N)\,,\ \ j = 1,\dots,|\Lambda_{N,\delta}|\}$$

the set of eigenvalues of the matrix B_N. Observe that, by elementary matrix calculus, we have

$$\sup_j\,|\lambda_j(N)| \leqslant \|B_N\| \equiv \sup_{x\in\Lambda_{N,\delta}}\sum_{y\in\Lambda_{N,\delta}}|B_N(x,y)|\,. \tag{3.3.60}$$

From (3.3.58),

$$\begin{aligned}\log P^\varepsilon\left(\phi^p_{N,\delta}\right) &= \frac{p}{2}\sum_i \log\left(1+\lambda_i\right) - \frac{1}{2}\sum_i \log\left(1+p\lambda_i\right)\\ &\leqslant \frac{p}{2}|\Lambda_{N,\delta}|\log\left(1+\|B_N\|\right) - \frac{1}{2}|\Lambda_{N,\delta}|\log\left(1-p\|B_N\|\right).\end{aligned}$$

A similar computation for (3.3.59) shows

$$\begin{aligned}&\log P^{N,\varepsilon}\left(\phi^{-p}_{N,\delta}\right)\\ &\quad\leqslant\ -\frac{p}{2q}|\Lambda_{N,\delta}|\log\left(1-\|B_N\|\right) - \frac{1}{2}|\Lambda_{N,\delta}|\log\left(1-\frac{p}{q}\|B_N\|\right).\end{aligned}$$

Corresponding lower bounds on $\log P^{\varepsilon}\left(\phi^{p}_{N,\delta}\right)$ and $\log P^{N,\varepsilon}\left(\phi^{-p}_{N,\delta}\right)$ can be obtained in the same way and we see that in order to prove (3.3.55) it is sufficient to show

$$\lim_{N\to\infty} \|B_N\| = 0\,. \tag{3.3.61}$$

We have

$$\|B_N\| \leqslant \|D_N\|\,\|C_N^{-1}\ -\ D_N^{-1}\|\,,$$

where the norm $\|\cdot\|$ is defined as in (3.3.60). From the absolute summability of G^{ε}, see (3.3.15), it follows that for each $\varepsilon > 0$

$$\sup_{N\in\mathbb{Z}_+} \|D_N\| \leqslant \frac{1}{\varepsilon}\,.$$

Then (3.3.61) will follow from

$$\lim_{N\to\infty} \|C_N^{-1}\ -\ D_N^{-1}\| = 0\,. \tag{3.3.62}$$

Thanks to (3.3.13) and Lemma 3.6 we can write, for $x, y \in \Lambda_{N,\delta/2}$

$$C_N^{-1}(x,y) - D_N^{-1}(x,y) = (1+\varepsilon)\left[\tilde{p}^{\varepsilon,N}_{\Lambda_{N,\delta/2}}(x,y) - \tilde{p}^{\varepsilon}_{\Lambda_{N,\delta/2}}(x,y)\right]\,.$$

It then follows from Lemma 3.7 that

$$\|D_N^{-1} - C_N^{-1}\| \leqslant 2(1+\varepsilon) \sup_{x\in\Lambda_{N,\delta/2}} \mathbb{P}_x\left(\tau_\varepsilon > \sigma_N\right)\,. \tag{3.3.63}$$

Now recall that

$$\mathbb{P}\left(\tau_\varepsilon > l\right) \leqslant \mathrm{e}^{-m(\varepsilon)\,l}\,, \quad l > 0\,.$$

We then estimate, for any sufficiently large integer n,

$$\begin{aligned}\mathbb{P}_x\left(\tau_\varepsilon > \sigma_N\right) &\leqslant \sum_{l=0}^{n} \mathbb{P}_x\left(\sigma_N = l\right) + \sum_{l=n+1}^{\infty} \mathbb{P}\left(\tau_\varepsilon > l\right) \\ &\leqslant n\,\mathbb{P}_x\left(\sigma_N \leqslant n\right) + \mathrm{e}^{-m(\varepsilon)\,n/2}\,.\end{aligned} \tag{3.3.64}$$

On the other hand, since $x \in \Lambda_{N,\delta/2}$ is at a distance at least $\delta N/2$ from Λ_N^{c}, for each fixed $n > 0$, $\delta > 0$,

$$\sup_{x\in\Lambda_{N,\delta/2}} \mathbb{P}_x\left(\sigma_N \leqslant n\right) \leqslant n\mathbb{P}_0\left(|\xi_1| \geqslant \frac{\delta N}{2n}\right) \to 0\,, \quad N\to\infty\,. \tag{3.3.65}$$

Since n is arbitrary in (3.3.64), this implies (3.3.62) and the Lemma follows. ☐

3.4 On the effect of boundary conditions

In this section we discuss the large deviations of $\tilde{R}_N$ under finite volume Gibbs measures with prescribed boundary conditions. Consider first the case of *zero boundary conditions*. We set $P_N^0 = \gamma_{\Lambda_N}^0$ for the corresponding Gibbs field. It is not hard to show that Theorem 3.1 holds if we replace P by P_N^0. One first checks that the rate functions must coincide, since

$$\lim_{N\to\infty} N^{-d} H_N(\mu|P_N^0) = h(\mu|P)\,, \quad \mu \in \mathcal{M}_1^S(\Omega)$$

as proved in Proposition 2.3. Moreover, for any closed set $F \subset \mathcal{M}_1(\Omega)$, $1 < p < \infty$, by Hölder's inequality we have

$$P_N^0\big(\tilde{R}_N \in F\big) \leqslant P\big(\tilde{R}_N \in F\big)^{1/q} P\big(|\varphi_N^0|^p\big)^{1/p}$$

where $q = p/(p-1)$ and we have defined the density

$$\varphi_N^0(\phi) \equiv \frac{\mathrm{d}P_N^0}{\mathrm{d}P_N}(\phi) = \det\big(G_N\Gamma_N^{-1}\big)^{1/2} \exp\left(-\frac{1}{2}\left\langle\phi, (\Gamma_N^{-1} - G_N^{-1})\phi\right\rangle_{\Lambda_N}\right).$$

From the proof of Proposition 2.3 we have

$$\limsup_{N\to\infty} N^{-d} \log P\big(|\varphi_N^0|^p\big) = 0\,, \quad p \in [1,\infty)\,,$$

and therefore the (weak) upper bound for P_N^0 follows from the upper bound for P and the arbitrariness of p. The same argument can be used to obtain the lower bound, since by Hölder's inequality one has

$$P_N^0\big(\tilde{R}_N \in G\big) \geqslant P\big(\tilde{R}_N \in G\big)^q P_N^0\big(|\varphi_N^0|^{-p}\big)^{-q/p}.$$

Let us discuss more general boundary conditions. We have

Proposition 3.9. *Let $\tau \in \Omega_J$, and assume*

$$\sigma(\tau) \equiv \lim_{N\to\infty} N^{-d} \log \frac{Z_N^\tau}{Z_N^0} = 0\,. \tag{3.3.66}$$

Then the empirical field $\tilde{R}_N$ satisfies the weak large deviations principle of Theorem 3.1 under the measure $P_N^\tau = \gamma_{\Lambda_N}^\tau$.

Proof. Let $\tau \in \Omega_J$ and

$$\psi_N^\tau = \frac{dP_N^\tau}{dP_N^0}\,.$$

From the above discussion we know that a sufficient condition is

$$\lim_{N\to\infty} N^{-d} \log P_N^0\left(|\psi_N^\tau|^p\right) = \lim_{N\to\infty} N^{-d} \log P_N^\tau\left(|\psi_N^\tau|^{-p}\right) = 0\,, \quad p \in [1,\infty)\,. \tag{3.3.67}$$

But this is equivalent to

$$\lim_{N\to\infty} N^{-d}\left\langle m^{\tau,N}, \Gamma_N^{-1} m^{\tau,N}\right\rangle = 0\,, \tag{3.3.68}$$

where $m^{\tau,N}$ stands for m^{τ,Λ_N}, cf. (2.2.5). From (2.2.7) we see that condition (3.3.68) is equivalent to the vanishing of the surface tension induced by τ, i.e. (3.3.66). □

Let us look at condition (3.3.66) in some specific models. Consider the massless free field, $d \geqslant 3$. We first observe that the effect of a linear tilt $\tau_x = u \cdot x$, for some $u \in \mathbb{R}^d$, is not negligeable on the thermodynamical level. Here one can compute

$$\sigma(\tau) = |u|^2/2\,.$$

On the other hand one can show that $\sigma(\tau) = 0$ is satisfied for all τ which have a strictly sublinear growth at infinity. More precisely, let $L > 0$ and $\beta \in (0,1)$ and define

$$\Omega_{L,\beta} = \left\{\tau \in \Omega :\ |\tau_x| \leqslant L(1+|x|^\beta)\,,\ x \in \mathbb{Z}^d\right\}\,.$$

Let $\tau \in \Omega_{L,\beta}$ and set

$$\chi_x^{\tau,N} = \sum_{z\notin\Lambda_N} p(x,z)\tau_z\,, \quad x \in \Lambda_N$$

so that $m^{\tau,N} = \Gamma_N \chi^{\tau,N}$. Then, using

$$\Gamma_N(x,y) \leqslant G(0,x-y) \leqslant \text{const.} \times\ |x-y|^{2-d}\,, \tag{3.3.69}$$

for $|x-y|$ large, we have

$$N^{-d}\left\langle m^{\tau,N}, \Gamma_N^{-1} m^{\tau,N}\right\rangle_N = N^{-d}\left\langle \chi^{\tau,N}, \Gamma_N \chi^{\tau,N}\right\rangle_N \leqslant \text{const.} \times\ L^2\, N^{2\beta-2}\,.$$

This shows that (3.3.66) holds *uniformly* for $\tau \in \Omega_{L,\beta}$. By Proposition 3.9 it follows that the large deviations estimates of Theorem 3.1 hold for P_N^τ, uniformly in the class of boundaries $\Omega_{L,\beta}$. The same argument can be used in the case of any random walk satisfying (3.3.69). Also, the above reasoning shows that the statement holds for the α–stable walks discussed in part II of this work (cf. chapter 5), with the constraint $\beta < \alpha/2$.

Part II

Part two

Chapter 4

The symmetric α-stable process

4.1 The process

Let $\alpha \in (0, 2)$. The symmetric $\alpha-$stable process is defined as the Markov process on $\mathbb{R}^d$ whose transition kernel has density

$$q_\alpha(t, x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-i(x-y)\cdot\xi} \, \mathrm{e}^{-t|\xi|^\alpha} \, \mathrm{d}\xi \,. \tag{4.4.1}$$

It is known that the paths $\omega : \mathbb{R}_+ \to \mathbb{R}^d$ of the symmetric $\alpha-$stable process are right continuous and have left limits with probability one. We denote by $\tilde{\mathbb{P}}_x$ and $\tilde{\mathbb{E}}_x$ the probability measure and the corresponding expectation on the set of paths which start in $x \in \mathbb{R}^d$, so that, for any measurable set $A \subset \mathbb{R}^d$, for any $t \geqslant 0$

$$\tilde{\mathbb{P}}_x \left(\omega(t) \in A\right) = \tilde{\mathbb{E}}_x \left(1_A(\omega(t))\right) = \int_A q_\alpha(t, x - y) \, \mathrm{d}y$$

The case $\alpha = 2$ is singular, it gives brownian motion in $\mathbb{R}^d$ with $q_2(t, x)$ the centered Gaussian density with variance $2t$. In particular, for $\alpha = 2$ we may restrict to paths ω which are continuous.

On the other hand, for $\alpha \in (0, 2)$ the density q_α has heavy tails and the paths are no longer continuous, cf. [45], chapter 7. One has

$$\lim_{|x|\to\infty} |x|^{d+\alpha} q_\alpha(1, x) = c_{\alpha,d} \,, \tag{4.4.2}$$

where

$$c_{\alpha,d} = \left(\int_{\mathbb{R}^d} (1 - \cos \hat{\xi} \cdot x) \, |x|^{-d-\alpha} \, \mathrm{d}x \right)^{-1} \tag{4.4.3}$$

with $\hat{\xi}$ a unit vector in $\mathbb{R}^d$, cf. [2]. We shall assume $\alpha < d$ in order to have a *transient* process. In particular we may define the resolvent kernel

$$g_\alpha(x) = \int_0^\infty q_\alpha(t, x) \, \mathrm{d}t \tag{4.4.4}$$

A detailed account of the potential theory associated to the kernel g_α, also known as the M. Riesz kernel, can be found in the treatise [50]. We have

$$g_\alpha(x) = \omega_{\alpha,d}\, |x|^{\alpha-d}, \tag{4.4.5}$$

where

$$\omega_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2})} .$$

The Fourier transform of g_α is given by

$$\hat{g}_\alpha(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\, g_\alpha(x)\ \mathrm{d}x = |\xi|^{-\alpha}$$

It follows

$$g_\beta * g_{\beta'}(x) = \int_{\mathbb{R}^d} g_\beta(x-y) g_{\beta'}(y)\, \mathrm{d}y = g_{\beta+\beta'}(x) \tag{4.4.6}$$

for any $\beta, \beta' \in (0, d/2)$.

4.2 Dirichlet form and capacity

The *Dirichlet Form* of the symmetric α-stable process is given by

$$\mathcal{E}(\psi,\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{-\alpha}(x-y)\,(\psi(x)-\psi(y))^2\ \mathrm{d}x\ \mathrm{d}y\,, \tag{4.4.7}$$

where we have introduced the kernel $g_{-\alpha}(x) \equiv c_{\alpha,d}|x|^{-d-\alpha}$, with $c_{\alpha,d}$ the constant given in (4.4.2). Its domain is given by

$$\mathcal{D}_\mathcal{E} = \left\{\psi \in L^2(\mathbb{R}^d) :\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{-\alpha}(x-y)\,(\psi(x)-\psi(y))^2\ \mathrm{d}x\ \mathrm{d}y < \infty\right\} . \tag{4.4.8}$$

In terms of Fourier transforms we have

$$\mathcal{E}(\psi,\psi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^2\, |\xi|^\alpha\ \mathrm{d}\xi\,, \tag{4.4.9}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ, cf. chapter 1 of [50]. Following Example 1.5.2 of [37] we can describe the extended domain of $\mathcal{E}$ as the range of the "square root" of g_α, i.e.

$$\overline{\mathcal{D}}_\mathcal{E} = \left\{\psi = g_{\alpha/2} * \varphi\,,\ \ \varphi \in L^2(\mathbb{R}^d)\right\}$$

and, for $\psi = g_{\alpha/2} * \varphi$ we have

$$\mathcal{E}(\psi,\psi) = (\varphi,\varphi)\,, \tag{4.4.10}$$

where $(\cdot,\cdot)$ denotes scalar product in $L^2(\mathbb{R}^d)$. Notice that $\mathcal{D}_\mathcal{E} \subset \overline{\mathcal{D}}_\mathcal{E}$, but $\psi \in \overline{\mathcal{D}}_\mathcal{E}$ is not necessarily in $L^2(\mathbb{R}^d)$, because g_α is only locally in $L^1(\mathbb{R}^d)$. In fact, $\mathcal{D}_\mathcal{E} = \overline{\mathcal{D}}_\mathcal{E} \cap L^2(\mathbb{R}^d)$, and $\overline{\mathcal{D}}_\mathcal{E}$ is the space of $\psi \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that ψ is a tempered distributions and $\hat{\psi}(\xi)$ is square integrable w.r.t. $|\xi|^\alpha \, d\xi$. One can also show, see Lemma 4.4 below, that $\overline{\mathcal{D}}_\mathcal{E} \subset L^{p_0}(\mathbb{R}^d)$ for $p_0 = 2d/(d-\alpha)$. Since the process is transient, the Dirichlet space $\overline{\mathcal{D}}_\mathcal{E}$ with inner product $\mathcal{E}(\cdot,\cdot)$ is a Hilbert space, [37].

The *capacity* of a compact set $\Gamma \subset \mathbb{R}^d$ is defined by the variational formula

$$\operatorname{cap}_\alpha(\Gamma) = \inf\left\{\mathcal{E}(\psi,\psi), \quad \psi \in \overline{\mathcal{D}}_\mathcal{E}, \;\; \psi = 1 \text{ a.e. on } \Gamma\right\} . \tag{4.4.11}$$

We may define the capacity of an arbitrary measurable set $A \subset \mathbb{R}^d$ by

$$\operatorname{cap}_\alpha(A) = \sup_{\substack{\Gamma \subset A \\ \Gamma \text{ compact}}} \operatorname{cap}_\alpha(\Gamma) .$$

The following definitions play an important role in potential theory. A set whose capacity is zero is called *polar*. A property holds *quasi everywhere* (q.e.) on $\mathbb{R}^d$ if it holds everywhere apart from polar sets. Two functions are said to be quasi-equivalent if they are equal quasi-everywhere. A function $f : \mathbb{R}^d \to \mathbb{R}$ is quasi-continuous if there exists a sequence of open sets $\mathcal{O}_n \subset \mathbb{R}^d$ such that f is continuous on $\mathbb{R}^d \setminus \mathcal{O}_n$ for all n and $\operatorname{cap}_\alpha(\mathcal{O}_n) \searrow 0$, $n \to \infty$. It is well known that a set of capacity zero must have zero Lebesgue measure, and that any $f \in \overline{\mathcal{D}}_\mathcal{E}$ has a quasi-continuous representative which is uniquely specified up to quasi-equivalence, cf. e.g. [68], Theorem 3.13, or [37], Theorem 2.1.7. Following [68], we shall make the following *convention*: when speaking of functions $f \in \overline{\mathcal{D}}_\mathcal{E}$ we assume (if necessary) to take their quasi-continuous representative. This enables us to simplify notations and avoid many repetitions.

4.3 Potentials, hitting distributions and balayage

Let $\mathcal{M}$ denote the set of (signed) Radon measures on $\mathbb{R}^d$. The *potential* of $\mu \in \mathcal{M}$ is defined by

$$U^\mu_\alpha(x) = \int_{\mathbb{R}^d} g_\alpha(x-y)\mu(\,dy) ,$$

where $U^\mu_\alpha(x)$ is regarded as a function on the set of $x \in \mathbb{R}^d$ for which the integral is finite. To $\mu \in \mathcal{M}$ we associate the quadratic functional

$$E_\alpha(\mu,\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_\alpha(x-y)\mu(\,dx)\mu(\,dy) \tag{4.4.12}$$

which is called the energy of μ. Measures of finite energy are denoted

$$\mathcal{M}_0 = \{\mu \in \mathcal{M} : \quad E_\alpha(|\mu|,|\mu|) < \infty\} ,$$

where $|\mu| = \mu^+ + \mu^-$ is the total variation of $\mu = \mu^+ - \mu^-$. For $\mu, \nu \in \mathcal{M}_0$, we may use (4.4.12) to define $E_\alpha(\mu, \nu)$, by polarization. If $\mu \in \mathcal{M}_0$, then by [37], Theorem 2.2.2, $U_\alpha^\mu \in \overline{\mathcal{D}}_\mathcal{E}$ and

$$\mathcal{E}(U_\alpha^\mu, U_\alpha^\mu) = E_\alpha(\mu, \mu)\,. \tag{4.4.13}$$

Balayage

Given a potential $f = U_\alpha^\mu$, $\mu \in \mathcal{M}_0$, and a compact set $\Gamma \subset \mathbb{R}^d$, there exists a unique measure $\nu \in \mathcal{M}_0$ such that $\mathrm{Supp}[\nu] \subset \Gamma$ and $U_\alpha^\nu = f$ q.e. on Γ. ν is called the *balayage* (or sweeping out) of μ on Γ, and $f_\Gamma = U_\alpha^\nu$ is called the *reduced function* of the potential f on Γ. For a proof of the above we refer to chapter 4 of [50], or chapter 2 of [37]. We point out below the probabilistic interpretation of these quantities.

Hitting distributions

Given a measurable set $A \subset \mathbb{R}^d$, we define the *hitting distribution* of A by

$$H_A(x, E) = \tilde{\mathbb{P}}_x\left(\omega(\sigma_A) \in E\right)\,, \tag{4.4.14}$$

where E is any measurable set and σ_A denotes the hitting time of A,

$$\sigma_A = \inf\left\{t \geqslant 0,\ \omega(t) \in A\right\}\,. \tag{4.4.15}$$

H_A defines a probability kernel on $\mathbb{R}^d$ and, for a measurable function f, we write

$$H_A f(x) = \int_{\mathbb{R}^d} f(y) H_A(x, \mathrm{d}y)\,. \tag{4.4.16}$$

Then $H_A f = f$ q.e. on Γ. Moreover, from [37], Theorem 4.3.2, we know that if $f = U_\alpha^\mu$, $\mu \in \mathcal{M}_0$, is a potential, and Γ is a compact, then $H_\Gamma f$ is identified with f_Γ, the reduced function of f on Γ, i.e. $H_\Gamma f = U_\alpha^\nu$ where ν is the balayage of μ on Γ. If e.g. $f = 1$ the above facts can be used to show that the infimum in (4.4.11) is attained at the function

$$p_\Gamma(x) = \tilde{\mathbb{P}}_x(\sigma_\Gamma < \infty)\,. \tag{4.4.17}$$

Indeed, (4.4.17) is only a particular case of Theorem 4.3, see below.

When f is not a potential, we cannot say that $H_\Gamma f$ is the potential of a measure $\nu \in \mathcal{M}_0$ with $\mathrm{Supp}[\nu] \subset \Gamma$, but thanks to the *spectral synthesis* result, see Theorem 2.3.2 of [37], $H_\Gamma f \in \overline{\mathcal{D}}_\mathcal{E}$ can be approximated by a sequence of potentials $U_\alpha^{\nu_n}$ such that $\mathrm{Supp}[\nu_n] \subset \Gamma$ and $U_\alpha^{\nu_n}$ converges to $H_\Gamma f$ in $(\overline{\mathcal{D}}_\mathcal{E}, \mathcal{E}(\cdot,\cdot))$. We shall need these facts in the proof of Theorem 4.3. The intuitive picture to have in mind is that for any $f \in \overline{\mathcal{D}}_\mathcal{E}$, $H_\Gamma f$ is a function which agrees with f on Γ and is $\alpha-$harmonic outside of Γ. By Theorem 4.3.2 in [37], we have

$$\mathcal{E}(H_\Gamma f, g) = 0\,, \quad f, g \in \overline{\mathcal{D}}_\mathcal{E}\,,\ g = 0 \text{ q.e. on } \Gamma\,.$$

4.4 Embedded process

Let us now consider the $\alpha-$stable process *embedded* in the unit cube

$$V \equiv \left[-\frac{1}{2}, \frac{1}{2}\right]^d .$$

The set V plays the role of the macroscopic box in the following chapters. However, it should be clear that for the purposes of this chapter, V may be replaced by an arbitrary closed, bounded subset of $\mathbb{R}^d$ with regular boundary.

We first introduce the integral operator on $L^2(V)$ given by

$$\mathcal{G}_V \phi(x) = \int_V g_\alpha(x-y)\phi(y)\,\mathrm{d}y\,, \quad x \in V\,. \tag{4.4.18}$$

We have

Lemma 4.1. *The operator $\mathcal{G}_V : L^2(V) \to L^2(V)$ is compact and nonnegative.*

Proof. For each $\delta > 0$ define the non singular kernels

$$g_\alpha^\delta(x) = \begin{cases} \omega_{\alpha,d}|x|^{\alpha-d} & \text{if } |x| > \delta \\ \omega_{\alpha,d}\delta^{\alpha-d} & \text{if } |x| \leqslant \delta \end{cases} \tag{4.4.19}$$

Then $g_\alpha^\delta(x-y)$ is in $L^2(V \times V)$ (as a function of two variables) and the corresponding integral operator on V $\mathcal{G}_V^\delta$ is Hilbert-Schmidt, therefore compact. We show that $\mathcal{G}_V$ is compact by proving $\mathcal{G}_V^\delta \to \mathcal{G}_V$ in the operator norm topology, as $\delta \to 0$ (cf. e.g. [9], Th. VI.1). To this end define

$$f^\delta = g_\alpha - g_\alpha^\delta$$

Let $\phi \in L^2(V)$ and call $\tilde{\phi}$ the function which agrees with ϕ in V and vanishes outside of V. We have

$$\mathcal{G}_V\phi - \mathcal{G}_V^\delta\phi = f^\delta * \tilde{\phi}, \quad \text{a.e. on } V.$$

Therefore

$$\|\mathcal{G}_V\phi - \mathcal{G}_V^\delta\phi\|_{L^2(V)} \leqslant \|f^\delta\|_{L^1(\mathbb{R}^d)}\|\tilde{\phi}\|_{L^2(\mathbb{R}^d)} = \|f^\delta\|_{L^1(\mathbb{R}^d)}\|\phi\|_{L^2(V)}$$

By dominated convergence $\|f^\delta\|_{L^1(\mathbb{R}^d)} \to 0$, as $\delta \to 0$. This proves that $\mathcal{G}_V$ is compact. To prove non-negativity observe that, by (4.4.6), for any $\phi \in L^2(V)$

$$(\phi, \mathcal{G}_V\phi) = \int_{\mathbb{R}^d} \tilde{\phi}(x)\left(g_\alpha * \tilde{\phi}\right)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \left|\left(g_{\alpha/2} * \tilde{\phi}\right)(x)\right|^2 \mathrm{d}x$$

$\square$

4.4.1 Embedded Dirichlet form

By Lemma 4.1 $\mathcal{G}_V$ has eigenvalues $\{\lambda_1 \geqslant \lambda_2 \geqslant \dots\} \subset \mathbb{R}^+$. We call $\{e_n\}$ the corresponding $L^2(V)$-normalized eigenfunctions, and denote $(\cdot,\cdot)_V$ the scalar product in $L^2(V)$. Let also $\mathcal{G}_V^{1/2}$ denote the square root of the operator $\mathcal{G}_V$, i.e.

$$\mathcal{G}_V^{1/2}\phi = \sum_n \lambda_n^{1/2}\,(\phi, e_n)_V\, e_n\,, \quad \phi \in L^2(V)\,. \tag{4.4.20}$$

Note that $\mathcal{G}_V^{1/2}$ is also a nonnegative compact operator.

The Dirichlet form $\mathcal{E}_V$ of the embedded process is defined by

$$\mathcal{E}_V(\psi,\psi) = \sum_n \frac{1}{\lambda_n}(\psi, e_n)_V^2, \qquad \psi \in L^2(V) \tag{4.4.21}$$

with domain

$$\mathcal{D}_{\mathcal{E}_V} = \left\{\psi \in L^2(V) :\ \mathcal{E}_V(\psi,\psi) < \infty\right\}. \tag{4.4.22}$$

Lemma 4.2. *We have*

$$\begin{aligned} \mathcal{D}_{\mathcal{E}_V} &= \left\{\psi = \mathcal{G}_V^{1/2}\phi,\ \ \phi \in L^2(V)\right\}, \\ \mathcal{E}_V(\psi,\psi) &= (\phi,\phi)_V\,, \quad \phi = \mathcal{G}_V^{-1/2}\psi\,. \end{aligned} \tag{4.4.23}$$

Moreover, for any $\psi \in \mathcal{D}_{\mathcal{E}_V}$ and any dense subset $\mathcal{S}_V$ of $L^2(V)$,

$$\begin{aligned} \mathcal{E}_V(\psi,\psi) &= \sup_{f\in\mathcal{S}_V} \{2\,(f,\psi)_V - (f, \mathcal{G}_V f)_V\} \\ &= \sup_{f\in\mathcal{S}_V} \left\{\frac{(f,\psi)_V^2}{(f,\mathcal{G}_V f)_V}\right\}. \end{aligned} \tag{4.4.24}$$

Finally, the function $\phi \to \mathcal{E}_V(\phi,\phi)$ is lower semi-continuous and has compact level sets in $L^2(V)$.

Proof. We first observe that (4.4.23) is an easy consequence of the spectral representations (4.4.21) and (4.4.20). In fact, defining the (unbounded) operator

$$\mathcal{G}_V^{-1/2}\phi = \sum_n \lambda_n^{-1/2}\,(\phi, e_n)_V\, e_n\,,$$

we see that the domain of $\mathcal{G}_V^{-1/2}$ coincides with $\mathcal{D}_{\mathcal{E}_V}$. Using again the spectral theorem, the first line in (4.4.24) follows from non-negativity of $\mathcal{G}_V$ and the fact that the function $x(2a - \lambda x)$, for fixed $a \in \mathbb{R}$ and $\lambda > 0$, is maximized at $x = a/\lambda$. Moreover, by the continuity of $\mathcal{G}_V$, the choice of the dense set $\mathcal{S}_V$ is arbitrary. The second variational formula in (4.4.24) follows by changing f into βf in the first line of (4.4.24) and optimizing over $\beta \in \mathbb{R}$. The lower semi-continuity of $\mathcal{E}_V$ is a straightforward consequence of (4.4.24). In particular, the level sets

$$\Phi_L = \left\{\mathcal{G}_V^{1/2}\varphi\,,\ \ \varphi \in L^2(V)\,,\ (\varphi,\varphi)_V^2 \leqslant L\right\},$$

are closed. But since $\mathcal{G}_V^{1/2}$ is compact, Φ_L is also relatively compact in $L^2(V)$, and thus compact. ☐

4.4.2 A variational formula

Let H_V be the hitting distribution of V, cf. (4.4.14). The following theorem shows the deep connection between the problem of balayage on V and the embedded process. In particular, it shows that $\mathcal{E}_V$ coincides with the "balayaged Dirichlet form" introduced in [56].

Theorem 4.3. *For any $f \in \mathcal{D}_{\mathcal{E}_V}$, we have*

$$\begin{aligned}\mathcal{E}_V(f,f) &= \mathcal{E}(H_V f, H_V f) \\ &= \inf\left\{\mathcal{E}(\psi,\psi),\ \ \psi \in \overline{\mathcal{D}}_{\mathcal{E}}\,,\ \ \psi = f \text{ a.e. on } V\right\}\end{aligned} \tag{4.4.25}$$

Proof. Let us define

$$\overline{\mathcal{E}}_V(f) = \inf\left\{\mathcal{E}(\psi,\psi),\ \ \psi \in \overline{\mathcal{D}}_{\mathcal{E}}\,,\ \ \psi = f \text{ a.e. on } V\right\}\,.$$

Let $\psi \in \overline{\mathcal{D}}_{\mathcal{E}}$ such that $\psi = g_{\alpha/2} * \varphi$, with $\varphi \in L^2(\mathbb{R}^d)$ as in (4.4.10). Then, for any $h \in L^2(\mathbb{R}^d)$

$$\mathcal{E}(\psi,\psi) = (\varphi,\varphi) \geqslant 2(\varphi,h) - (h,h)\,.$$

By choosing $h = g_{\alpha/2} * \phi$, $\phi \in L^2(\mathbb{R}^d)$, from (4.4.6) we obtain

$$\mathcal{E}(\psi,\psi) \geqslant \sup_{\phi}\left\{2(\psi,\phi) - (\phi, g_\alpha * \phi)\right\}\,, \tag{4.4.26}$$

where the supremum is taken over $\phi \in L^2(\mathbb{R}^d)$ such that $g_{\alpha/2} * \phi$ is in $L^2(\mathbb{R}^d)$. Restricting (4.4.26) to ϕ's vanishing out of V and using (4.4.24), we see that

$$\mathcal{E}(\psi,\psi) \geqslant \mathcal{E}_V(f,f)\,, \quad \psi = f \text{ a.e. on } V, \tag{4.4.27}$$

and therefore $\overline{\mathcal{E}}_V(f) \geqslant \mathcal{E}_V(f,f)$.

Since $H_V f = f$ q.e. (and therefore a.e.), we always have

$$\overline{\mathcal{E}}_V(f) \leqslant \mathcal{E}(H_V f, H_V f)\,.$$

To finish the proof of the theorem it is then sufficient to show

$$\mathcal{E}(H_V f, H_V f) = \mathcal{E}_V(f,f)\,. \tag{4.4.28}$$

Let us first prove (4.4.28) when f is a potential, i.e. assume there exists $\mu \in \mathcal{M}_0$ such that $f = U^\mu_\alpha$ a.e. on V (this is the case if e.g. $f \in \mathcal{C}^{d+2}(V)$ by Lemma 1.1 in [50]). Then $H_V f = U^\nu_\alpha$ where ν is the balayage of μ on V, cf. previous discussion. By (4.4.13)

$$\mathcal{E}(H_V f, H_V f) = \mathcal{E}(U^\nu_\alpha, U^\nu_\alpha) = E_\alpha(\nu,\nu)\,. \tag{4.4.29}$$

Observe now that for any $\lambda \in \mathcal{M}_0$, $E_\alpha(\nu-\lambda,\nu-\lambda) \geqslant 0$, and therefore

$$E_\alpha(\nu,\nu) = \sup\left\{2E_\alpha(\nu,\lambda) - E_\alpha(\lambda,\lambda)\,,\ \ \lambda \in \mathcal{M}_0,\ \mathrm{Supp}[\lambda] \subset V\right\}.$$

Any $\lambda \in \mathcal{M}_0$ supported on V can be approximated by absolutely continuous measures $\lambda_n(\,\mathrm{d}x) = \phi_n(x)\,\mathrm{d}x$, $\phi_n \in \mathcal{C}(V)$, (cf. [50], Lemma 1.2), so that

$$\begin{aligned} 2E_\alpha(\nu,\lambda_n) - E_\alpha(\lambda_n,\lambda_n) &= 2(U_\alpha^\nu,\phi_n)_V - (\phi_n,\mathcal{G}_V\phi_n)_V \\ &\to 2E_\alpha(\nu,\lambda) - E_\alpha(\lambda,\lambda)\,,\quad n\to\infty\,. \end{aligned}$$

Thus, using $U_\alpha^\nu = f$ a.e. on V and the variational principle (4.4.24), we see that

$$E_\alpha(\nu,\nu) = \sup_{\phi\in\mathcal{C}(V)} \left\{2(f,\phi)_V - (\phi,\mathcal{G}_V\phi)_V\right\} = \mathcal{E}_V(f,f)\,. \tag{4.4.30}$$

This, together with (4.4.29), proves the claim when f is a potential. In order to treat the general case, recall that $H_V f \in \overline{\mathcal{D}}_\mathcal{E}$ is approximated by a sequence of potentials $U_\alpha^{\nu_n}$ such that $\mathrm{Supp}[\nu_n] \subset V$ and $U_\alpha^{\nu_n}$ is $\mathcal{E}$-convergent to $H_V f$. Then, by (4.4.27) we see that the functions $f_n = U_\alpha^{\nu_n} 1_V$ are $\mathcal{E}_V$-converging to f and by (4.4.30) we conclude

$$\begin{aligned} \mathcal{E}(H_V f, H_V f) &= \lim_{n\to\infty} \mathcal{E}(U_\alpha^{\nu_n}, U_\alpha^{\nu_n}) \\ &= \lim_{n\to\infty} E_\alpha(\nu_n,\nu_n) = \lim_{n\to\infty} \mathcal{E}_V(f_n,f_n) = \mathcal{E}_V(f,f)\,. \end{aligned} \tag{4.4.31}$$

This concludes the proof of (4.4.28). □

4.4.3 A Sobolev inequality

We shall need the following application of the so called Hardy-Littlewood-Sobolev theorem of fractional integration, cf. [72]. Let $\|\cdot\|_p$, $1<p<\infty$, denote the usual L^p-norm and set

$$\|f\|_{p,V} = \left(\int_V |f(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

We denote $L^p(\mathbb{R}^d)$ and $L^p(V)$ the corresponding Banach spaces.

Lemma 4.4. *Let $\alpha \in (0, 2\wedge d)$ and $p_0 = 2d/(d-\alpha)$. Then $\overline{\mathcal{D}}_\mathcal{E} \subset L^{p_0}(\mathbb{R}^d)$ and there exists a constant $K_\alpha < \infty$ such that*

$$\|\psi\|_{p_0} \leqslant K_\alpha\sqrt{\mathcal{E}(\psi,\psi)}\,,\quad \psi \in \overline{\mathcal{D}}_\mathcal{E}\,. \tag{4.4.32}$$

Moreover, $\mathcal{D}_{\mathcal{E}_V} \subset L^{p_0}(V)$ and

$$\|\phi\|_{p_0,V} \leqslant K_\alpha\sqrt{\mathcal{E}_V(\phi,\phi)}\,,\quad \phi \in \mathcal{D}_{\mathcal{E}_V}\,. \tag{4.4.33}$$

Proof. Let $g_\beta(x) = \omega_{\beta,d}|x|^{\beta-d}$ be the Riesz kernel. Then from [72], chapter 5, Theorem 1, we have the inequality

$$\|g_\beta * \varphi\|_p \leqslant K_{p,q}\|\varphi\|_q\,, \quad \varphi \in L^q(\mathbb{R}^d)\,,$$

for any $\beta < d$, $1 < p, q < \infty$ satisfying $1/p = 1/q - \beta/d$, with some constant $K_{p,q} < \infty$. In particular, choosing $q = 2$ and $\beta = \alpha/2$, we see that (4.4.32) follows from (4.4.10). To prove (4.4.33) observe that for any $\phi \in \mathcal{D}_{\mathcal{E}_V}$ and any $\psi \in \overline{\mathcal{D}}_{\mathcal{E}}$ such that $\psi = \phi$ a.e. on V, (4.4.32) implies

$$\|\phi\|_{p_0,V} \leqslant K_\alpha \sqrt{\mathcal{E}(\psi,\psi)}\,,$$

and the conclusion follows from Theorem 4.3. □

4.5 Killed process

Let us denote ζ_t, $t \geqslant 0$, the coordinate process generated by the symmetric α-stable density (4.4.1). Given our unit box V, let

$$\tau_V = \inf\{t \geqslant 0, \zeta_t \notin V\}\,. \tag{4.4.34}$$

We add an extra point ∂, and define the process

$$\zeta_t^V = \begin{cases} \zeta_t & \text{if } t < \tau_V \\ \partial & \text{if } t \geqslant \tau_V \end{cases} \tag{4.4.35}$$

The process ζ^V is called the symmetric α-stable process killed upon exiting V. It is known, cf. [37, 68], that its Dirichlet form is given by

$$\mathcal{E}_V^0(\psi,\psi) = \mathcal{E}(\tilde\psi,\tilde\psi)\,, \tag{4.4.36}$$

where $\psi \in L^2(V)$ and $\tilde\psi \in L^2(\mathbb{R}^d)$ agrees with ψ on V and vanishes on V^c, with domain

$$\mathcal{D}_{\mathcal{E}_V^0} = \{\psi \in \mathcal{D}_{\mathcal{E}} : \quad \psi = 0 \text{ a.e. on } V^c\} \tag{4.4.37}$$

Let us define the corresponding semigroup and the potential. The semigroup is defined by

$$P_t^V\psi(x) = \tilde{\mathbb{E}}_x\left[\psi(\zeta_t)\,;\ t < \tau_V\right] \tag{4.4.38}$$

where $x \in V$. Then P_t^V is a compact operator in $L^2(V)$ with eigenvalues $e^{-\mu_k t}$, $k = 1, 2, \dots$ and $0 < \mu_1 < \mu_2 < \dots$, cf. Theorem 2.5 in [17]. In particular, the Green operator

$$\mathcal{G}_V^0 = \int_0^\infty P_t^V\,\mathrm{d}t \tag{4.4.39}$$

is a bounded, compact operator in $L^2(V)$, with eigenvalues $\mu_1^{-1} > \mu_2^{-1} > \ldots \geqslant 0$. We then have the variational formulas for $\psi \in \mathcal{D}_{\mathcal{E}_V^0}$

$$\begin{aligned}\mathcal{E}_V^0(\psi,\psi) &= \sup_{f\in\mathcal{S}_V} \left\{2\,(f,\psi)_V \,-\, (f,\mathcal{G}_V^0 f)_V\right\}\\ &= \sup_{f\in\mathcal{S}_V} \left\{\frac{(f,\psi)_V^2}{(f,\mathcal{G}_V^0 f)_V}\right\}\,,\end{aligned} \tag{4.4.40}$$

where $\mathcal{S}_V$ is any dense subset of $L^2(V)$, see Lemma 4.2. Also, by Lemma 4.2 the Dirichlet form $\mathcal{E}_V^0$ has compact level sets in $L^2(V)$.

We close this section with a computation of $\mathcal{E}(f,f)$ when f is the indicator function of a Euclidean ball.

Lemma 4.5. *Let $f = 1_{B_\delta}$, $\delta > 0$, $B_\delta = \{x \in \mathbb{R}^d :\ |x| < \delta\} \subset V$. Then,*

$$\mathcal{E}(f,f) < \infty\,, \quad \text{if } \alpha \in (0,1)\,,\ d \geqslant 1\,, \tag{4.4.41}$$

while

$$\mathcal{E}(f,f) = \infty\,, \quad \text{if } \alpha \in [1,2)\,,\ d \geqslant 2\,,\ \text{or } \alpha = 2\,,\ d \geqslant 3\,. \tag{4.4.42}$$

Proof. We first consider the case $\alpha \in (0,1)$, $d = 1$. Then the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{-\delta}^{\delta} \cos x\xi \,\mathrm{d}x = 2\,\frac{\sin\delta\xi}{\xi}\,, \quad \xi \in \mathbb{R}\,.$$

From (4.4.9) we see that $\mathcal{E}(f,f) < \infty$, and therefore (4.4.41) follows for $d = 1$. Let now $d \geqslant 2$ and $\alpha \in (0,2)$. Then, for any $\xi \in \mathbb{R}^d$, using the Bessel function (5.5.21), we have

$$\begin{aligned}\hat{f}(\xi) &= \int_{B_\delta} \mathrm{e}^{i\xi\cdot x}\,\mathrm{d}x\\ &= \int_0^\delta \rho^{d-1}\frac{1}{(\rho|\xi|)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(\rho|\xi|)\,\mathrm{d}\rho\\ &= |\xi|^{-d}\int_0^{\delta|\xi|} s^{d/2} J_{\frac{d-2}{2}}(s)\,\mathrm{d}s\,.\end{aligned}$$

From the asymptotic expansion for Bessel functions (cf. e.g. [24], chapter 15) one has, for large t

$$\int_0^t s^{d/2} J_{\frac{d-2}{2}}(s)\,\mathrm{d}s = \sqrt{\frac{2}{\pi}}\, t^{\frac{d-1}{2}} \cos\left(t - (d+1)\pi/4\right) + O(t^{\max(\frac{d-3}{2},0)})\,. \tag{4.4.43}$$

For large $|\xi|$ we then have

$$|\xi|^{d+1}|\hat{f}(\xi)|^2 = \frac{2\delta^{d-1}}{\pi}\cos^2\left(\delta|\xi| - (d+1)\pi/4\right) \,+\, o(1)\,. \tag{4.4.44}$$

Then both claims follow from (4.4.44) and (4.4.9). $\square$

Chapter 5

The stable random walk

In the rest of part II we shall consider the massless field P defined in chapter 1, with the following specific choice of the random walk p.

5.1 Definitions

For each $\alpha \in (0, 2 \wedge d)$, we consider the jump process on $\mathbb{Z}^d$ defined by a symmetric, homogeneous stochastic matrix p_α,

$$\begin{gathered} p_\alpha(x,y) = p_\alpha(0, y-x) = p_\alpha(y,x)\,, \quad x, y \in \mathbb{Z}^d\,, \\ \sum_{x \in \mathbb{Z}^d} p_\alpha(0,x) = 1\,. \end{gathered} \tag{5.5.1}$$

We make the following assumption.
Assumption H1.

- there exists a function $\rho_\alpha : [0,\infty) \to [0,1]$ such that

$$p_\alpha(x) = \rho_\alpha(|x|) \quad x \in \mathbb{Z}^d\,. \tag{5.5.2}$$

- there exists a constant $v_{\alpha,d} \in (0,\infty)$ such that

$$\lim_{t\to\infty} t^{d+\alpha} \rho_\alpha(t) = v_{\alpha,d}\,. \tag{5.5.3}$$

Note that if p_α satisfies H1 then it is automatically irreducible. As in chapter 1, we set

$$\hat{p}_\alpha(\theta) = \sum_{x \in \mathbb{Z}^d} p_\alpha(0,x)\, \mathrm{e}^{i\, x\cdot\theta}\,, \quad \theta \in \mathbb{T}^d\,, \tag{5.5.4}$$

and the Green function associated to p_α is given by

$$G(x,y) = \sum_{n=0}^{\infty} p_\alpha^n(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\mathrm{e}^{i\,(y-x)\cdot\theta}}{1 - \hat{p}_\alpha(\theta)}\, \mathrm{d}\theta\,. \tag{5.5.5}$$

Lemma 5.1. *Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$. Assume p_α satisfies H1. Then the random walk p_α is transient and (5.5.5) is well defined. Moreover, letting $\gamma_{\alpha,d} = v_{\alpha,d}/c_{\alpha,d}$, cf. (4.4.3) and (5.5.3), we have*

$$\lim_{|\theta|\to 0} |\theta|^{-\alpha}(1 - \hat{p}_\alpha(\theta)) = \gamma_{\alpha,d}\,. \tag{5.5.6}$$

Proof. The transience follows from integrability of $(1-\hat{p}_\alpha)^{-1}$, cf. [71], which in turn is a consequence of (5.5.6) since $\alpha < d$. To prove (5.5.6) we follow Example 2 in section 8 of [71], in which the claim was proven for $d = 1$. Let $\varepsilon > 0$ and set $\theta = \varepsilon\hat{\theta}$, with $|\hat{\theta}| = 1$. We write

$$|\theta|^{-\alpha}(1 - \hat{p}_\alpha(\theta)) = \varepsilon^d \sum_{x\in\mathbb{Z}^d} |x|^{d+\alpha} p_\alpha(0,x) \frac{1 - \cos \varepsilon x \cdot \hat{\theta}}{|\varepsilon x|^{d+\alpha}}\,.$$

From Riemann integration, (5.5.3) and (4.4.3), we see that the above expression converges, as $\varepsilon \to 0$, to

$$v_{\alpha,d} \int_{\mathbb{R}^d} (1 - \cos \hat{\theta}\cdot x)\, |x|^{-d-\alpha}\, \mathrm{d}x = v_{\alpha,d}/c_{\alpha,d}\,. \quad \square$$

Remark 5.1. Let $\xi_1, \xi_2, \dots$ be the random walk generated by p_α, and denote $\phi_{\alpha,n}$ the characteristic function of the $\mathbb{R}^d$-valued random variable

$$\bar{\xi}_n = n^{-1/\alpha}\xi_n\,.$$

By (5.5.6) we then have

$$\phi_{\alpha,n}(\theta) = \hat{p}_\alpha(n^{-1/\alpha}\theta)^n \to \exp\left(-\gamma_{\alpha,d}|\theta|^\alpha\right), \quad n \to \infty\,, \tag{5.5.7}$$

which implies the invariance principle

$$\bar{\xi}_n \to^{(d)} U_\alpha\,, \quad n \to \infty\,, \tag{5.5.8}$$

where (d) stands for convergence in distribution and U_α is an $\mathbb{R}^d$-valued random variable distributed according to the symmetric α-stable density $q_\alpha(t = \gamma_{\alpha,d}, x)$, cf. (4.4.1). In this sense we say that the random walk p_α is in the *domain of attraction* of the symmetric α-stable law[1].

Convention: From now on we shall also assume

$$\gamma_{\alpha,d} = 1\,. \tag{5.5.9}$$

This is done merely for notational convenience. Indeed (5.5.9) amounts to fix $v_{\alpha,d} = c_{\alpha,d}$ in H1 or, equivalently, it corresponds to a different tuning of the time scale for

[1] One can actually prove the convergence of the random walk to the symmetric α-stable process on the space of right continuous paths with left limits, when the latter is given the Skorokhod topology, see [69]. We shall not need this stronger result in the following.

the limiting stable process. It is also immediate to see that $\gamma_{\alpha,d}$ will enter in the rate functions (6.6.8) and (6.6.9) only as a prefactor in front of the Dirichlet form $\mathcal{E}$. We shall also need:

Assumption H2.

- Let $g_\alpha(z) = \omega_{\alpha,d}|z|^{\alpha-d}$, $z \in \mathbb{R}^d$, be the Riesz kernel defined in (4.4.5). Then, for any $\eta \in \mathbb{R}^d$ with $|\eta| = 1$, we have

$$\lim_{|x|\to\infty} |x|^{d-\alpha} G(0,x) = g_\alpha(\eta)\,. \tag{5.5.10}$$

Remark 5.2. It might appear at first sight that the result (5.5.10) simply follows from (5.5.6), since a change of variables in (5.5.5) yields

$$G(0,x) = |x|^{\alpha-d} \frac{1}{(2\pi)^d} \int_{|x|\mathbb{T}^d} \frac{\mathrm{e}^{i\,\xi\cdot x/|x|}}{(1-\hat{p}_\alpha(\xi/|x|))|x|^\alpha}\,\mathrm{d}\xi\,. \tag{5.5.11}$$

But the passage to the limit under the integral is not obvious. In the context of irreducible random walks with finite range (5.5.10) is well known to hold with $\alpha = 2$, and it can be shown to follow from the local limit theorem with error bounds (see section 7 and section 25 of [71]). In our setting we shall rely on the analogous results of [74] for stable random walks. The interested reader is also referred to section 5 of [55] where (5.5.10) is proven for a general class of stable random walks in the regime $\alpha > (d-1)/2$. In the next subsection we discuss several examples for which (5.5.10) can be shown to hold. It seems to be an open question whether (5.5.10) always follows from assumption H1.

The case $\alpha = 2$

The case $\alpha = 2$ is often not treated explicitly in this work, but the techniques we use certainly apply to it. To treat the $\alpha = 2$ case explicitly one defines the stable random walk p_2 assuming that (5.5.2) is satisfied and replacing (5.5.3) by the finite variance condition

$$\sum_{x\in\mathbb{Z}^d} |x|^2 p_2(0,x) < \infty\,. \tag{5.5.12}$$

Then, provided H2 is satisfied, with $\alpha = 2$ and g_2 the Newtonian potential, all results of this and the following chapters hold in this case. The finite range case (a special case of (5.5.12)) was studied in [5].

5.2 Examples

We shall be working under assumptions H1 and H2 for the rest of the time. We call *stable random walk* any transition p_α satisfying (5.5.1) and assumptions H1,H2. In order to check assumption H2 in specific cases, we recall a basic result of [74]. Given a function $f : \mathbb{Z}^d \to \mathbb{R}$, denote $\nabla_i f$, $i = 1, \ldots, d$, its discrete gradient,

$$\nabla_i f(x) = f(x + e_i) - f(x)\,,$$

with e_i denoting the unit vector in the positive i-th direction. Putting together Corollary 3-A and 3-B of [74] we obtain:

Lemma 5.2. *Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$. Assume p_α satisfies H1 and assume there exists a constant $c \in (0, \infty)$ such that for any $x \in \mathbb{Z}^d_+$ with $|x| > c$, we have*

$$(-1)^d \nabla_1 \cdots \nabla_d\, p_\alpha(0, x) \geqslant 0\,. \tag{5.5.13}$$

Then (5.5.10) holds.

Note that in dimension $d = 1$ the condition (5.5.13) simply means that $p_\alpha(0, x)$ is monotone decaying after a certain distance. Let us discuss some more examples:

1. Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$, and define

$$p_\alpha(0, x) = b(a + |x|^2)^{-(d+\alpha)/2}\,, \quad x \in \mathbb{Z}^d\,, \tag{5.5.14}$$

with positive constants a, b compatible with (5.5.1). Assumption H1 is obviously satisfied. In order to check condition (5.5.13) let us consider $p_\alpha(0, x)$ in (5.5.14) as a function defined for any $x \in \mathbb{R}^d$. Let $r = |x|$ and $p_\alpha(r) = p_\alpha(0, x)$. Then

$$\frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, p_\alpha(0, x) = x_1 \cdots x_d\, D_r^d\, p_\alpha(r) \tag{5.5.15}$$

where we have introduced the differential operator

$$D_r = \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\,.$$

It is easily seen that

$$D_r^d\, p_\alpha(r) = (-1)^d 2^{-d}\, b(a + r^2)^{-(3d+\alpha)/2} \prod_{k=0}^{d-1} (d + \alpha + 2k)\,.$$

Restricting to $\mathbb{R}^d_+$ and integrating along each coordinate x_i, $i = 1, \ldots, d$ on an interval of length 1 we obtain (5.5.13). By Lemma 5.2 this implies p_α satisfies H2.

2. Example 1 can be generalized by letting b depend (smoothly) on $r = |x|$ in (5.5.14). Let $b(r) \geqslant 0$, $b(r) \to b > 0$ as $r \to \infty$ and assume

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}\, b(r) \to 0\,, \quad r \to \infty\,, \quad k = 1, \ldots, d\,. \tag{5.5.16}$$

To see that (5.5.16) is sufficient for p_α to satisfy H2, we observe that

$$D_r^d\, p_\alpha(r) = \sum_{k=0}^{d} \binom{d}{k} \left(D_r^{d-k}(a+r^2)^{-(d+\alpha)/2}\right) \left(D_r^k\, b(r)\right) . \tag{5.5.17}$$

Now from (5.5.16) it is easy to see that $r^k\, D_r^k\, b(r) \to 0$ when $r \to \infty$ for any $k = 1, \ldots, d$. Therefore the first term ($k = 0$) in (5.5.17) dominates when r is large. As before, this implies that for sufficiently large $|x|$ (5.5.13) is satisfied, and H2 follows from Lemma 5.2.

3. Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$, and consider the discretized α-stable process

$$p_\alpha(0,x) = Z^{-1} q_\alpha(1,x)\,. \tag{5.5.18}$$

Here $q_\alpha(1,x)$ is the symmetric α-stable density at time $t = 1$, cf. (4.4.1), and

$$Z = \sum_{x\in\mathbb{Z}^d} q_\alpha(1,x)\,, \quad \hat{p}_\alpha(\theta) = Z^{-1} \sum_{y\in\mathbb{Z}^d} \exp\left(-|\theta + 2\pi y|^\alpha\right).$$

In this case H1 follows from (4.4.2) with $\gamma_{\alpha,d} = Z^{-1}$. In the case $d = 1$, H2 follows from Lemma 5.2 and the monotonicity of q_α, or directly from the method of [55]. For $d \geqslant 2$, H2 holds as a consequence of Lemma 5.2 and the following identity for stable densities[2].

Lemma 5.3. *Let $d \geqslant 2$ and $\alpha \in (0, 2)$. Let $q_\alpha^{(d)}$ be the isotropic stable density (4.4.1) in dimension d. Then, for any $x \in \mathbb{R}^d$,*

$$\frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, q_\alpha^{(d)}(1,x) = (-1)^d\, x_1 \cdots x_d\, (2\pi)^d\, q_\alpha^{(3d)}(1,x)\,. \tag{5.5.19}$$

Proof. Let $r = |x|$ and set $q_\alpha^{(d)}(r) = q_\alpha^{(d)}(1,x)$. From (5.5.15) it is clear that the claim (5.5.19) will follow from the identity

$$D_r\, q_\alpha^{(d)}(r) = -(2\pi) q_\alpha^{(d+2)}(r)\,. \tag{5.5.20}$$

To prove (5.5.20) we write

$$q_\alpha^{(d)}(r) = (2\pi)^{-d/2} \int_0^\infty \rho^{d-1} \mathrm{e}^{-\rho^\alpha} \frac{1}{(r\rho)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(r\rho)\, \mathrm{d}\rho$$

where we have introduced the usual Bessel function of first kind

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu - \frac{1}{2}} \mathrm{e}^{itz}\, \mathrm{d}t\,. \tag{5.5.21}$$

[2] The argument of Lemma 5.3 is due to O. Zeitouni.

Using the known identity (see e.g. [1], 9.1.30)

$$D_z z^{-\nu} J_\nu(z) = -z^{-(\nu+1)} J_{\nu+1}(z)\,,$$

we conclude

$$\begin{aligned} D_r\, q_\alpha^{(d)}(r) &= (2\pi)^{-d/2} \int_0^\infty \rho^{d+1} \mathrm{e}^{-\rho^\alpha} D_{r\rho} \frac{1}{(r\rho)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(r\rho)\, \mathrm{d}\rho \\ &= -2\pi\, q_\alpha^{(d+2)}(r)\,. \end{aligned}$$

□

Remark 5.3. In the above lemma the function $\mathrm{e}^{-\rho^\alpha}$ plays no essential role. In particular, this means that any probability density q which is given by the Fourier transform of a (sufficiently rapidly decaying) radial function which is independent of d will satisfy (5.5.19). Now, if it also satisfies $q(x) \sim |x|^{-d-\alpha}$ then we may set $p_\alpha(0,x) = \text{const.} \times q(x)$, and both H1 and H2 are satisfied.

Remark 5.4. As the following limit theorems show, it is not essential that the random walk be isotropic as assumed in H1. It is only necessary that it is asymptotically isotropic, in the sense that (5.5.3) and (5.5.10) hold with constants which are independent of the direction in which the limit is taken. In particular, all results of the following chapters hold for such random walks. The following asymptotically isotropic stable random walk was considered in [6],

$$p_\alpha(x,y) = \int_V q_\alpha(1, z + (x-y)^+)\, \mathrm{d}z \tag{5.5.22}$$

where $V \equiv [-1/2, 1/2]^d$ and $(x)^+ \equiv (|x_1|, \ldots, |x_d|)$. Then (5.5.3) is a consequence of (4.4.2) and H2 follows from Lemma 5.2 and Lemma 5.3.

5.3 Limit theorems

In the rest of this section we exploit the convergence (5.5.10) to prove that certain quadratic forms associated to the lattice random walk converge, after rescaling, to the forms of the corresponding stable process in $\mathbb{R}^d$. The latter were defined in chapter 4. Recall also the notation $\langle\cdot,\cdot\rangle_\Lambda$ for scalar products in $l^2(\Lambda)$. The following convergence results were first obtained in [5], in the local case. The generalization of Lemma 5.4 to our setting is straightforward. On the other hand, the capacity-order result of Lemma 5.5 appears to be non trivial, since Theorem 4.3 is essential.

Recall the definitions of the embedded Dirichlet form $\mathcal{E}_V$ and the integral operator $\mathcal{G}_V$ for the rescaled unit box $V = [-1/2, 1/2]^d$, cf. chapter 4. We also write $G^{-1} = J = \mathbb{1} - p$.

Lemma 5.4. *Let f be a Riemann integrable function on V and let $h \in \mathcal{C}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Define $f_N(x) = f(x/N)$, for $x \in \Lambda_N$, and $h_N(x) = h(x/N)$, for $x \in \mathbb{Z}^d$, then*

$$\lim_{N\to\infty} N^{-d-\alpha} \langle f_N, G_N f_N \rangle_{\Lambda_N} = (f, \mathcal{G}_V f)_V \,,$$
$$\lim_{N\to\infty} N^{\alpha-d} \left\langle h_N, G^{-1} h_N \right\rangle_{\mathbb{Z}^d} = \mathcal{E}(h,h) \,.$$

In particular, for any $h \in L^2(\mathbb{R}^d)$, such that $h1_{V^c} = 0$ a.e., and $h1_V \in \mathcal{C}^1(V)$, we have

$$\lim_{N\to\infty} N^{\alpha-d} \left\langle h_N, \Gamma_N^{-1} h_N \right\rangle_{\Lambda_N} = \mathcal{E}(h,h) \,.$$

Proof. From Riemann integration and (5.5.10)

$$\begin{aligned}
&\lim_{N\to\infty} N^{-d-\alpha} \langle f_N, G_N f_N \rangle_{\Lambda_N} \\
&\quad = \lim_{M\to\infty} \lim_{N\to\infty} N^{-2d} \sum_{\substack{x,y\in\Lambda_N: \\ |x-y|>M}} f(x/N) N^{d-\alpha} G(x-y) f(y/N) \\
&\quad = \lim_{N\to\infty} N^{-2d} \sum_{x,y\in\Lambda_N, x\neq y} f(x/N) g_\alpha(x/N - y/N) f(y/N) \\
&\quad = (f, \mathcal{G}_V f)_V \,.
\end{aligned}$$

To prove the second identity we write

$$\begin{aligned}
&\lim_{N\to\infty} N^{-d+\alpha} \left\langle h_N, G^{-1} h_N \right\rangle_{\mathbb{Z}^d} \\
&\quad = \frac{1}{2} \lim_{M\to\infty} \lim_{N\to\infty} N^{-2d} \sum_{\substack{x,y\in\mathbb{Z}^d: \\ |x-y|>M}} N^{d+\alpha} p_\alpha(x-y) \big(h(x/N) - h(y/N)\big)^2 \\
&\quad = \frac{1}{2} c_{\alpha,d} \lim_{N\to\infty} N^{-2d} \sum_{x,y\in\mathbb{Z}^d, x\neq y} |x-y|^{-d-\alpha} \big(h(x/N) - h(y/N)\big)^2 \\
&\quad = \mathcal{E}(h,h) \,,
\end{aligned}$$

where we have used (5.5.3) with $v_{\alpha,d} = c_{\alpha,d}$, cf. (5.5.9), and the expression (4.4.7) for $\mathcal{E}(\cdot,\cdot)$. Finally, the last assertion is an immediate consequence of the previous one, since if $h1_{V^c} = 0$ a.e. we have

$$\left\langle h_N, \Gamma_N^{-1} h_N \right\rangle_{\Lambda_N} = \left\langle h_N, G^{-1} h_N \right\rangle_{\mathbb{Z}^d} \,.$$

□

Lemma 5.5. *Let $f \in \mathcal{C}^1(V)$ and define $f_N(\cdot) = f(\cdot/N)$ on Λ_N. Then*

$$\lim_{N\to\infty} N^{-d+\alpha} \left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} = \mathcal{E}_V(f,f) \tag{5.5.23}$$

Proof. Recall that

$$\left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} = \sup \left\{ 2 \left\langle f_N, \phi_N \right\rangle_{\Lambda_N} - \left\langle \phi_N, G_N \phi_N \right\rangle_{\Lambda_N}, \phi_N \in \mathbb{R}^{\Lambda_N} \right\}$$

Therefore

$$N^{-d+\alpha} \left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} \geqslant 2N^{-d} \left\langle f_N, \phi_N \right\rangle_{\Lambda_N} - N^{-d-\alpha} \left\langle \phi_N, G_N \phi_N \right\rangle_{\Lambda_N},$$

for any $\phi \in \mathcal{C}(V)$, with $\phi_N(\cdot) = \phi(\cdot/N)$. By Lemma 5.4 and (4.4.24), we see that

$$\liminf_{N\to\infty} N^{-d+\alpha} \left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} \geqslant \mathcal{E}_V(f, f).$$

On the other hand, for each $h \in \mathcal{C}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, with $h = f$ on V

$$\begin{aligned}\left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} &\leqslant \sup_{\phi \in \ell^2(\mathbb{Z}^d)} \left\{ 2 \left\langle h_N, \phi \right\rangle_{\mathbb{Z}^d} - \left\langle \phi, G\phi \right\rangle_{\mathbb{Z}^d} \right\} \\ &= \left\langle h_N, G^{-1} h_N \right\rangle_{\mathbb{Z}^d}.\end{aligned}$$

Using Lemma 5.4 and Theorem 4.3, we see that

$$\begin{aligned}&\limsup_{N\to\infty} N^{-d+\alpha} \left\langle f_N, G_N^{-1} f_N \right\rangle_{\Lambda_N} \\ &\qquad \leqslant \inf \left\{ \mathcal{E}(h,h), \quad h \in \mathcal{C}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad h = f \text{ a.e. on } V \right\} \\ &\qquad = \inf \left\{ \mathcal{E}(h,h), \quad h \in \overline{\mathcal{D}}_{\mathcal{E}}, \ h = f \text{ a.e. on } V \right\} \\ &\qquad = \mathcal{E}_V(f,f).\end{aligned}$$

□

Remark 5.5. As a consequence of Lemma 5.5 we obtain the convergence of *capacity.* Choosing $f = 1_V$, we have

$$\lim_{N\to\infty} N^{-d+\alpha} \mathrm{cap}_N(\Lambda_N) = \mathrm{cap}_\alpha(V) \tag{5.5.24}$$

where

$$\mathrm{cap}_N(\Lambda_N) = \left\langle 1_N, G_N^{-1} 1_N \right\rangle_{\Lambda_N}$$

is the capacity of the box Λ_N for the random walk (cf. e.g. section 25 of [71]), while

$$\mathrm{cap}_\alpha(V) = \mathcal{E}_V(1_V, 1_V)$$

is the capacity of V for the α-stable process, cf. (4.4.11) and Theorem 4.3.

Chapter 6

Capacity-order large deviations

6.1 Main results

Recall the definition of $P = P_\alpha$, the reference measure corresponding to the stable random walk defined in chapter 5. We shall sometimes omit the subscript α in the following.

We start from the idea introduced in [5] to use the following convenient representation for the set of shift invariant Gibbs states. Namely, from Theorem 2.2[1] and an obvious reparametrization argument, the set $\mathcal{G}_2^S$ of shift invariant Gibbs measures with finite second moments can be written

$$\mathcal{G}_2^S = \left\{ \int_V \gamma_{\phi(x)} \, \mathrm{d}x \, , \quad \phi \in L^2(V) \right\}, \tag{6.6.1}$$

where V is the rescaled box

$$V \equiv \left[-\frac{1}{2}, \frac{1}{2} \right]^d ,$$

and $\gamma_{\phi(x)}$ stands for the Gaussian field with constant mean $\phi(x)$ and covariance G. Note that in the above decomposition, apart from the case of extremal Gibbs states ($\phi =$ const.), the profile ϕ need not be unique. Recall the definition (4.4.21) of the Dirichlet form $\mathcal{E}_V$ of the embedded process. For $\mu \in \mathcal{M}_1(\Omega)$, the free rate function is defined by

$$\mathcal{C}_\alpha(\mu) = \frac{1}{2} \inf \left\{ \mathcal{E}_V(\phi, \phi) \, , \; \phi \in L^2(V) : \; \mu = \int_V \gamma_{\phi(x)} \, \mathrm{d}x \right\}, \tag{6.6.2}$$

where the infimum over empty sets is set equal to $+\infty$. We prove the following strong large deviations principle for R_N.

Theorem 6.1. *$\mathcal{C}_\alpha$ is lower semi-continuous, it has compact level sets, and the following bounds hold*

[1] Apparently, the necessity of Theorem 2.2 had been overlooked in [5].

- *for any closed set* $F \subset \mathcal{M}_1(\Omega)$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}_\alpha(\mu) \tag{6.6.3}$$

- *for any open set* $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} \mathcal{C}_\alpha(\mu)\,. \tag{6.6.4}$$

The next result is a strong large deviations principle for the empirical field of the zero boundary condition measure $P^0_N = P^0_{\alpha,N}$. The rate function is now expressed in terms of the Dirichlet form $\mathcal{E}^0_V$ of the killed process, see (4.4.36). The rate function is given by

$$\mathcal{C}^0_\alpha(\mu) = \frac{1}{2}\,\inf\Big\{\mathcal{E}^0_V(\psi,\psi),\ \psi \in L^2(V):\ \mu = \int_V \gamma_{\psi(x)}\,\mathrm{d}x\Big\}. \tag{6.6.5}$$

Theorem 6.2. $\mathcal{C}^0_\alpha$ *is lower semi-continuous, it has compact level sets, and the following bounds hold*

- *for any closed set* $F \subset \mathcal{M}_1(\Omega)$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}^0_\alpha(\mu) \tag{6.6.6}$$

- *for any open set* $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} \mathcal{C}^0_\alpha(\mu)\,. \tag{6.6.7}$$

Remark 6.1. Let us denote $\mathcal{E}(\psi,\psi)$, the Dirichlet form of the symmetric α-stable process on $\mathbb{R}^d$, with extended domain $\overline{\mathcal{D}}_{\mathcal{E}}$. The rate functions $\mathcal{C}_\alpha$ and $\mathcal{C}^0_\alpha$ may be written in the following way. We know that $\mathcal{E}^0_V(\phi,\phi)$ is given by $\mathcal{E}(\psi,\psi)$ where $\psi \in \mathcal{D}_{\mathcal{E}}$ agrees with ϕ on V and $\psi = 0$ a.e. on $V^c = \mathbb{R}^d \setminus V$. Therefore

$$\mathcal{C}^0_\alpha(\mu) = \frac{1}{2}\,\inf\Big\{\mathcal{E}(\psi,\psi),\ \psi \in \overline{\mathcal{D}}_{\mathcal{E}}:\ \psi 1_{V^c} = 0\ \text{ a.e. and }\ \mu = \int_V \gamma_{\psi(x)}\,\mathrm{d}x\Big\}. \tag{6.6.8}$$

On the other hand, by Theorem 4.3 we may write $\mathcal{E}_V(\phi,\phi)$ as an infimum over $\mathcal{E}(\psi,\psi)$ for $\psi \in \mathcal{D}_{\mathcal{E}}$ which agree with ϕ on V. We then have

$$\mathcal{C}_\alpha(\mu) = \frac{1}{2}\,\inf\Big\{\mathcal{E}(\psi,\psi),\ \psi \in \overline{\mathcal{D}}_{\mathcal{E}}:\ \mu = \int_V \gamma_{\psi(x)}\,\mathrm{d}x\Big\}. \tag{6.6.9}$$

The above expressions show that $\mathcal{C}^0_\alpha(\mu) \geqslant \mathcal{C}_\alpha(\mu)$ for any $\mu \in \mathcal{M}_1(\Omega)$. Moreover, they can be used to compute the rate at which R_N concentrates around the extremal Gibbs states γ_t, $t \in \mathbb{R}$. Indeed, for the infinite field problem we have

$$\mathcal{C}_\alpha(\gamma_t) = \frac{t^2}{2}\,\mathrm{cap}_\alpha(V)\,, \tag{6.6.10}$$

where $\mathrm{cap}_\alpha(V)$ is the *capacity* of the unit box V for the symmetric α-stable process (see chapter 4). In particular, $\mathcal{C}_\alpha(\gamma_t) < \infty$. On the other hand we expect that the cost of such a deviation in the case of zero boundary condition is at least of the "surface" order N^{d-1} for each $t \neq 0$. This would imply $\mathcal{C}^0_\alpha(\gamma_t) = +\infty$ for $\alpha > 1$, $t \neq 0$. In Lemma 4.5 we have shown that if we replace V by a Euclidean ball B we have

$$\mathcal{E}(1_B, 1_B) = \begin{cases} +\infty & \alpha \in [1,2)\,,\ d \geqslant 2 \\ < \infty & \alpha \in (0,1)\,,\ d \geqslant 1 \end{cases} \tag{6.6.11}$$

6.2 Strategy of proofs

We turn to a description of the main ideas involved in the proof of Theorem 6.1 and Theorem 6.2.

Rate function

The properties of the rate functions $\mathcal{C}_\alpha$ and $\mathcal{C}^0_\alpha$ on $\mathcal{M}_1(\Omega)$ are a consequence of the corresponding properties of the Dirichlet forms $\mathcal{E}_V$ and $\mathcal{E}^0_V$, and the continuity of the map

$$L^2(V) \ni \phi \to \gamma(\phi) \equiv \int_V \gamma_{\phi(x)}\,\mathrm{d}x \in \mathcal{M}_1(\Omega)\,. \tag{6.6.12}$$

This is discussed in Proposition 6.5 below.

Lower bounds

The main idea of the lower bound is to use the convergence pointed out in Lemma 5.5, which is seen to yield convergence of relative entropies at capacity-order. Namely, given a smooth profile $\psi \in \mathcal{C}^\infty(V)$, consider the Gaussian measure $\gamma^{\psi_N} \in \mathcal{M}_1(\mathbb{R}^{\Lambda_N})$, defined by

$$\begin{aligned} \gamma^{\psi_N}(\phi_x) = \psi_N(x) = \psi(x/N)\,, \\ \gamma^{\psi_N}(\phi_x\phi_y) - \gamma^{\psi_N}(\phi_x)\gamma^{\psi_N}(\phi_y) = G(x,y)\,, \quad x, y \in \Lambda_N\,. \end{aligned} \tag{6.6.13}$$

Its Radon-Nykodin derivative w.r.t. P on the box Λ_N is given by

$$\varphi_N(\phi) = \frac{d\gamma^{\psi_N}}{dP_N}(\phi) = \exp\Big(-\frac{1}{2}\,\big\langle \psi_N, G_N^{-1}\psi_N\big\rangle_{\Lambda_N} + \big\langle \psi_N, G_N^{-1}\phi\big\rangle_{\Lambda_N}\Big) \tag{6.6.14}$$

Therefore

$$H_N(\gamma^{\psi_N}|P) = \gamma^{\psi_N}(\log\varphi_N) = \frac{1}{2}\,\big\langle \psi_N, G_N^{-1}\psi_N\big\rangle_{\Lambda_N}\,, \tag{6.6.15}$$

and by Lemma 5.5

$$\lim_{N\to\infty} N^{\alpha-d} H_N(\gamma^{\psi_N}|P) = \frac{1}{2}\,\mathcal{E}_V(\psi,\psi)\,. \tag{6.6.16}$$

The classical change of measure argument then shows that all we have to do to establish the lower bound is check that

$$\lim_{N\to\infty} \gamma^{\psi_N}\big(R_N \in G\big) = 1\,,$$

for any open set G containing the measure $\gamma(\psi)$, see (6.6.12). We refer to Proposition 6.5 below.

Upper bounds: The profile measures

The proof of the upper bounds is based on the following analysis of ergodic properties of "level 2" profiles. We define the profile measures

$$Y_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \delta_{x/N} \otimes \delta_{\phi_x}\,. \tag{6.6.17}$$

Y_N is a random variable with values in $\mathcal{M}_1(V\times\mathbb{R})$, the set of probability measures on the product $V\times\mathbb{R}$. We consider the rate function

$$\mathcal{I}_\alpha(\nu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_V(\phi,\phi) & \text{if } \nu = dx\otimes\mathcal{N}(\phi(x),\sigma^2),\ \ \phi\in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{6.6.18}$$

where $\nu\in\mathcal{M}_1(V\times\mathbb{R})$ and $\mathcal{N}(m,\sigma^2)$ stands for the normal distribution on $\mathbb{R}$ with mean m and variance $\sigma^2 = G(0,0)$. We shall prove the following strong large deviations principle for Y_N under the infinite volume field P_α.

Theorem 6.3. *The function $\mathcal{I}_\alpha$ is lower semi-continuous and has compact level sets. Moreover, the profiles Y_N satisfy the estimates*

- *for any closed set $F\subset\mathcal{M}_1(V\times\mathbb{R})$,*

$$\limsup_{N\to\infty} N^{\alpha-d}\log P_\alpha\big(Y_N\in F\big) \leqslant -\inf_{\nu\in F}\mathcal{I}_\alpha(\nu) \tag{6.6.19}$$

- *for any open set $G\subset\mathcal{M}_1(V\times\mathbb{R})$,*

$$\liminf_{N\to\infty} N^{\alpha-d}\log P_\alpha\big(Y_N\in G\big) \geqslant -\inf_{\nu\in G}\mathcal{I}_\alpha(\nu)\,. \tag{6.6.20}$$

In section 3 we show how to derive the upper bound (6.6.3) from the upper bound (6.6.19).

The strategy behind the proof of Theorem 6.3 is the same as in [5]. We first consider the signed measures X_N on V, defined by

$$X_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \phi_x \delta_{x/N} \,. \tag{6.6.21}$$

Since X_N is a Gaussian variable, a large deviations principle under P is established in a standard way, see Proposition 7.5. We then make a continuous transformation which sends X_N into a smooth profile $\phi_{N,\varepsilon} \in L^2(V)$, $\varepsilon > 0$, given by

$$\phi_{N,\varepsilon}(x) = N^{-d} \sum_{z \in \Lambda_N} \rho_\varepsilon(x - z/N)\phi_z \tag{6.6.22}$$

where $\rho_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ is a smooth probability density which tends to a delta function as $\varepsilon \to 0$. We then define the level 2 profile measures

$$\Phi_\varepsilon(X_N) = \mathrm{d}x \otimes \mathcal{N}(\phi_{N,\varepsilon}(x), \sigma^2) \in \mathcal{M}_1(V \times \mathbb{R}) \,. \tag{6.6.23}$$

A large deviations principle for $X_{N,\varepsilon}$ is obtained via contraction principle on the variables X_N, for each fixed $\varepsilon > 0$. The crucial point is to show that as ε goes to zero the variables $\Phi_\varepsilon(X_N)$ and Y_N are exponentially well approximated on the scale $N^{d-\alpha}$. The main ingredient in the approximation is a conditioning argument which fully exploits the Gaussian nature of the problem. Following [5], we choose a sub-lattice $\mathcal{L} = L\mathbb{Z}^d$, with scale $L > 0$ and consider the Gaussian fields

$$\zeta_x = P(\phi_x | \mathcal{F}_\mathcal{L}) \,, \quad \tau_x = \phi_x - \zeta_x \,, \quad x \in \mathbb{Z}^d \,, \tag{6.6.24}$$

where $\mathcal{F}_\mathcal{L}$ denotes the σ-algebra generated by the variables ϕ_x, $x \in \mathcal{L}$. The fields τ and ζ are independent. Moreover, we have a random walk representation for the τ-field:

$$P(\tau_x \tau_y) = G^L(x, y) \,, \quad x, y \in \mathbb{Z}^d \,, \tag{6.6.25}$$

where G^L is the Green function of the random walk which is killed upon hitting the sub-lattice $\mathcal{L}$. In particular, τ is a weakly coupled field. These properties of the conditional fields are discussed in appendix A. We then choose the scale $L = \log N$ and introduce the auxiliary profiles

$$Z_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \delta_{x/N} \otimes \mathcal{N}(\zeta_x, \sigma^2) \,. \tag{6.6.26}$$

Using the weak coupling property of the field τ we prove that Z_N approximates exponentially both the profiles Y_N and $\Phi_\varepsilon(X_N)$, see Proposition 7.2 and Lemma 7.7.

Let us finally discuss the profiles Y_N under the measure $P^0_{\alpha,N}$, the zero boundary conditions field. We define the rate function

$$\mathcal{I}^0_\alpha(\nu) = \begin{cases} \frac{1}{2}\,\mathcal{E}^0_V(\phi,\phi) & \text{if } \nu = dx \otimes \mathcal{N}(\phi(x), \sigma^2),\ \phi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{6.6.27}$$

Theorem 6.4. *The function $\mathcal{I}_\alpha^0$ is lower semi-continuous, it has compact level sets, and*

- *for any closed set $F \subset \mathcal{M}_1(V \times \mathbb{R})$,*
$$\limsup_{N\to\infty} N^{\alpha-d} \log P_{\alpha,N}^0\big(Y_N \in F\big) \leqslant -\inf_{\nu\in F} \mathcal{I}_\alpha^0(\nu) \tag{6.6.28}$$
- *for any open set $G \subset \mathcal{M}_1(V \times \mathbb{R})$,*
$$\liminf_{N\to\infty} N^{\alpha-d} \log P_{\alpha,N}^0\big(Y_N \in G\big) \geqslant -\inf_{\nu\in G} \mathcal{I}_\alpha^0(\nu)\,. \tag{6.6.29}$$

Again, the upper bound in Theorem 6.2 will follow from the upper bound (6.6.28).

The proof of Theorem 6.4 will be accomplished through the following main steps. We introduce the measures $P_k^{N,L}$, $L > 0$ and $k > 1$, defined as the harmonic crystal corresponding to a random walk with *soft* killing proportional to L/N^α in the region $W_{k,N} = \Lambda_{kN} \setminus \Lambda$, i.e.

$$\frac{\mathrm{d}P_k^{N,L}}{\mathrm{d}P}(\phi) = \big(Z_{k,N}^L\big)^{-1} \exp\Big(-\frac{1}{2}\,LN^{-\alpha}\sum_{x\in W_{k,N}} \phi_x^2\Big)\,.$$

We denote $G_k^{N,L}$ the corresponding Green function. The first step then consists in proving a large deviations principle for Y_N under $P_k^{N,L}$, for each fixed $L > 0$ and $k > 1$. This will be obtained essentially as an application of Varadhan's lemma, cf. Proposition 8.1. Note that the choice of the scaling $LN^{-\alpha}$ is precisely the one required. The second step is the exponential approximation between $P_k^{N,L} \circ Y_N^{-1}$ and $P_{k,N}^0 \circ Y_N^{-1}$ when L goes to infinity. Here $P_{k,N}^0$ stands for the harmonic crystal embedded in the large box Λ_{kN} with zero boundary conditions on $W_{k,N}$. Note that in the local case $P_{k,N}^0$ is not distinguished from P_N^0 as soon as $k > 1$. The approximation is achieved by means of a probabilistic estimate on the top eigenvalues of the difference of covariances $\Gamma_k^{N,L} \equiv G_k^{N,L} - \Gamma_{k,N}$, cf. Proposition 8.2. Here we use the explicit representation

$$\Gamma_k^{N,L}(x,y) = \mathbb{E}_x\left[\sum_{n=\tau_{k,N}}^{\infty} 1_{\{y\}}(\xi_n) \exp\Big(-\log\big(1+LN^{-\alpha}\big)\sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l)\Big)\right],$$

where $\tau_{k,N}$ is the hitting time

$$\tau_{k,N} = \inf\left\{n \geqslant 1\,,\ \xi_n \in W_{k,N}\right\}\,.$$

At this point we would be done in the local case. In the *non-local* case obtained from stable random walks, we shall prove that, as $k \to \infty$, the field $P_{k,N}^0 \circ Y_N^{-1}$ is exponentially equivalent to $P_N^0 \circ Y_N^{-1}$.

We now proceed to prove Theorem 6.1 and Theorem 6.2. Large deviations for the profiles Y_N will be proven in chapter 7 and chapter 8.

6.3 Infinite volume field

We recall the definition (6.6.2) of the rate function $\mathcal{C}_\alpha$.

Proposition 6.5. *The function $\mathcal{C}_\alpha$ is lower semi-continuous and has compact level sets. Moreover, for any open set $G \subset \mathcal{M}_1(\Omega)$,*

$$\liminf_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in G\big) \geqslant - \inf_{\mu\in G} \mathcal{C}_\alpha(\mu)\,. \tag{6.6.30}$$

Proof. Let us first show that the level sets

$$\big\{\mu \in \mathcal{M}_1(\Omega) : \ \mathcal{C}_\alpha(\mu) \leqslant L\big\}\,, \quad L \geqslant 0 \tag{6.6.31}$$

are compact. Consider the function $\gamma(\phi)$ defined in (6.6.12). It is not difficult to show that $\phi \to \gamma(\phi)$ is continuous as a function from $L^1(V)$ to $\mathcal{M}_1(\Omega)$. In particular, (6.6.12) is continuous. On the other hand, by Lemma 4.2 the function

$$L^2(V) \ni \psi \to \mathcal{E}_V(\psi, \psi)$$

is lower semi-continuous and has compact level sets. It follows that the level sets of $\mathcal{C}_\alpha$ are the image under $\gamma(\cdot)$ of the level sets of $\mathcal{E}_V$ and this implies compactness. In particular, the sets (6.6.31) are closed, so that $\mathcal{C}_\alpha$ is lower semi-continuous.

We turn to the proof of (6.6.30). Here we follow [5]. We may assume that $G \cap \mathcal{G}_2^S \neq \emptyset$, otherwise the lower bound (6.6.30) is trivially satisfied. To prove (6.6.30) it is therefore sufficient to show

$$\liminf_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in G\big) \geqslant - \frac{1}{2}\,\mathcal{E}_V(\psi, \psi) \tag{6.6.32}$$

for any $\psi \in L^2(V)$ such that $\gamma(\psi) \in G$. Moreover, we may restrict ourselves to $\psi \in \mathcal{C}^\infty(V)$. In fact, from the continuity of the map (6.6.12) and the semi-continuity of $\mathcal{E}_V$, it follows that for any $\varepsilon > 0$, for any $\phi \in L^2(V)$ and for any open set G containing $\gamma(\phi)$, we can find $\psi \in \mathcal{C}^\infty(V)$ such that $\gamma(\psi) \in G$ and

$$\mathcal{E}_V(\psi, \psi) \leqslant \mathcal{E}_V(\phi, \phi) + \varepsilon\,.$$

The classical argument (e.g. Lemma 5.4.21 in [23]) shows that (6.6.32) follows from (6.6.16) and

$$\lim_{N\to\infty} \gamma^{\psi_N}\big(R_N \in G\big) = 1\,, \tag{6.6.33}$$

for any open set G containing $\gamma(\psi)$. Note that if ψ is a constant, then (6.6.33) is a straightforward consequence of the ergodic theorem. Moreover, (6.6.33) can be extended to piecewise constant functions. As in [5], Theorem 2.9, we may now approximate the probabilities in (6.6.33) by the corresponding probabilities obtained using piecewise constant approximations of ψ. The conclusion is that (6.6.33) holds for any $\psi \in \mathcal{C}^\infty(V)$. We omit the details, since the argument given in [5] carries over unchanged. $\square$

Let us now show that the upper bound of Theorem 6.3 implies the upper bound of Theorem 6.1.

Proposition 6.6. *Assume that the profiles Y_N satisfy Theorem 6.3. Then for any closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}_\alpha(\mu) \tag{6.6.34}$$

We shall need the following simple lemma. Let μ be the centered Gaussian law on $\mathbb{R}^n$, $n \in \mathbb{Z}_+$, with positive definite covariance matrix

$$\Gamma = \Gamma\,(i,j)_{i,j=1,\dots,n},$$

and denote by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0$ the eigenvalues of Γ.

Lemma 6.7. *For any $\beta > 0$ such that $\beta\lambda_1 < 1$, we have*

$$\log \mu\big(\exp \frac{1}{2}\,\beta \sum_{i=1}^n \phi_i^2\big) \leqslant \frac{1}{2}\,\beta\,\mathrm{Tr}\,\Gamma + \frac{1}{2}\,\beta^2 W(\beta,\lambda_1)\,\mathrm{Tr}\,\Gamma^2 \tag{6.6.35}$$

where

$$W(\beta,\lambda_1) = \frac{1}{2} - \log\left(1-\beta\lambda_1\right). \tag{6.6.36}$$

Proof. Direct computation yields

$$\mu\big(\exp \frac{1}{2}\,\beta \sum_{i=1}^n \phi_i^2\big) = \det\big(1-\beta\Gamma\big)^{-\frac{1}{2}} = \prod_{j=1}^n \big(1-\beta\lambda_j\big)^{-\frac{1}{2}}.$$

Using

$$\sum_{j=1}^n \lambda_j = \mathrm{Tr}\,\Gamma\,, \quad \sum_{j=1}^n \lambda_j^2 = \mathrm{Tr}\,\Gamma^2\,,$$

and the definition (6.6.36) of W, we have

$$\begin{aligned}
&\log \mu\big(\exp \frac{1}{2}\,\beta \sum_{i=1}^n \phi_i^2\big) \\
&= -\frac{1}{2}\sum_{j=1}^n \log\big(1-\beta\lambda_j\big) = \frac{1}{2}\sum_{j=1}^n\sum_{k=1}^\infty \frac{(\beta\lambda_j)^k}{k} \\
&= \frac{1}{2}\,\beta\,\mathrm{Tr}\,\Gamma + \frac{1}{2}\sum_{j=1}^n\sum_{k=2}^\infty \frac{(\beta\lambda_j)^k}{k} \\
&\leqslant \frac{1}{2}\,\beta\,\mathrm{Tr}\,\Gamma + \frac{1}{2}\,\beta^2\sum_{k=0}^\infty \frac{(\beta\lambda_1)^k}{k+2}\,\mathrm{Tr}\,\Gamma^2 \\
&\leqslant \frac{1}{2}\,\beta\,\mathrm{Tr}\,\Gamma + \frac{1}{2}\,\beta^2 W(\beta,\lambda_1)\,\mathrm{Tr}\,\Gamma^2\,.
\end{aligned}$$

$\square$

Proof of Proposition 6.6

We start with a proof of the exponential tightness of R_N. We claim that there exist $c > 0$, $L_0 > 0$, such that, for any $L > L_0$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \notin \mathcal{K}_L\big) \leqslant -cL, \tag{6.6.37}$$

where $\mathcal{K}_L$ is the compact set

$$\mathcal{K}_L = \big\{\mu \in \mathcal{M}_1^S(\Omega)\,,\ \ \mu(\phi_0^2) \leqslant L\big\}\,. \tag{6.6.38}$$

In order to prove (6.6.37), notice that for any $a > 0$

$$\begin{aligned} P\big(R_N \notin \mathcal{K}_L\big) &\leqslant P\big(N^{-d} \sum_{x\in\Lambda_N} \phi_x^2 > L\big) \\ &\leqslant \exp(-aL)\, P\big(\exp aN^{-d} \sum_{x\in\Lambda_N} \phi_x^2\big)\,. \end{aligned} \tag{6.6.39}$$

From Lemma 6.7 we obtain

$$\log P\big(R_N \notin \mathcal{K}_L\big) \leqslant -aL + aN^{-d}\operatorname{Tr} G_N + 2a^2 N^{-2d} W(2aN^{-d}, \lambda_1) \operatorname{Tr} G_N^2\,, \tag{6.6.40}$$

where $\lambda_1 = \lambda_1(N)$ is the maximal eigenvalue of G_N and we have set $\beta = 2aN^{-d}$ in Lemma 6.7. Recalling (5.5.10) it is easy to check:

$$\lambda_1(N) \leqslant \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} G(x,y) \leqslant O(N^\alpha) \tag{6.6.41}$$

$$\operatorname{Tr} G_N^2 = \sum_{x,y\in\Lambda_N} G(x,y)^2 \leqslant o(N^{d+\alpha})\,. \tag{6.6.42}$$

Choosing now $a = a(N) = 2cN^{d-\alpha}$ in (6.6.40), for $c > 0$ sufficiently small, we see that $W(2aN^{-d}, \lambda_1)$ is uniformly bounded in N and the last term in the r.h.s. is $o(N^{d-\alpha})$. Using $\operatorname{Tr} G_N = N^d\sigma^2$, we have

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \notin \mathcal{K}_L\big) \leqslant -2c(L-\sigma^2) \tag{6.6.43}$$

which implies (6.6.37) for all sufficiently large L.

We turn to the proof of (6.6.34). Once the exponential tightness is established we may assume F to be a compact set in (6.6.34), see e.g. Lemma 2.1.5 of [23]. Therefore, given $\rho > 0$, we can find $\mu_i \in F$, $i = 1, \dots, n$, such that

$$F \subset \bigcup_{i=1}^n B_\rho(\mu_i)\,,$$

where, for each $\mu \in \mathcal{M}_1(\Omega)$, $B_\rho(\mu)$ is the Prohorov ball of radius ρ around μ. Then

$$\begin{aligned}\limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in F\big) \\ \leqslant \max_{i=1,\dots,n} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in B_\rho(\mu_i)\big)\,. \end{aligned} \tag{6.6.44}$$

This shows that to establish (6.6.34) is now sufficient to prove that, for any $\mu \in F$,

$$\lim_{\rho\searrow 0} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in B_\rho(\mu)\big) \leqslant -\,\mathcal{C}_\alpha(\mu)\,. \tag{6.6.45}$$

Observe that in view of (3.3.4) and (2.2.35), (6.6.45) is always satisfied for $\mu \notin \mathcal{G}_2^S$. We may therefore assume $\mu = \gamma(\psi)$ for some $\psi \in L^2(V)$, cf. (6.6.12). Let $\pi_0 : \Omega \to \mathbb{R}$ be the canonical projection onto the origin, $\pi_0\phi = \phi_0$. Notice that

$$\mu \circ \pi_0^{-1} = \gamma(\psi) \circ \pi_0^{-1} = \int_V \mathcal{N}(\psi(x), \sigma^2)\,\mathrm{d}x\,. \tag{6.6.46}$$

Consider the level 2 empirical measure

$$L_N(\phi) = R_N(\phi) \circ \pi_0^{-1} = N^{-d} \sum_{x\in\Lambda_N} \delta_{\phi_x} \in \mathcal{M}_1(\mathbb{R})\,.$$

In particular,

$$L_N(\phi)(f) = Y_N(\phi)(1_V \otimes f)\,, \quad f \in \mathcal{C}_b(\mathbb{R})\,.$$

Let B^1_ρ denote the Prohorov ball of radius ρ in $\mathcal{M}_1(\mathbb{R})$. Then if $R_N \in B_\rho(\mu)$, there exists $\rho' > 0$ such that $L_N \in B^1_{\rho'}(\mu \circ \pi_0^{-1})$, with $\rho' \to 0$, as $\rho \to 0$. Applying the contraction principle (see e.g. [21], Theorem 4.2.1) to the variables Y_N and using Theorem 6.3, we see that

$$\begin{aligned}\lim_{\rho\searrow 0} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in B_\rho(\mu)\big) \\ \leqslant \lim_{\rho'\searrow 0} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(L_N \in B^1_{\rho'}(\mu \circ \pi_0^{-1})\big) \leqslant -\,\mathcal{J}(\mu)\,,\end{aligned}$$

where

$$\mathcal{J}(\mu) = \inf\big\{\mathcal{I}(\nu),\ \nu \in \mathcal{M}_1(V \times \mathbb{R}) :\ \nu(1_V \otimes \cdot\,) = \mu \circ \pi_0^{-1}\big\}.$$

By the definition (6.6.18) of $\mathcal{I}$ and (6.6.46), we see that $\mathcal{J}(\mu) = \mathcal{C}_\alpha(\mu)$. This concludes the proof of (6.6.45). □

6.4 Zero boundary conditions field

We turn to the proof of Theorem 6.2. Recall the definition (6.6.5) of the rate function $\mathcal{C}^0_\alpha$.

Proposition 6.8. *The function $\mathcal{C}^0_\alpha$ is lower semi-continuous and has compact level sets. Moreover, for any open set $G \subset \mathcal{M}_1(\Omega)$,*

$$\liminf_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} \mathcal{C}^0_\alpha(\mu)\,. \tag{6.6.47}$$

Proof. Observe that the map

$$L^2(V) \ni \psi \to \mathcal{E}^0_V(\psi,\psi)$$

is lower semi-continuous and has compact level sets, see (4.4.40) and the discussion in chapter 4. Then the statements about the rate function follow just as in the proof of Proposition 6.5. To prove the lower bound (6.6.47) we also proceed in analogy with the proof of Proposition 6.5, cf. (6.6.32). We now have to replace (6.6.15) with

$$H_N(\gamma^{0,\psi_N}|P^0_{\alpha,N}) = \frac{1}{2}\,\big\langle \psi_N, \Gamma_N^{-1}\psi_N\big\rangle_{\Lambda_N}\,, \tag{6.6.48}$$

where γ^{0,ψ_N} is the Gaussian field on Λ_N with mean ψ_N and covariance Γ_N. By Lemma 5.4 and (4.4.36)

$$\lim_{N\to\infty} N^{\alpha-d} H_N(\gamma^{0,\psi_N}|P^0_{\alpha,N}) = \frac{1}{2}\,\mathcal{E}^0_V(\psi,\psi)\,. \tag{6.6.49}$$

As before, the lower bound now follows from standard arguments. □

A repetition of the proof of Proposition 6.6 yields the following

Proposition 6.9. *Assume that the profiles Y_N satisfy Theorem 6.4. Then for any closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\limsup_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}^0_\alpha(\mu)\,. \tag{6.6.50}$$

Chapter 7

Large deviations for profiles

In this chapter we prove large deviations for the profiles Y_N, i.e. Theorem 6.3. Let us first introduce some notations. For every bounded Lipshitz continuous function f on V or on $\mathbb{R}$, we write

$$\|f\|_{BL} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} .$$

We define the metric d_* by

$$\begin{aligned} d_*(\mu, \nu) = \sup\big\{ | \langle \nu - \mu, f \otimes g \rangle | \,, \\ f \in \mathcal{C}(V),\ \|f\|_{BL} \leqslant 1\,,\ \ g \in \mathcal{C}(\mathbb{R}),\ \|g\|_{BL} \leqslant 1 \big\}\,, \end{aligned} \tag{7.7.1}$$

where $\mu, \nu \in \mathcal{M}_1(V \times \mathbb{R})$ and we have used the notation

$$\langle \mu, f \otimes g \rangle = \int_V \int_{\mathbb{R}} f(x) g(u) \mu(\,\mathrm{d}x,\,\mathrm{d}u)\,.$$

Recall that the weak topology is compatible with the Wasserstein metric, which is obtained from (7.7.1) by replacing $f \otimes g$ with $h \in \mathcal{C}(V \times \mathbb{R})$ such that $\|h\|_{BL} \leqslant 1$, cf. [28]. Since the functions $f \otimes g$ are separating in $\mathcal{M}_1(V \times \mathbb{R})$, one concludes that d_* metrizes the weak topology on $\mathcal{M}_1(V \times \mathbb{R})$.

The main step in the proof of Theorem 6.3 is a large deviations principle for the profiles Z_N, with rate $\mathcal{I}_\alpha$. These are defined by

$$Z_N(\zeta) = N^{-d} \sum_{x \in \Lambda_N} \delta_{x/N} \otimes \mathcal{N}(\zeta_x, \sigma^2)\,, \tag{7.7.2}$$

where $\zeta_x = P(\phi_x | \mathcal{F}_{\mathcal{L}})$ is the field defined in (6.6.24) with $\mathcal{L} = L\mathbb{Z}^d$. We shall prove:

Proposition 7.1. *The profiles Z_N satisfy the estimates*

- *for any closed set $F \subset \mathcal{M}_1(V \times \mathbb{R})$,*

$$\limsup_{N \to \infty} N^{\alpha - d} \log P_\alpha\big(Z_N \in F\big) \leqslant -\inf_{\nu \in F} \mathcal{I}_\alpha(\nu) \tag{7.7.3}$$

- *for any open set* $G \subset \mathcal{M}_1(V \times \mathbb{R})$,

$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(Z_N \in G\big) \geqslant -\inf_{\nu\in G} \mathcal{I}_\alpha(\nu)\,. \tag{7.7.4}$$

We will also prove that Y_N and Z_N are exponentially equivalent.

Proposition 7.2. *For any* $\beta > 0$*, for any* $f \in \mathcal{C}(V)$ *and* $g \in \mathcal{C}(\mathbb{R})$ *such that* $\|g\|_{BL} < \infty$*, we have*

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(|\,\langle Y_N - Z_N, f\otimes g\rangle\,| \geqslant \beta\big) = -\infty\,. \tag{7.7.5}$$

Assuming the validity of Propositions 7.1 and 7.2, whose proof will be given in the following sections, we are in a position to prove Theorem 6.3.

7.1 Large Deviations for Y_N

Let us first check the properties of the rate function $\mathcal{I}_\alpha$, cf. (6.6.18). Define the continuous map $\Phi : L^2(V) \to \mathcal{M}_1(V\times\mathbb{R})$,

$$\Phi(\psi) = \,\mathrm{d}x \otimes \mathcal{N}(\psi(x), \sigma^2)\,. \tag{7.7.6}$$

That $\mathcal{I}_\alpha$ is lower semi-continuous and has compact level sets then follows from the continuity of Φ and the arguments of Proposition 6.5.

We turn to the proof of the upper bound (6.6.19). Let us first show that the profiles Y_N are exponentially tight. Define the compact sets

$$\mathcal{R}_L = \big\{\mu \in \mathcal{M}_1(V\times\mathbb{R}),\;\; \mu(1_V \otimes z^2) \leqslant L\big\}\;,\quad L > 0\,. \tag{7.7.7}$$

Then the arguments used in Proposition 6.6 imply that there exists $c \in (0,\infty)$ such that

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(Y_N \notin \mathcal{R}_L\big) = -\,c\,L$$

for all sufficiently large L. Then, in proving the upper bound (6.6.19), we may restrict to compact sets $F \subset \mathcal{M}_1(V\times\mathbb{R})$. Recalling the proof of Proposition 6.6, we see that it is sufficient to show that for any $\mu\in F$ we can find a sequence $\mathcal{U}_\rho(\mu)$, $\rho > 0$, of neighbourhoods of μ such that

$$\limsup_{\rho\searrow 0}\limsup_{N\to\infty} N^{\alpha-d} \log P\big(Y_N \in \mathcal{U}_\rho(\mu)\big) \leqslant -\mathcal{I}_\alpha(\mu)\,. \tag{7.7.8}$$

For any $\rho > 0$, we may choose functions $u_i \in \mathcal{C}_b(V\times\mathbb{R})$, $i = 1,\dots,n$, and $\varepsilon > 0$ such that

$$\begin{aligned}\mathcal{U}_\rho(\mu) = \{\nu \in \mathcal{M}_1(V\times\mathbb{R}) :\; &|\nu(u_i) - \mu(u_i)| < \varepsilon\,,\;\; i = 1,\dots,n\}\\ \text{and}\qquad |\nu(u_i) - \mu(u_i)| < 2\varepsilon\,,&\;\; i = 1,\dots,n \quad\Rightarrow\quad \nu \in B_\rho(\mu)\end{aligned}$$

where $B_\rho(\mu)$ denotes the ball of radius ρ around μ w.r.t. the metric d_*. Moreover, by a density argument we may further assume the functions u_i to be (a linear combination of functions) of the form $f \otimes g$, with $f \in \mathcal{C}(V)$ and $g \in \mathcal{C}(\mathbb{R})$ with $\|g\|_{BL} < \infty$. At this point it is easy to show that Proposition 7.2 implies

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(Y_N \in \mathcal{U}_\rho(\mu)\big) \leqslant \limsup_{N\to\infty} N^{\alpha-d} \log P\big(Z_N \in B_\rho(\mu)\big) . \qquad (7.7.9)$$

From (7.7.9) and the upper bound of Proposition 7.1 for Z_N, we arrive at (7.7.8). The lower bound can be proven in a similar way. □

7.2 Exponential equivalence of Y_N and Z_N

We turn to the proof of Proposition 7.2. Let $f \in \mathcal{C}(V)$ and $g \in \mathcal{C}(\mathbb{R})$ with $\|g\|_{BL} < \infty$. We define, for any $\gamma > 0$,

$$\overline{g}_\gamma(m) = \int_{\mathbb{R}} g(r)\, \mathcal{N}(m, \gamma^2)(\,\mathrm{d}r) . \qquad (7.7.10)$$

We have,

$$|\,\langle f \otimes g, Y_N - Z_N\rangle\,| \leqslant N^{-d}\Big|\sum_{x\in\Lambda_N} f(x/N)\big(g(\phi_x) - \overline{g}_\sigma(\zeta_x)\big)\Big| .$$

Using the independence of ζ and $\tau = \phi - \zeta$ (see appendix A), we can estimate uniformly in the field ζ

$$\begin{aligned} &P\big(|\,\langle f \otimes g, Y_N - Z_N\rangle\,| \geqslant \beta\big) \\ &\quad\leqslant \sup_{\bar\zeta\in\mathbb{R}^{\Lambda_N}} P\Big(N^{-d}\Big|\sum_{x\in\Lambda_N} f(x/N)\big(g(\bar\zeta_x + \tau_x) - \overline{g}_\sigma(\bar\zeta_x)\big)\Big| \geqslant \beta\Big) . \end{aligned} \qquad (7.7.11)$$

Now define

$$\gamma_x^2 = P(\tau_x^2) = G^L(x,x) , \quad x \in \Lambda_N .$$

From Lemma A.4, we see that

$$N^{-d} \sum_{x\in\Lambda_N} |\gamma_x \,-\, \sigma| \to 0 ,$$

and therefore we can replace $\overline{g}_\sigma$ with $\overline{g}_{\gamma_x}$ in (7.7.11) with a negligible error term as $N \to \infty$. Let us define

$$F_N(\tau) = N^{-d} \sum_{x\in\Lambda_N} f(x/N) g(\bar\zeta_x + \tau_x)$$

as a function of the field $\{\tau_x\}_{x\in\Lambda_N}$. In this way

$$P(F_N) = N^{-d} \sum_{x\in\Lambda_N} f(x/N)\, \overline{g}_{\gamma_x}(\bar\zeta_x) .$$

We need the following estimate which was proven in [5] (see Lemma A.4 there).

Lemma 7.3. *Let F be a continuous function on $\mathbb{R}^n$ such that*

$$\delta_i(F) = \sup_{\substack{x,y\in\mathbb{R}^n:\\ x_j=y_j,\forall j\neq i}} \frac{|F(x)-F(y)|}{|x-y|}$$

is finite, for all $i=1,\dots,n$. Let $\mu\in\mathcal{M}_1(\mathbb{R}^n)$ be a centered Gaussian measure with a positive definite covariance matrix $\Gamma(i,j)$. Then

$$\mu\left(\exp\left(F-\mu(F)\right)\right) \leqslant \exp\Big(\frac{1}{2}\sum_{i,j=1}^{n}\delta_i(F)\Gamma(i,j)\delta_j(F)\Big)\,. \tag{7.7.12}$$

An application of Lemma 7.3 to the function F_N yields

$$P\big(\exp\left(F_N-P(F_N)\right)\big) \leqslant \exp\Big(\frac{1}{2}N^{-2d}\|f\|_\infty^2\|g\|_{BL}^2\sum_{x,y\in\Lambda_N}G^L(x,y)\Big)\,. \tag{7.7.13}$$

Thanks to (7.7.13) and (7.7.11), after applying Chebyshev's inequality we get, for any $a>0$,

$$\begin{aligned}&\log P\big(\pm\,\langle f\otimes g, Y_N-Z_N\rangle\,\geqslant\beta\big)\\ &\qquad\leqslant\,-\,a\,\beta^2\,+\,a^2N^{-2d}\|f\|_\infty^2\|g\|_{BL}^2\sum_{x,y\in\Lambda_N}G^L(x,y)\,.\end{aligned}$$

Using the estimate of Lemma A.4 with $L=\log N$, and setting $a=a(N)=N^{d-\alpha+\delta}$, with any $0<\delta<\alpha/2$, we see that

$$\limsup_{N\to\infty}N^{\alpha-d}\log P\big(\pm\,\langle f\otimes g, Y_N-Z_N\rangle\,\geqslant\beta\big)=-\infty\,.$$

This implies (7.7.5). □

7.3 Large deviations for Z_N

The proof of Proposition 7.1 will be organized into several lemmas. As explained in section 6.2, we first derive a large deviations principle for level 1 profiles.

7.3.1 Level 1 profiles

Let $\mathcal{M}(V)$ be the set of all finite signed measures on V. By the F. Riesz' representation theorem, the set $\mathcal{M}(V)$ is the dual of $\mathcal{C}(V)$, the Banach space of continuous functions on V with the supremum norm. With the total variation norm $\|\cdot\|_{\mathrm{var}}$, $\mathcal{M}(V)$ becomes a Banach space. In order to prove our results it is essential to choose a weaker topology on $\mathcal{M}(V)$. Consider the convex sets

$$\mathcal{M}_r(V)=\big\{\mu\in\mathcal{M}(V),\ \ \|\mu\|_{\mathrm{var}}\leqslant r\big\}\,,\quad r\,>\,0\,,$$

so that

$$\mathcal{M}(V) \;=\; \bigcup_{r>0} \mathcal{M}_r(V)$$

We equip $\mathcal{M}(V)$ with the weak-$*$ topology, i.e. the weakest topology such that all the maps

$$\mu \to \mu(f)\,, \quad f \in \mathcal{C}(V)$$

are continuous. In this way, the topology inherited by each $\mathcal{M}_r(V)$ is the usual weak topology of measures and the sets $\mathcal{M}_r(V)$ are compact.

The level 1 profiles

$$X_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \phi_x \delta_{x/N}\,,$$

are Gaussian random variables with values in $\mathcal{M}(V)$. Let $\mathcal{G}_V$ be the operator defined in (4.4.18), and define the convex function

$$J(\mu) = \frac{1}{2} \sup_{f\in\mathcal{C}(V)} \left\{2\mu(f) - (f, \mathcal{G}_V f)_V\right\}. \tag{7.7.14}$$

Lemma 7.4. *The function $J : \mathcal{M}(V) \to [0,\infty]$ is lower semi-continuous and has compact level sets. Moreover, it satisfies*

$$J(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_V(\psi,\psi) & \text{if } \mu(\,\mathrm{d}x) = \psi(x)\,\mathrm{d}x\,,\ \psi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{7.7.15}$$

Proof. The semi-continuity of J is a straightforward consequence of the definition (7.7.14). Let us prove (7.7.15). Let $\psi \in L^2(V)$ with $\mu(\,\mathrm{d}x) = \psi(x)\,\mathrm{d}x$. Then,

$$J(\mu) = \frac{1}{2}\,\mathcal{E}_V(\psi,\psi)$$

is a consequence of the variational expression for $\mathcal{E}_V$, see (4.4.24). An alternative expression for J is obtained by replacing f in (7.7.14) with βf, and then optimizing over $\beta \in \mathbb{R}$. One has

$$J(\mu) = \frac{1}{2} \sup_{f\in\mathcal{C}(V)} \left\{\frac{|\mu(f)|^2}{(f,\mathcal{G}_V f)_V}\right\}.$$

This shows that if $J(\mu) < \infty$, then there exists $C_\mu \in (0,\infty)$, such that

$$|\mu(f)|^2 \leqslant C_\mu (f, \mathcal{G}_V f)_V \leqslant C_\mu \|\mathcal{G}_V\| (f,f)_V^2\,, \quad f \in \mathcal{C}(V)\,,$$

and therefore $f \to \mu(f)$ can be extended to a continuous functional on $L^2(V)$, and this implies there exists $\psi \in L^2(V)$ such that

$$\mu(f) = (\psi, f)_V\,, \quad f \in L^2(V)\,.$$

This completes the proof of (7.7.15). Compactness of the level sets of J now follows from the properties of $\mathcal{E}_V$, see Lemma 4.2, and the continuity of the map

$$L^2(V) \ni \psi \;\to\; \psi(x)\,\mathrm{d}x \;\in\; \mathcal{M}_1(\mathbb{R})\,.$$

□

Lemma 7.5. *The profile measures X_N satisfy the bounds*

- *for any closed set $F \subset \mathcal{M}(V)$,*
$$\limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(X_N \in F\big) \leqslant\; - \inf_{\mu\in F} J(\mu) \tag{7.7.16}$$
- *for any open set $G \subset \mathcal{M}(V)$,*
$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(X_N \in G\big) \geqslant\; - \inf_{\mu\in G} J(\mu)\,. \tag{7.7.17}$$

Proof. In view of (7.7.15) we see that the lower bound (7.7.17) follows from the same arguments of Proposition 6.5. To prove the upper bound we use the asymptotics of the logarithmic moment generating function. Since X_N is Gaussian we have, for $f \in \mathcal{C}(V)$,

$$\begin{aligned}\log P\big(\exp N^{d-\alpha} X_N(f)\big) &= \frac{1}{2}\,N^{2d-2\alpha} P\big(X_N(f)^2\big)\\ &= \frac{1}{2}\,N^{-2\alpha}\,\langle f_N, G_N f_N\rangle_{\Lambda_N}\;.\end{aligned}$$

Thanks to Lemma 5.4, we obtain, for any $f \in \mathcal{C}(V)$

$$\lim_{N\to\infty} N^{\alpha-d} \log P\big(\exp N^{d-\alpha} X_N(f)\big) = \frac{1}{2}\,(f, \mathcal{G}_V f)_V \tag{7.7.18}$$

The Legendre transform of (7.7.18) coincides, by definition (7.7.14), with the function J. Using now the standard argument (cf. Theorem 4.5.3 of [21]) we see that (7.7.16) is satisfied for all compact sets F.

In order to extend the upper bound to all closed sets $F \subset \mathcal{M}(V)$ we prove that the X_N are exponentially tight. Namely, there exist constants $c > 0$ and $r_0 > 0$, such that for any $r > r_0$

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(X_N \notin \mathcal{M}_r\big) \leqslant\; - cr^2\,. \tag{7.7.19}$$

To prove (7.7.19) observe that

$$X_N \notin \mathcal{M}_r \;\Rightarrow\; N^{-d} \sum_{x\in\Lambda_N} |\phi_x| > r \;\Rightarrow\; N^{-d} \sum_{x\in\Lambda_N} \phi_x^2 > r^2\,.$$

Therefore (7.7.19) can be proven with the same argument used in Proposition 6.6, cf. (6.6.39)-(6.6.43). The proof of the Lemma is complete. □

7.3.2 Level 2 smooth profiles

Let ρ be a smooth probability density with support contained in the unit ball of $\mathbb{R}^d$, and define $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$, $\varepsilon > 0$. For each $\mu \in \mathcal{M}(V)$, $\varepsilon > 0$, we form the function $\mu_\varepsilon \in \mathcal{C}(V)$,

$$\mu_\varepsilon(x) = \rho_\varepsilon * \mu(x) = \int_V \rho_\varepsilon(x-y)\mu(\,\mathrm{d}y)\,.$$

We define

$$\Phi_\varepsilon : \mathcal{M}(V) \to \mathcal{M}_1(V \times \mathbb{R})\,, \quad \Phi_\varepsilon(\mu) = \Phi(\mu_\varepsilon)\,,$$

where Φ is defined by (7.7.6). Then Φ_ε is continuous for each $\varepsilon > 0$. From Proposition 7.5 and the contraction principle (see e.g. Theorem 4.2.1 of [21]) we arrive at the following Lemma.

Lemma 7.6. *For each $\varepsilon > 0$, the measures $\Phi_\varepsilon(X_N)$ satisfy the bounds*

- *for any closed set $F \subset \mathcal{M}_1(V \times \mathbb{R})$,*

$$\limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(\Phi_\varepsilon(X_N) \in F\big) \leqslant -\inf_{\nu\in F} J_\varepsilon(\nu) \tag{7.7.20}$$

- *for any open set $G \subset \mathcal{M}_1(V \times \mathbb{R})$,*

$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(\Phi_\varepsilon(X_N) \in G\big) \geqslant -\inf_{\nu\in G} J_\varepsilon(\nu)\,, \tag{7.7.21}$$

where

$$J_\varepsilon(\nu) = \inf\big\{J(\mu)\,, \quad \mu \in \mathcal{M}(V) : \ \Phi_\varepsilon(\mu) = \nu\,\big\}\,. \tag{7.7.22}$$

7.3.3 Exponential approximation

Recall the definition (7.7.1) of the metric d_*.

Lemma 7.7. *For any $\beta > 0$,*

$$\lim_{\varepsilon\searrow 0} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(d_*(\Phi_\varepsilon(X_N), Z_N) \geqslant \beta\big) = -\infty\,. \tag{7.7.23}$$

Proof. Let $f \in \mathcal{C}(V)$, $g \in \mathcal{C}(\mathbb{R})$ be given, with $\|f\|_{BL} \leqslant 1$, $\|g\|_{BL} \leqslant 1$. By definition,

$$\langle f \otimes g, \Phi_\varepsilon(X_N) - Z_N\rangle = \int_V f(x)\overline{g}_\sigma(\phi_{N,\varepsilon}(x))\,\mathrm{d}x - N^{-d} \sum_{x\in\Lambda_N} f(x/N)\overline{g}_\sigma(\zeta_x)\,, \tag{7.7.24}$$

where $\overline{g}_\sigma(m)$ has been defined in (7.7.10) and we have introduced the random profile

$$\phi_{N,\varepsilon}(x) = N^{-d} \sum_{y\in\Lambda_N} \rho_\varepsilon(x - y/N)\phi_y\,, \quad x \in V\,,\, \phi \in \Omega\,. \tag{7.7.25}$$

We can rewrite (7.7.24) as

$$N^{-d} \sum_{z\in\Lambda_N} \int_V \mathrm{d}x \big[f(x/N + z/N)\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N + z/N)) - f(z/N)\,\overline{g}_\sigma(\zeta_z)\big]\,.$$

Using the assumptions on f, g and the smoothness of $\overline{g}_\sigma$, one easily checks the following estimates, valid uniformly on $x \in V$ and $z \in \Lambda_N$:

$$|\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N + z/N))|\,|f(x/N + z/N) - f(z/N)| \leqslant N^{-1}\,,$$
$$|f(z/N)|\,|\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N + z/N)) - \overline{g}_\sigma(\phi_{N,\varepsilon}(z/N))| \leqslant C\varepsilon^{-d-1}N^{-d-1} \sum_{y\in\Lambda_N} |\phi_y|\,,$$
$$|f(z/N)|\,|\overline{g}_\sigma(\phi_{N,\varepsilon}(z/N)) - \overline{g}_\sigma(\zeta_z)| \leqslant C|\eta^\varepsilon_z|\,,$$

where $C < \infty$ is independent of N, ε, and we have defined the field

$$\eta^\varepsilon_z = \zeta_z - \phi_{N,\varepsilon}(z/N)\,, \quad z \in \mathbb{Z}^d\,. \tag{7.7.26}$$

Thanks to these bounds we have

$$d_*(\Phi_\varepsilon(X_N), Z_N) \leqslant N^{-1} + CN^{-d} \sum_{z\in\Lambda_N} \big(|\eta^\varepsilon_z| + N^{-1}\varepsilon^{-d-1}|\phi_z|\big) \tag{7.7.27}$$

We first observe that for any $\beta > 0$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(N^{-d-1} \sum_{z\in\Lambda_N} |\phi_z| \geqslant \beta\big) = -\infty\,. \tag{7.7.28}$$

This is an obvious consequence of the argument used in the proof of (7.7.19). The proof of the lemma will then be completed by showing

$$\lim_{\varepsilon\searrow 0} \limsup_{N\to\infty} N^{\alpha-d} \log P\big(N^{-d} \sum_{z\in\Lambda_N} |\eta^\varepsilon_z| \geqslant \beta\big) = -\infty\,, \quad \beta > 0\,. \tag{7.7.29}$$

Let

$$\Gamma^\varepsilon_N(y, z) = P(\eta^\varepsilon_y \eta^\varepsilon_z)\,, \quad y, z \in \Lambda_N\,,$$

denote the covariance of the centered Gaussian field η^ε on Λ_N, and let $\lambda_1(N, \varepsilon)$ be its largest eigenvalue. By Chebyshev's inequality and Lemma 6.7, for any $a > 0$ we have the bound

$$\begin{aligned} &\log P\big(N^{-d} \sum_{z\in\Lambda_N} |\eta^\varepsilon_z| \geqslant \beta\big) \\ &\qquad\leqslant\; -\, a\,\beta^2 \;+\; a\,N^{-d}\,\mathrm{Tr}\,\Gamma^\varepsilon_N \;+\; 2a^2\,N^{-2d}\,\mathrm{Tr}\,(\Gamma^\varepsilon_N)^2\, W(2aN^{-d}, \lambda_1)\,, \end{aligned} \tag{7.7.30}$$

where W is defined by (6.6.36). We first claim that, for each $\varepsilon > 0$,

$$\lim_{N\to\infty} N^{-d}\,\mathrm{Tr}\,\Gamma_N^\varepsilon = 0\,. \tag{7.7.31}$$

Indeed,

$$\mathrm{Tr}\,\Gamma_N^\varepsilon = \sum_{z\in\Lambda_N} P(\eta_z^{\varepsilon 2}) \leqslant 2\sum_{z\in\Lambda_N} P(\zeta_z^2) + 2\sum_{z\in\Lambda_N} P(\phi_{N,\varepsilon}(z/N)^2)\,.$$

The second term above is easily seen to be at most $O(N^\alpha)$, while from Lemma A.4 with $L = \log N$, we have

$$N^{-d}\sum_{z\in\Lambda_N} P(\zeta_z^2) = N^{-d}\sum_{z\in\Lambda_N}\big[G(z,z) - G^L(z,z)\big] \to 0\,, N\to\infty\,.$$

This proves (7.7.31). It is also not difficult, cf. (6.6.42), to see that

$$\mathrm{Tr}\,(\Gamma_N^\varepsilon)^2 \leqslant o(N^{d+\alpha})\,. \tag{7.7.32}$$

Let us define, for each $\varepsilon > 0$

$$\gamma(\varepsilon) \;=\; \limsup_{N\to\infty} N^{-\alpha}\max_{x\in\Lambda_N}\sum_{z\in\Lambda_N}|\Gamma_N^\varepsilon(x,z)|\,. \tag{7.7.33}$$

We claim that

$$\lim_{\varepsilon\searrow 0}\gamma(\varepsilon) = 0\,. \tag{7.7.34}$$

We postpone for a moment the proof of this last fact, and show that (7.7.34) is in fact sufficient to complete the proof of Lemma 7.7. Let us assume $\gamma(\varepsilon) > 0$ for each $\varepsilon > 0$ (the proof is simpler otherwise) and fix

$$a = a(N,\varepsilon) = \gamma(\varepsilon)^{-1/2}N^{d-\alpha}\,. \tag{7.7.35}$$

With this choice of a, we see that (7.7.34) implies, for large N,

$$a\,N^{-d}\lambda_1 \;\leqslant\; \gamma(\varepsilon)^{-1/2}N^{-\alpha}\max_{x\in\Lambda_N}\sum_{z\in\Lambda_N}|\Gamma_N^\varepsilon(x,z)| \;\leqslant\; 2\gamma(\varepsilon)^{1/2}\,,$$

which shows that $W(2aN^{-d},\lambda_1(N,\varepsilon))$ is uniformly bounded in (7.7.30). Inserting (7.7.35) in (7.7.30) and using (7.7.31) and (7.7.32), we have the upper bound

$$\limsup_{N\to\infty} N^{\alpha-d}\log P\Big(N^{-d}\sum_{z\in\Lambda_N}|\eta_z^\varepsilon| \geqslant \beta\Big) \leqslant\; -\gamma(\varepsilon)^{-1/2}\beta^2\,,$$

which implies (7.7.29).

It remains to prove (7.7.34). Let us define the field

$$\tilde{\eta}^\varepsilon_z = \zeta_z - N^{-d} \sum_{y\in\Lambda_N} \rho_\varepsilon(z/N - y/N)\zeta_y = \eta^\varepsilon_z + N^{-d} \sum_{y\in\Lambda_N} \rho_\varepsilon(z/N - y/N)\tau_y$$

where $\tau = \phi - \zeta$ is the centered Gaussian field with covariance G^L. From the independence of ζ and τ we have

$$\Gamma^\varepsilon_N(x,y) = P(\tilde{\eta}^\varepsilon_x \tilde{\eta}^\varepsilon_y) + N^{-2d} \sum_{z,z'\in\Lambda_N} \rho_\varepsilon(x/N - z/N)\rho_\varepsilon(y/N - z'/N) G^L(z,z')\,.$$

Moreover, we easily see from Lemma A.4 that

$$\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |G^L(x,y)| = 0$$

and therefore (7.7.34) will follow from

$$\lim_{\varepsilon\searrow 0} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |P(\tilde{\eta}^\varepsilon_x \tilde{\eta}^\varepsilon_y)| = 0\,. \tag{7.7.36}$$

Letting $G^\zeta = G - G^L$, and using $P(\zeta_x\zeta_y) = G^\zeta(x,y)$, we have

$$\begin{aligned} P(\tilde{\eta}^\varepsilon_x \tilde{\eta}^\varepsilon_y) &= G^\zeta(x,y) \\ &- N^{-d} \sum_{z\in\Lambda_N} \left[\rho_\varepsilon(y/N - z/N) G^\zeta(x,z) + \rho_\varepsilon(x/N - z/N) G^\zeta(y,z)\right] \\ &+ N^{-2d} \sum_{z,z'\in\Lambda_N} \rho_\varepsilon(x/N - z/N)\rho_\varepsilon(y/N - z'/N) G^\zeta(z,z') \end{aligned} \tag{7.7.37}$$

Again we may neglect the contribution of G^L to this sums. It is therefore sufficient to prove (7.7.36) by replacing G^ζ with G in (7.7.37). At this point we use (5.5.10) to write

$$G(x,y) = g_\alpha(x-y) + R(x-y)$$

with $g_\alpha(x) = \omega_{\alpha,d}|x|^{\alpha-d}$, and

$$\lim_{|x|\to\infty} |x|^{d-\alpha}|R(x)| = 0\,.$$

In particular

$$\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |R(x-y)| = 0\,,$$

and we are left with the contribution of g_α alone in (7.7.37). By Riemann integration (see Lemma 5.4),

$$\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |\Gamma_N^\varepsilon(x,y)| \leqslant \max_{x\in V} \int_V \big|g_\alpha(x-y) - 2\rho_\varepsilon * g_\alpha(x-y) + \rho_\varepsilon * \rho_\varepsilon * g_\alpha(x-y)\big| \,\mathrm{d}y\,.$$

Since the densities ρ_ε converge to the Dirac mass, a dominated convergence argument implies the claim (7.7.36). This completes the proof of (7.7.34). □

To conclude the proof of Proposition 7.1, we use a classical exponential approximation result, cf. Theorem 4.2.16 of [21]. The latter shows that if Lemma 7.6 and Lemma 7.7 hold, then Z_N satisfies the *weak* Large deviations principle with rate function given by

$$\tilde{J}(\nu) \equiv \sup_{\delta>0} \liminf_{\varepsilon\searrow 0} \inf_{\nu'\in B_\delta(\nu)} J_\varepsilon(\nu')\,, \tag{7.7.38}$$

where $B_\delta(\nu)$ is the ball of radius δ around ν w.r.t. the d_* metric in $\mathcal{M}_1(V\times\mathbb{R})$. In order to extend the upper bound to all closed sets we observe that the variable Z_N are exponentially tight (choose e.g. the compact sets (7.7.7)). To conclude Proposition 7.1 we only have to check that (7.7.38) yields the correct rate function.

Lemma 7.8. *Let $\nu\in\mathcal{M}_1(V\times\mathbb{R})$. Then*

$$\mathcal{I}_\alpha(\nu) = \tilde{J}(\nu) \tag{7.7.39}$$

Proof. Suppose $\mathcal{I}_\alpha(\nu)<\infty$. Then there exists $\phi\in L^2(V)$, with $\nu=\Phi(\phi)$ and $\mathcal{I}_\alpha(\nu)=\frac{1}{2}\mathcal{E}_V(\phi,\phi)$, cf. (6.6.18) and (7.7.6). We define

$$\nu^{(\varepsilon)} \equiv \Phi(\rho_\varepsilon * \phi)\,.$$

From the continuity of Φ and the fact that $\rho_\varepsilon * \phi \to \phi$ in $L^2(V)$ as $\varepsilon\searrow 0$, it follows that for each $\delta>0$, $\nu^{(\varepsilon)}\in B_\delta(\nu)$ for all sufficiently small $\varepsilon>0$. Moreover, if $\mu\in\mathcal{M}(V)$ is defined by $\mu(\mathrm{d}x)=\phi(x)\,\mathrm{d}x$, then $\Phi_\varepsilon(\mu)=\nu^{(\varepsilon)}$. These observations show that

$$\inf_{\nu'\in B_\delta(\nu)} J_\varepsilon(\nu') \leqslant J(\mu) = \frac{1}{2}\,\mathcal{E}_V(\phi,\phi)\,.$$

We have proven

$$\mathcal{I}_\alpha(\nu) \geqslant \tilde{J}(\nu)\,.$$

To prove the converse, assume $\tilde{J}(\nu)<\infty$. Then, by compactness, we can find a sequence $\{\mu^{(\varepsilon)}\}\subset\mathcal{M}(V)$, $\varepsilon\to 0$, such that $J(\mu^{(\varepsilon)})\to\tilde{J}(\nu)$ and $\Phi_\varepsilon(\mu^{(\varepsilon)})\to\nu$. In particular there exists a sequence $\{\phi^{(\varepsilon)}\}\subset L^2(V)$ with $\mu^{(\varepsilon)}(\mathrm{d}x)=\phi^{(\varepsilon)}(x)\,\mathrm{d}x$, and

$$\frac{1}{2}\,\mathcal{E}_V(\phi^{(\varepsilon)},\phi^{(\varepsilon)}) \to \tilde{J}(\nu)\,.$$

This implies that $\{\phi^{(\varepsilon)}\}$ is a relatively compact family in $L^2(V)$. Indeed we can write, cf. Lemma 4.2, $\phi^{(\varepsilon)} = \mathcal{G}_V^{1/2}\varphi^{(\varepsilon)}$, with $\varphi^{(\varepsilon)}$ a bounded family in $L^2(V)$, and $\mathcal{G}_V^{1/2}$ a compact operator. Let $\phi \in L^2(V)$ be such that, passing to a subsequence, $\phi^{(\varepsilon)} \to \phi$. It follows

$$\Phi_\varepsilon(\mu^{(\varepsilon)}) = \Phi(\rho_\varepsilon * \phi^{(\varepsilon)}) \to \Phi(\phi) = \nu\,.$$

By the lower semi-continuity of $\mathcal{E}_V$, we now have

$$\mathcal{I}_\alpha(\nu) = \frac{1}{2}\,\mathcal{E}_V(\phi,\phi) \leqslant \liminf_{\varepsilon\searrow 0} \frac{1}{2}\,\mathcal{E}_V(\phi^{(\varepsilon)},\phi^{(\varepsilon)}) = \tilde{J}(\nu)\,.$$

The proof of Lemma 7.8 is complete. □

Chapter 8

Profiles with zero boundary conditions

In this chapter we prove large deviations for the profiles Y_N under the zero boundary condition field $P_N^0 = P_{\alpha,N}^0$, i.e. Theorem 6.4. Before starting, we first need a slight extension of the results of the previous chapter on the infinite volume field. Let $k > 0$ and define

$$V_k = [-k/2, k/2]^d\,, \quad \Lambda_{kN} = NV_k \cap \mathbb{Z}^d\,.$$

Let also

$$Y_N^{(k)}(\phi) = N^{-d} \sum_{x \in \Lambda_{kN}} \delta_{x/N} \otimes \delta_{\phi_x}\,. \tag{8.8.1}$$

$Y_N^{(k)}$ is a $\mathcal{F}_{\Lambda_{kN}}$-measurable random variable with values in $\mathcal{M}^{(k)}(V_k \times \mathbb{R})$, the space of nonnegative measures on the product $V_k \times \mathbb{R}$ with total variation norm bounded by k^d, equipped with the weak topology. Then the proof of Theorem 6.3 shows that for each integer $k \geqslant 1$, the profiles $Y_N^{(k)}$ satisfy the strong large deviation principle at the order $N^{d-\alpha}$ (from now on simply the strong $N^{d-\alpha}$-LDP), with rate function

$$\mathcal{I}^{(k)}(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi) & \text{if } \mu = \,\mathrm{d}x \otimes \mathcal{N}(\varphi(x), \sigma^2),\ \ \varphi \in L^2(V_k) \\ +\infty & \text{otherwise} \end{cases} \tag{8.8.2}$$

where now μ is a measure in $\mathcal{M}^{(k)}(V_k \times \mathbb{R})$, and $\mathcal{E}_{V_k}$ stands for the Dirichlet form embedded in V_k, i.e. (cf. Theorem 4.3)

$$\mathcal{E}_{V_k}(\varphi,\varphi) = \inf\left\{\mathcal{E}(\phi,\phi),\ \ \phi \in \overline{\mathcal{D}}_{\mathcal{E}}\,,\ \ \phi = \varphi \text{ a.e. on } V_k\right\}\,, \quad \varphi \in L^2(V_k)\,.$$

8.1 Soft killing measures

For each $L > 0$, $k \geqslant 1$, we define the measure $P_k^{L,N}$ by

$$\frac{\mathrm{d}P_k^{N,L}}{\mathrm{d}P}(\phi) = \left(Z_k^{N,L}\right)^{-1} \exp\left(-\frac{1}{2} LN^{-\alpha} \sum_{x \in W_{k,N}} \phi_x^2\right), \tag{8.8.3}$$

where

$$W_{k,N} \equiv \Lambda_{kN} \setminus \Lambda_N .$$

Then $P_k^{N,L}$ is the centered Gaussian field with covariance $G_k^{N,L}$, given by the Green function of the stable random walk on $\mathbb{Z}^d$, with soft killing on the set $W_{k,N}$ only,

$$G_k^{N,L}(x,y) = \mathbb{E}_x \sum_{n=0}^{\infty} 1_y(\xi_n) \exp\left(-\log\left(1 + LN^{-\alpha}\right) \sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l)\right). \tag{8.8.4}$$

The above expression can be easily obtained as in chapter 3, see (3.3.39). Recall the definition of Φ given by (7.7.6).

Proposition 8.1. *For each $L > 0$, $k > 1$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P_k^{N,L}$ with rate function*

$$\mathcal{I}_{k,L}(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{J}_{k,L}(\psi,\psi) & \text{if } \mu = \Phi(\psi),\ \ \psi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{8.8.5}$$

where $\mu \in \mathcal{M}_1(V \times \mathbb{R})$ and

$$\mathcal{J}_{k,L}(\psi,\psi) \equiv \inf_{\substack{\varphi \in L^2(V_k): \\ \varphi 1_V = \psi \text{ a.e.}}} \left\{\mathcal{E}_{V_k}(\varphi,\varphi) + L \int_{V_k \setminus V} \varphi(x)^2\,\mathrm{d}x\right\}. \tag{8.8.6}$$

Proof. The proof uses the LDP for $Y_N^{(k)}$ discussed above, and Varadhan's Lemma. Note in fact that

$$\frac{1}{2} LN^{-\alpha} \sum_{x \in W_{k,N}} \phi_x^2 = \frac{1}{2} LN^{d-\alpha} \left\langle Y_N^{(k)}, f\right\rangle, \tag{8.8.7}$$

with

$$f(x,r) = 1_{V_k \setminus V}(x) \otimes r^2, \quad x \in V_k,\, r \in \mathbb{R}. \tag{8.8.8}$$

However, a straightforward application of Varadhan's Lemma is not allowed since the function (8.8.8) is not continuous. We first have to deal with the following approximation.

Let $\varepsilon > 0$ and define the sets

$$V^{\varepsilon,\pm} = [-(1\pm\varepsilon)/2, (1\pm\varepsilon)/2]^d\,, \quad \Lambda_N^{\varepsilon,\pm} = NV^{\varepsilon,\pm}\cap\mathbb{Z}^d\,.$$

Let $f_0^{\varepsilon,-}\in\mathcal{C}(V_k)$ be the function which is zero on V, 1 on $V_k\setminus V^{\varepsilon,+}$ and interpolates linearly in the set $V^{\varepsilon,+}\setminus V$. Let also $f_0^{\varepsilon,+}\in\mathcal{C}(V_k)$ be the function which is zero on $V^{\varepsilon,-}$, 1 on $V_k\setminus V$ and interpolates linearly in $V\setminus V^{\varepsilon,-}$. We define

$$f^{\varepsilon,\pm}(x,r) = f_0^{\varepsilon,\pm}(x)\otimes r^2\,, \quad x\in V_k\,, r\in\mathbb{R}\,. \tag{8.8.9}$$

We also define the probability measures $P_{\varepsilon,\pm}^{N,L}$ on Ω by

$$\frac{\mathrm{d}P_{\varepsilon,\pm}^{N,L}}{\mathrm{d}P} = \left(Z_{\varepsilon,\pm}^{N,L}\right)^{-1}\exp\left(-\frac{1}{2}LN^{d-\alpha}\left\langle Y_N^{(k)}, f^{\varepsilon,\pm}\right\rangle\right). \tag{8.8.10}$$

From Varadhan's Lemma (see e.g. Theorem 4.3.1 in [21]) we have[1]

$$\begin{aligned}
\lim_{N\to\infty} N^{\alpha-d}\log Z_{\varepsilon,\pm}^{N,L} &= -\inf_{\mu\in\mathcal{M}_1(V\times\mathbb{R})}\left\{\mathcal{I}^{(k)}(\mu) + \frac{L}{2}\left\langle\mu, f_0^{\varepsilon,\pm}\right\rangle\right\}\\
&= -\frac{\sigma^2 L}{2}\int_{V_k} f_0^{\varepsilon,\pm}(x)\,\mathrm{d}x + \frac{1}{2}\inf_{\varphi\in L^2(V_k)}\left\{\mathcal{E}_{V_k}(\varphi,\varphi) + L\int_{V_k} f_0^{\varepsilon,\pm}(x)\varphi(x)^2\,\mathrm{d}x\right\}\\
&= -\frac{\sigma^2 L}{2}\int_{V_k} f_0^{\varepsilon,\pm}(x)\,\mathrm{d}x
\end{aligned} \tag{8.8.11}$$

Also, it follows (cf. e.g. Exercise 4.3.11 in [21]) that for each $L>0$ and $\varepsilon>0$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P_{\varepsilon,\pm}^{N,L}$ with rate

$$\mathcal{I}_{k,L}^{\varepsilon,\pm}(\mu) = \frac{1}{2}\inf_{\substack{\varphi\in L^2(V_k):\\ \mu=\Phi(\varphi 1_V)}}\left\{\mathcal{E}_{V_k}(\varphi,\varphi) + L\int_{V_k} f_0^{\varepsilon,\pm}(x)\varphi(x)^2\,\mathrm{d}x\right\}. \tag{8.8.12}$$

To complete the proof of the proposition, note that

$$\left\langle Y_N, f^{\varepsilon,-}\right\rangle \leqslant \langle Y_N, f\rangle \leqslant \left\langle Y_N, f^{\varepsilon,+}\right\rangle\,.$$

In order to prove the large deviations upper bound, let $F\subset\mathcal{M}_1(V\times\mathbb{R})$ be a closed set. We have

$$P_k^{N,L}\big(Y_N\in F\big) \leqslant \frac{Z_{\varepsilon,-}^{N,L}}{Z_{\varepsilon,+}^{N,L}} P_{\varepsilon,-}^{N,L}\big(Y_N\in F\big)\,. \tag{8.8.13}$$

Therefore,

$$\begin{aligned}
&\limsup_{N\to\infty} N^{\alpha-d}\log P_k^{N,L}\big(Y_N\in F\big)\\
&\qquad\leqslant -\inf_{\mu\in F}\mathcal{I}_{k,L}^{\varepsilon,-}(\mu) + \frac{\sigma^2 L}{2}\int_{V_k}\left(f_0^{\varepsilon,+}(x) - f_0^{\varepsilon,-}(x)\right)\,\mathrm{d}x\,.
\end{aligned} \tag{8.8.14}$$

[1] Note that the functions $f^{\varepsilon,\pm}$ are not bounded, but the exponential tightness of the sets (7.7.7) shows that there is no difficulty in applying Varadhan's Lemma.

Since the last term in the r.h.s. of (8.8.14) vanishes as $\varepsilon \to 0$, in order to prove the upper bound we only have to show

$$\inf_{\mu\in F} \mathcal{I}^{\varepsilon,-}_{k,L}(\mu) \nearrow \inf_{\mu\in F} \mathcal{I}_{k,L}(\mu)\,, \quad \varepsilon \to 0\,. \tag{8.8.15}$$

Observe that we may assume there exists $\varphi \in L^2(V_k)$ such that $\Phi(\varphi 1_V) \in F$ and (by compactness) the infimum in (8.8.12) is attained at φ, otherwise both terms in (8.8.15) are infinite and the upper bound follows from (8.8.14). With $\mu = \Phi(\varphi 1_V) \in F$, we have

$$\begin{aligned}\mathcal{I}^{\varepsilon,-}_{k,L}(\mu) &= \frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi) + \frac{L}{2}\int_{V_k} f_0^{\varepsilon,-}(x)\varphi(x)^2\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi) + \frac{L}{2}\int_{V_k\setminus V}\varphi(x)^2\,\mathrm{d}x - \frac{L}{2}\int_{V^{\varepsilon,+}\setminus V}\varphi(x)^2\,\mathrm{d}x\,.\end{aligned} \tag{8.8.16}$$

We use now the Sobolev inequality of Lemma 4.4, applied to the Dirichlet form $\mathcal{E}_{V_k}$. Let $v(\varepsilon)$ denote the volume of $V^{\varepsilon,+}\setminus V$. Then, letting $p_0 = 2d/(d-\alpha)$, by Hölder's inequality and (4.4.33) we have

$$\int_{V^{\varepsilon,+}\setminus V}\varphi(x)^2\,\mathrm{d}x \leqslant v(\varepsilon)^{1/q}\|\varphi\|^2_{p_0,V_k} \leqslant\; v(\varepsilon)^{1/q}\,K_\alpha^2\,\mathcal{E}_{V_k}(\varphi,\varphi)\,, \tag{8.8.17}$$

where

$$q = p_0/2/(p_0/2-1)$$

. Letting $c(\varepsilon) = v(\varepsilon)^{1/q}K_\alpha^2 L$, (8.8.16) and (8.8.17) imply

$$\mathcal{I}^{\varepsilon,-}_{k,L}(\mu) \geqslant (1-c(\varepsilon))\mathcal{I}_{k,L}(\mu)\,, \tag{8.8.18}$$

for any μ such that $\mu = \Phi(\varphi 1_V)$ for some $\varphi \in L^2(V_k)$. Taking the infimum over such $\mu \in F$, (8.8.18) implies the claim (8.8.15) since $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. The lower bound can be proven by the same argument (inverting the role of the functions $f^{\varepsilon,\pm}$). ☐

8.2 Exponential approximations

Let us consider the field

$$\omega_x = \omega(\phi)_x = \phi_x - P_k^{N,L}(\phi_x|\mathcal{F}_{W_{k,N}})\,, \quad x\in\Lambda_N\,,\ k>1\,. \tag{8.8.19}$$

Observe that $P_k^{N,L}\circ\omega^{-1}$ is the centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with covariance $\Gamma_{k,N}$, the Green function of the random walk which is killed upon hitting $W_{k,N}$. We define the profiles of the ω-field,

$$\mathcal{W}_N(\phi) = Y_N(\omega) = N^{-d}\sum_{x\in\Lambda_N}\delta_{x/N}\otimes\delta_{\omega_x}\,. \tag{8.8.20}$$

Then

$$P_k^{N,L} \circ \mathcal{W}_N^{-1} = P_{k,N}^0 \circ Y_N^{-1}\,, \quad L > 0\,,\ k > 1\,, \tag{8.8.21}$$

where $P_{k,N}^0$ is the centered Gaussian measure on $\mathbb{R}^{\Lambda_N}$ with covariance $\Gamma_{k,N}$. The zero boundary condition field $P_N^0 = P_{\alpha,N}^0$ of Theorem 6.4 shall be recovered by letting $k \to \infty$ in the end. Recall the definition (7.7.1) of the metric d_*.

Proposition 8.2. *For any $k > 1$, $\beta > 0$*

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{\alpha-d} \log P_k^{N,L}\big(d_*(Y_N, \mathcal{W}_N) \geqslant \beta\big) = -\infty\,. \tag{8.8.22}$$

Proof. Let η_x be the field

$$\eta_x = \phi_x - \omega_x = P_k^{N,L}(\phi_x | \mathcal{F}_{W_{k,N}})\,, \quad x \in \Lambda_N\,. \tag{8.8.23}$$

As in the proof of (7.7.23), the claim (8.8.22) will follow from

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{\alpha-d} \log P_k^{N,L}\big(N^{-d} \sum_{x\in\Lambda_N} |\eta_x| \geqslant \beta\big) = -\infty\,, \quad \beta > 0\,. \tag{8.8.24}$$

Observe that η is the centered Gaussian field with covariance $\Gamma_k^{N,L} = G_k^{N,L} - \Gamma_{k,N}$ (see (8.8.4)), i.e. for $x, y \in \Lambda_N$

$$\Gamma_k^{N,L}(x,y) = \mathbb{E}_x\left[\sum_{n=\tau_{k,N}}^{\infty} 1_y(\xi_n) \exp\left(-\log\left(1 + LN^{-\alpha}\right) \sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l)\right)\right]\,, \tag{8.8.25}$$

where

$$\tau_{k,N} = \inf\left\{n \geqslant 1\,,\ \xi_n \in W_{k,N}\right\}\,. \tag{8.8.26}$$

Then from the proof of (7.7.23) we see that it will be enough to prove that, for each $k > 1$,

$$\lim_{N\to\infty} N^{-d-\alpha}\, \mathrm{Tr}\left(\Gamma_k^{N,L}\right)^2 = 0\,, \quad L > 0\,, \tag{8.8.27}$$

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N} \Gamma_k^{N,L}(x,x) = 0\,, \tag{8.8.28}$$

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} \Gamma_k^{N,L}(x,y) = 0\,. \tag{8.8.29}$$

Since $\Gamma_k^{N,L}(x,y) \leqslant G(x,y)$, it is clear that (8.8.27) simply follows from (6.6.42). In order to prove (8.8.28) and (8.8.29) we may proceed as follows. For each $\delta \in (0, 1/4)$, recall the boxes

$$\Lambda_N^{\delta,-} = \Lambda_{(1-2\delta)N}\,, \quad \Lambda_N^{\delta,+} = \Lambda_{(1+2\delta)N}$$

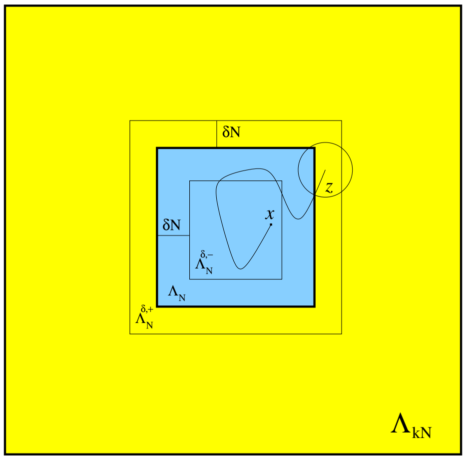

FIGURE 8.1. The sets $\Lambda_N^{\delta,-}$,$\Lambda_N^{\delta,+}$ and $W_{k,N} = \Lambda_{kN} \setminus \Lambda_N$.

so that a site in $\Lambda_N^{\delta,-}$ has a distance at least δN from any site in Λ_N^c. We also choose $\delta > 0$, such that $\Lambda_N^{\delta,+} \subset \Lambda_{kN}$. It is clear that it is sufficient to prove (8.8.24) with Λ_N replaced by $\Lambda_N^{\delta,-}$, for any $\delta > 0$. In particular, it will be enough to replace (8.8.28) and (8.8.29) by

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N^{\delta,-}} \Gamma_k^{N,L}(x,x) = 0\,, \tag{8.8.30}$$

$$\lim_{L\to\infty} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N^{\delta,-}} \sum_{y\in\Lambda_N^{\delta,-}} \Gamma_k^{N,L}(x,y) = 0\,, \quad \delta > 0\,. \tag{8.8.31}$$

Consider the random walk $\xi_0, \xi_1, \xi_2, \ldots$ starting at $\xi_0 = x \in \Lambda_N^{\delta,-}$, and let $\tau_{k,N}$ be as in (8.8.26). Define also the function

$$\Psi_{N,L}(\xi,n) = \log\left(1 + LN^{-\alpha}\right) \sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l)\,, \quad n \in \mathbb{Z}_+\,. \tag{8.8.32}$$

Then by (8.8.4) and (8.8.25), using the strong Markov property we have,

$$\begin{aligned}\Gamma_k^{N,L}(x,y) &= \mathbb{E}_x \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n) \exp\left(-\Psi_{N,L}(\xi,n)\right) \right] \\ &= \mathbb{E}_x G_k^{N,L}(\xi_{\tau_{k,N}}, y)\,, \quad x, y \in \Lambda_N^{\delta,-}\,. \end{aligned} \tag{8.8.33}$$

Let us call s_N the time of first come back to $\Lambda_N^{\delta,-}$ after exiting Λ_N. By the strong Markov property, from (8.8.33) we have

$$\Gamma_k^{N,L}(x,y) = \mathbb{E}_x \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) G_k^{N,L}(\xi_{s_N}, y) \right] \tag{8.8.34}$$

Using $G_k^{N,L}(z,y) \leqslant \sigma^2$, this shows that

$$\Gamma_k^{N,L}(x,x) \leqslant \sigma^2\, \mathbb{E}_x\, \mathbb{E}_{\xi_{\tau_{k,N}}} \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right), \quad x \in \Lambda_N^{\delta,-}\,. \tag{8.8.35}$$

On the other hand, letting

$$\Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) \equiv \sum_{y \in \Lambda_N^{\delta,-}} \Gamma_k^{N,L}(x,y)\,, \quad G_k^{N,L}(z, \Lambda_N^{\delta,-}) \equiv \sum_{y \in \Lambda_N^{\delta,-}} G_k^{N,L}(z,y)$$

we see from (8.8.34) that

$$\Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) = \mathbb{E}_x \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) G_k^{N,L}(\xi_{s_N}, \Lambda_N^{\delta,-})\right].$$

Since (cf. (5.5.10))

$$\sup_N \max_{x \in \Lambda_N^{\delta,-}} N^{-\alpha} G(x, \Lambda_N^{\delta,-}) < \infty\,,$$

using $G_k^{N,L} \leqslant G$, there exists a constant $C < \infty$ such that for any $N \in \mathbb{Z}_+$ we have

$$N^{-\alpha}\Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) \leqslant C\, \mathbb{E}_x\, \mathbb{E}_{\xi_{\tau_{k,N}}} \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right), \quad x \in \Lambda_N^{\delta,-}\,. \tag{8.8.36}$$

We may restrict to the case $\xi_{\tau_{k,N}} \in \Lambda_N^{\delta,+} \cap W_{k,N}$ in (8.8.36), i.e. we may exclude that the walk enters $W_{k,N}$ by hitting $W_{k,N} \setminus \Lambda_N^{\delta,+}$. Indeed,

$$\max_{x \in \Lambda_N^{\delta,-}} \mathbb{P}_x\left(\xi_{\tau_{k,N}} \in W_{k,N} \setminus \Lambda_N^{\delta,+}\right) \to 0\,, \quad N \to \infty\,.$$

In particular, (8.8.35) and (8.8.36) show that both claims (8.8.30) and (8.8.31) will follow from the following lemma.

Lemma 8.3. *For any $\delta > 0$,*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \max_{z \in \Lambda_N^{\delta,+} \cap W_{k,N}} \mathbb{E}_z \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) = 0\,. \tag{8.8.37}$$

Proof. We are going to exploit the fact that $z \in W_{N,L}$ is at least at distance δN from any site $x \in \Lambda_N^{\delta,-}$. Let us define the first exit time for the Euclidean ball $B_{\delta N}(\xi_0)$ of radius δN around ξ_0

$$s_N^\delta = \inf\{n \geqslant 1 :\ |\xi_n - \xi_0| \geqslant \delta N\} \tag{8.8.38}$$

Note that by construction $B_{\delta N}(z) \cap \Lambda_N^{\delta,-} = \emptyset$ if $\xi_0 = z \in \Lambda_N^{\delta,+} \cap W_{k,N}$, therefore in this case $s_N \geqslant s_N^\delta$. We claim that

$$\limsup_{N\to\infty} \mathbb{P}_0\left(s_N^\delta < \varepsilon N^\alpha\right) \to 0\,, \quad \varepsilon \to 0\,. \tag{8.8.39}$$

Assuming the validity of (8.8.39), we may (by the arbitrariness of ε) rule out the possibility that ξ_n visits $\Lambda_N^{\delta,-} \cup \Lambda_{kN}^c$ up to time $[\varepsilon N^\alpha]$. In particular the lemma will follow from

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{z\in\Lambda_N^c} \mathbb{E}_z \left[\exp\Big(-\frac{1}{2}\, L N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \Big) \right] = 0 \tag{8.8.40}$$

for all $\varepsilon > 0$, where we use the fact that for $LN^{-\alpha} \leqslant 1$, $\log(1 + LN^{-\alpha}) \geqslant \frac{1}{2}\, LN^{-\alpha}$. To prove (8.8.40), it is sufficient to show that for all $\varepsilon > 0$,

$$\limsup_{N\to\infty} \sup_{z\in\Lambda_N^c} \mathbb{P}_z \Big(N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \leqslant \gamma \Big) \to 0\,, \quad \gamma \to 0\,. \tag{8.8.41}$$

We are left with the proof of (8.8.39) and (8.8.41).

Proof of (8.8.39). By a well known inequality of Levy (cf. e.g. [73], Theorem 1.4.15), we have

$$\begin{aligned} \mathbb{P}_0\left(s_N^\delta < \varepsilon N^\alpha\right) &= \mathbb{P}_0\left(\max_{1\,\leqslant\, n\,\leqslant\, [\varepsilon N^\alpha]} |\xi_n| \geqslant \delta N \right) \\ &\leqslant 2d\, \mathbb{P}_0\left(|\xi_{[\varepsilon N^\alpha]}| \geqslant \delta N/d \right) \end{aligned} \tag{8.8.42}$$

Let $\bar\xi(n) = n^{-1}\xi_{[n^\alpha]}$. Then (8.8.42) implies

$$\mathbb{P}_0\left(s_N^\delta < \varepsilon N^\alpha\right) \leqslant 2d\mathbb{P}_0\left(|\bar\xi(\varepsilon^{1/\alpha} N)| > \delta \varepsilon^{-1/\alpha}/d \right)\,. \tag{8.8.43}$$

We now use the invariance principle $\bar\xi(n) \to^{(d)} U_\alpha$, cf. (5.5.8), to write

$$\limsup_{N\to\infty} \mathbb{P}_0\left(s_N^\delta < \varepsilon N^\alpha\right) \leqslant 2d\, \mathcal{P}\left(|U_\alpha| > \delta\, \varepsilon^{-1/\alpha}/d \right) \leqslant \varepsilon\, c(d,\delta)\,, \tag{8.8.44}$$

for some finite constant $c(d,\delta)$, where $\mathcal{P} \in \mathcal{M}_1(\mathbb{R}^d)$ is the distribution of the symmetric α-stable variable U_α, with density $q_\alpha(1,x)$, cf. (4.4.1). This proves (8.8.39).

Proof of (8.8.41). We first observe that the probability appearing in (8.8.41) is maximized when $z \in \partial^+\Lambda_N$, with $\partial^+\Lambda_N$ denoting the set of sites $x \notin \Lambda_N$ at Euclidean distance $d(x,\Lambda_N) = 1$ from Λ_N. By symmetry, the situation may be further simplified as follows. Let $(\xi_n) = (\xi_n^1, \ldots, \xi_n^d)$ be the $d-$dimensional random walk vector and let J_T, $T \in \mathbb{Z}_+$, be the time spent up to time T on the half space corresponding to nonnegative values of the first coordinate, i.e. $J_T = \sum_{n=0}^T 1_{\mathbb{Z}_+}(\xi_n^1)$. Then, for any $z \in \Lambda_N^c$, letting $T = [\varepsilon N^\alpha] - 1$

$$\mathbb{P}_z \Big(N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \leqslant \gamma \Big) \leqslant \mathbb{P}_0\left(J_T/T \leqslant \gamma/\varepsilon\right)\,. \tag{8.8.45}$$

We now apply to ξ_n^1 the arcsine law for an arbitrary one-dimensional, symmetric random walk (cf. [30], chapter XII). Namely, the distribution of J_T/T converges, as $T \to \infty$, to the arcsine function

$$\mathbb{P}_0\left(J_T/T \leqslant \gamma/\varepsilon\right) \to \frac{2}{\pi} \arcsin \sqrt{\gamma/\varepsilon}\,, \quad T \to \infty\,. \tag{8.8.46}$$

From this and (8.8.45) the claim (8.8.41) follows immediately. $\square$

Proposition 8.4. *For each $k \geqslant 1$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P^0_{k,N}$ with rate function*

$$\mathcal{I}_k(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{J}_{k,\infty}(\psi,\psi) & \text{if } \mu = \Phi(\psi),\ \ \psi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{8.8.47}$$

where $\mu \in \mathcal{M}_1(V \times \mathbb{R})$ and

$$\mathcal{J}_{k,\infty}(\psi,\psi) \equiv \mathcal{E}_{V_k}(\tilde\psi,\tilde\psi)\,, \quad \tilde\psi \in L^2(V_k)\,, \ \ \tilde\psi = \begin{cases} \psi & \text{on } V \\ 0 & \text{on } V_k \setminus V \end{cases} \quad \text{a.e.} \tag{8.8.48}$$

Proof. We see from (8.8.21) and Propositions 8.1 and 8.2, that as $L \to \infty$, the measures $P_k^{N,L}$ are an exponentially good approximation of the measure $P^0_{k,N}$. Therefore, using the the classical exponential approximation lemma (cf. Theorem 4.2.16 of [21]), we have that Y_N satisfies the weak $N^{d-\alpha}$-LDP under the measure $P^0_{k,N}$, with rate given by

$$\tilde{\mathcal{I}}_k(\mu) = \sup_{\delta>0} \liminf_{L\to\infty} \inf_{\nu \in B_\delta(\mu)} \mathcal{I}_{k,L}(\nu)\,, \tag{8.8.49}$$

where $\mathcal{I}_{k,L}$ is the rate function defined in (8.8.5). As in the previous chapter, there is no difficulty in turning the weak LDP into a strong LDP. The proposition then follows if we prove

$$\mathcal{I}_k(\mu) = \tilde{\mathcal{I}}_k(\mu)\,, \quad \mu \in \mathcal{M}_1(V \times \mathbb{R})\,. \tag{8.8.50}$$

From (8.8.5) and (8.8.6) it is easily seen that $\mathcal{I}_{k,L}(\mu) \leqslant \mathcal{I}_k(\mu)$ and $\mathcal{I}_{k,L}(\mu)$ is increasing in L for fixed μ. It follows $\mathcal{I}_k(\mu) \geqslant \tilde{\mathcal{I}}_k(\mu)$, $\mu \in \mathcal{M}_1(V \times \mathbb{R})$. To prove the converse, assume $\mu \in \mathcal{M}_1(V \times \mathbb{R})$ and $\tilde{\mathcal{I}}_k(\mu) < \infty$. By lower semi-continuity, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{I}_{k,L}(\nu) \geqslant \mathcal{I}_{k,L}(\mu) - \varepsilon$, for any $\nu \in B_\delta(\mu)$. In particular, there exists $\psi \in L^2(V)$ such that $\mu = \Phi(\psi)$ and by (8.8.6)

$$\tilde{\mathcal{I}}_k(\mu) \geqslant \frac{1}{2} \liminf_{L\to\infty} \mathcal{J}_{k,L}(\psi,\psi) - \varepsilon\,. \tag{8.8.51}$$

By compactness we may now find a sequence $\varphi_{k,L} \in L^2(V_k)$, $L > 0$, such that $\varphi_{k,L} 1_V = \psi$ a.e. and

$$\mathcal{J}_{k,L}(\psi,\psi) = \mathcal{E}_{V_k}(\varphi_{k,L},\varphi_{k,L}) + L \int_{V_k \setminus V} \varphi_{k,L}(x)^2 \,\mathrm{d}x$$

This implies $\varphi_{k,L}1_{V_k\setminus V} \to 0$ as $L \to \infty$, in $L^2(V_k)$, and therefore $\varphi_{k,L} \to \varphi_{k,\infty}$ in $L^2(V_k)$ where $\varphi_{k,\infty}1_V = \psi$ a.e. and $\varphi_{k,\infty}1_{V_k\setminus V} = 0$ a.e. By lower semi-continuity we then have

$$\begin{aligned}\liminf_{L\to\infty} \mathcal{J}_{k,L}(\psi,\psi) &\geqslant \liminf_{L\to\infty} \mathcal{E}_{V_k}(\varphi_{k,L},\varphi_{k,L}) \\ &\geqslant \mathcal{E}_{V_k}(\varphi_{k,\infty},\varphi_{k,\infty}) = \mathcal{J}_{k,\infty}(\psi,\psi)\,.\end{aligned}$$

From (8.8.51) and the arbitrariness of ε, this implies $\mathcal{I}_k(\mu) \leqslant \tilde{\mathcal{I}}_k(\mu)$. □

8.3 Proof of Theorem 6.4

We are now in the position to conclude the proof of Theorem 6.4. We first prove that $P^0_{k,N}$ is an exponentially good approximation of P^0_N as $k \to \infty$. In order to do so, observe that the argument of Proposition 8.2 shows that it is sufficient to prove

$$\lim_{k\to\infty} \limsup_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N} \tilde{\Gamma}_{k,N}(x,x) = 0\,, \tag{8.8.52}$$

$$\lim_{k\to\infty} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} \tilde{\Gamma}_{k,N}(x,y) = 0\,, \tag{8.8.53}$$

where we have defined

$$\tilde{\Gamma}_{k,N}(x,y) \equiv \Gamma_{k,N}(x,y) - \Gamma_N(x,y)\,. \tag{8.8.54}$$

Recall that $\Gamma_{k,N}$ is the Green function of the random walk with killing on $W_{k,N} = \Lambda_{kN} \setminus \Lambda_N$ while Γ_N is the Green function of the random walk with killing on Λ_N^c. We then have

$$\tilde{\Gamma}_{k,N}(x,y) = \mathbb{E}_x\left[\sum_{n=\sigma_{k,N}}^{\infty} 1_y(\xi_n)\,;\ \sigma_{k,N} > \tau_{k,N}\right]\,, \tag{8.8.55}$$

where

$$\begin{aligned}\tau_{k,N} &= \inf\left\{n \geqslant 1\,,\ \xi_n \in W_{k,N}\right\}\,,\\ \sigma_{k,N} &= \inf\left\{n \geqslant 1\,,\ \xi_n \notin \Lambda_{kN}\right\}\,.\end{aligned}$$

Using the strong Markov property we can write

$$\tilde{\Gamma}_{k,N}(x,y) \leqslant \mathbb{E}_x \mathbb{E}_{\xi_{\sigma_{k,N}}} G(\xi_{\sigma_{k,N}},y) \leqslant \sup_{z\notin\Lambda_{kN}} G(z,y)\,, \quad x,y \in \Lambda_N\,.$$

Then using assumption H2, cf. (5.5.10), we have

$$N^{-d} \sum_{x\in\Lambda_N} \tilde{\Gamma}_{k,N}(x,x) \leqslant \sup_{x\in\Lambda_N} \sup_{z\notin\Lambda_{kN}} G(z,y) \leqslant C\ [(k-1)N]^{\alpha-d}\,,$$

for some $C < \infty$ as $N \to \infty$. This obviously implies (8.8.52). On the other hand, by the same argument,

$$N^{-\alpha} \max_{x \in \Lambda_N} \sum_{y \in \Lambda_N} \tilde{\Gamma}_{k,N}(x,y) \leqslant N^{d-\alpha}\, C\, [(k-1)N]^{\alpha-d} = C\,(k-1)^{\alpha-d}\,,$$

for all sufficiently large N, and (8.8.53) follows by letting $k \to \infty$.

From the preceding observations we know (cf. the proof of Proposition 8.4) that Y_N satisfies the strong $N^{d-\alpha}$-LDP under the measure P_N^0, with rate function given by

$$\tilde{\mathcal{I}}_\infty(\mu) = \sup_{\delta>0} \liminf_{k\to\infty} \inf_{\nu \in B_\delta(\mu)} \mathcal{I}_k(\nu)\,, \tag{8.8.56}$$

where $\mathcal{I}_k$ has been defined in (8.8.47). We are left with the proof of

$$\tilde{\mathcal{I}}_\infty(\mu) = \mathcal{I}_\alpha^0(\mu) \quad \mu \in \mathcal{M}_1(V \times \mathbb{R})\,, \tag{8.8.57}$$

where $\mathcal{I}_\alpha^0$ is the rate function defined in (6.6.27). Observing that $\mathcal{J}_{k,\infty}(\psi,\psi) \leqslant \mathcal{E}_V^0(\psi,\psi)$, for any $\psi \in L^2(V)$ (cf. (8.8.48) and (4.4.36)), it is clear that $\mathcal{I}_\alpha^0(\mu) \geqslant \tilde{\mathcal{I}}_\infty(\mu)$, $\mu \in \mathcal{M}_1(V \times \mathbb{R})$. To prove the converse, as in Proposition 8.4, it is sufficient to show that if $\psi \in L^2(V)$ is such that $\mathcal{J}_{k,\infty}(\psi,\psi) < \infty$, then (cf. (8.8.48))

$$\lim_{k\to\infty} \mathcal{J}_{k,\infty}(\psi,\psi) = \lim_{k\to\infty} \mathcal{E}_{V_k}(\tilde{\psi},\tilde{\psi}) = \mathcal{E}_V^0(\psi,\psi)\,. \tag{8.8.58}$$

Moreover, by a density argument, we may restrict to $\psi \in \mathcal{C}^1(V)$. In this case, using Lemma 5.4, we see that (8.8.58) follows if we can show

$$\lim_{k\to\infty} \lim_{N\to\infty} N^{\alpha-d} \sum_{x,y \in \Lambda_N} \psi(x/N)\psi(y/N) \left| G_{kN}^{-1}(x,y) - G^{-1}(x,y) \right| = 0\,, \tag{8.8.59}$$

where $G^{-1}(x,y) = \delta(x,y) - p(x,y)$ and G_{V_k} stands for the matrix $\{G(x,y)\}_{x,y\in\Lambda_{kN}}$. We recall that (cf. [13], Lemma A.1 and Proposition 3.1)

$$\sum_{y\in\Lambda_N} \left| G_{kN}^{-1}(x,y) - G^{-1}(x,y) \right| \leqslant \mathbb{P}_x\left(\xi_1 \notin \Lambda_{kN}\right)\,.$$

The r.h.s. above is the probability that, starting in $x \in \Lambda_N$, the first jump brings the particle outside Λ_{kN}, and is easily seen to be bounded by $C[(k-1)N]^{-\alpha}$, for sufficiently large N, with $C < \infty$ independent of k and N. Going back to (8.8.59), we then see that

$$\begin{aligned} N^{\alpha-d} &\sum_{x,y\in\Lambda_N} \psi(x/N)\psi(y/N) \left| G_{kN}^{-1}(x,y) - G^{-1}(x,y) \right| \\ &\leqslant \|\psi\|_\infty^2\, N^\alpha \sup_{x\in\Lambda_N} \mathbb{P}_x\left(\xi_1 \notin \Lambda_{kN}\right) \leqslant C\,\|\psi\|_\infty^2\,(k-1)^{-\alpha}\,, \end{aligned}$$

for all sufficiently large N. This proves (8.8.58) and completes the proof of Theorem 6.4.

Appendix A

Random walk representations

A.1 An expansion

We start with an elementary but very useful expansion for the covariance G_Λ of the marginal of P on $\Lambda \in \mathbb{F}$. The result is well known and is contained in the identities for hitting probabilities discussed in section 25 of Spitzer's book [71]. The proof given here is slightly different and can be generalized to treat some cases of non-ferromagnetic interactions, see [13].

Let G be the Green function of the random walk defined by the transitions $p(x, y)$, as in chapter 1. For $\Lambda \in \mathbb{F}$, define the stopping times

$$\sigma_\Lambda = \inf\{n \geqslant 0\,,\ \ \xi_n \notin \Lambda\}\ ,\ \ \tilde{\sigma}_\Lambda = \inf\{n \geqslant 1\,,\ \ \xi_n \in \Lambda\}\ . \tag{A.A.1}$$

Lemma A.1. *For each $x, y \in \Lambda$, we have*

$$G(x, y) = \sum_{n=0}^{\infty} \tilde{p}_\Lambda^n(x, y)\,, \tag{A.A.2}$$

where $\tilde{p}_\Lambda$ is given by

$$\tilde{p}_\Lambda(x, y) = \mathbb{P}_x\left(\xi_{\tilde{\sigma}_\Lambda} = y\right)\ . \tag{A.A.3}$$

Proof. Let us define the function

$$\psi_n(\xi) = \sum_{l=1}^{n} 1_\Lambda(\xi_l)\,.$$

Define also, for $x, y \in \Lambda$ and integers $0 \leqslant k \leqslant n$

$$\mathcal{A}(x, y; k, n) = \begin{cases} p(x, y) & n = k = 0 \\ \mathbb{P}_x\left[\xi_{n+1} = y\,;\ \psi_n(\xi) = k\right] & n \geqslant 1,\ k \leqslant n \end{cases}$$

We write

$$\begin{aligned}\sum_{n=1}^{\infty} p^n(x,y) &= \sum_{n=0}^{\infty} \mathbb{P}_x\left[\xi_{n+1}=y\right] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{A}(x,y;k,n) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \mathcal{A}(x,y;k,n)\,. \end{aligned} \tag{A.A.4}$$

We then have

$$G(x,y) = \delta(x,y) + \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \mathcal{A}(x,y;k,n)\,. \tag{A.A.5}$$

In order to finish the proof, we have to check the identity

$$\sum_{n=k}^{\infty} \mathcal{A}(x,y;k,n) = \tilde{p}_\Lambda^{k+1}(x,y)\,, \quad k=0,1\ldots \tag{A.A.6}$$

We observe that (A.A.6) is straightforward for $k=0$, since

$$\sum_{n=1}^{\infty} \mathcal{A}(x,y;0,n) = \mathbb{P}_x\left[\xi_{\tilde{\sigma}_\Lambda}=y\,;\ \tilde{\sigma}_\Lambda>1\right]\,.$$

To prove the induction step, we use the Markov property to write

$$\sum_{z\in\Lambda} \mathcal{A}(x,z;k,n)\tilde{p}_\Lambda(z,y) = \mathcal{A}(x,y;k+1,n+1)\,.$$

Therefore, for integers $k \geqslant 0$

$$\begin{aligned}\tilde{p}_\Lambda^{k+2}(x,y) &= \sum_{z\in\Lambda} \tilde{p}_\Lambda^{k+1}(x,z)\tilde{p}_\Lambda(z,y) \\ &= \sum_{z\in\Lambda}\sum_{n=k}^{\infty} \mathcal{A}(x,z;k,n)\tilde{p}_\Lambda(z,y) = \sum_{n=k+1}^{\infty} \mathcal{A}(x,y;k,n)\,.\end{aligned}$$

This completes the proof of (A.A.6). □

A.2 Conditioning and killing

The basic identities collected here are well known. They were derived in [70] for the case of the simple random walk, but the proof easily extends to the general case. We include it here for the sake of completeness.

We are going to determine the conditional measure obtained from P by fixing the heights ϕ_x, $x \notin \Lambda$ outside of a given (arbitrary) subset Λ of $\mathbb{Z}^d$. We shall write

$$P_\Lambda^\phi(\,\mathrm{d}\phi) = P(\,\mathrm{d}\phi \mid \phi_{\Lambda^c})$$

for the conditional field.

Lemma A.2. *Let $\Lambda \subset \mathbb{Z}^d$. Then P_Λ^ϕ is the Gaussian field with mean*

$$m_x(\phi) = P(\phi_x|\phi_{\Lambda^c}) = \sum_{y\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = y\right)\phi_y \tag{A.A.7}$$

and covariance

$$\Gamma_\Lambda(x,y) = \mathbb{E}_x\left[\sum_{n=1}^{\sigma_\Lambda - 1} 1_{\{y\}}(\xi_n)\right], \tag{A.A.8}$$

where σ_Λ is the exit time for Λ, cf. (A.A.1).

Remark A.1. It should be observed that

$$P(m_x^2) \leqslant \sum_{y\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = y\right) P(\phi_y^2) = \sigma^2\, \mathbb{P}_x\left(\sigma_\Lambda < \infty\right) < \infty$$

and therefore (A.A.7) is well defined for $P-$a.a. $\phi \in \Omega$.

Proof. That P_Λ^ϕ is Gaussian is a well known fact about Gaussian measures. Moreover, the mean $m_x(\phi)$ is characterized by the orthogonality in $L^2(P)$ expressed by

$$\phi_x - m_x(\phi) \quad \perp \quad \phi_z\,, \quad \forall\, z \in \Lambda^c\,. \tag{A.A.9}$$

To prove (A.A.7) it is then sufficient to show

$$G(x,z) = \sum_{y\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = y\right) G(y,z)\,, \quad \forall\, z \in \Lambda^c\,.$$

This is easily seen to follow from the Markov property.

It is well known that the covariance

$$C_\Lambda(x,y) \equiv P(\phi_x\phi_y|\phi_{\Lambda^c}) - P(\phi_x|\phi_{\Lambda^c})P(\phi_y|\phi_{\Lambda^c})$$

is independent of ϕ_{Λ^c}. Then, to prove $C_\Lambda = \Gamma_\Lambda$, using the orthogonality (A.A.9), we see that it is sufficient to check

$$P\left[(\phi_x - m_x)(\phi_y - m_y)\right] = \Gamma_\Lambda(x,y)\,. \tag{A.A.10}$$

Observe that by the Markov property

$$\begin{aligned} P(m_x m_y) &= \sum_{z,z'\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = z\right) \mathbb{P}_y\left(\xi_{\sigma_\Lambda} = z'\right) G(z,z') \\ &= \sum_{z'\notin\Lambda} G(x,z')\mathbb{P}_y\left(\xi_{\sigma_\Lambda} = z'\right) = \sum_{z'\notin\Lambda} \mathbb{P}_y\left(\xi_{\sigma_\Lambda} = z'\right) G(z',x)\,. \end{aligned}$$

On the other hand

$$
\begin{aligned}
G(x,y) - \Gamma_\Lambda(x,y) &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n)\,;\ \sigma_\Lambda \leqslant n\right] \\
&= \sum_{z\notin\Lambda} \mathbb{P}_x\left(\xi_{\sigma_\Lambda} = z\right) G(z,y)\,.
\end{aligned}
$$

Therefore it follows

$$
P\left[(\phi_x - m_x)(\phi_y - m_y)\right] = G(y,x) - P(m_x m_y) = \Gamma_\Lambda(y,x) = \Gamma_\Lambda(x,y)\,.
$$

□

A.2.1 Conditioning on a sub-grid

Let L be a positive integer and define the sub-grid $\mathcal{L} = L\mathbb{Z}^d$. Define

$$
\zeta_x = P_\alpha(\phi_x|\phi_{\mathcal{L}})\,, \quad x \in \mathbb{Z}^d\,, \tag{A.A.11}
$$

and let σ_L be the first time the random walks hits $\mathcal{L}$. Define also the Green function of the random walk which is killed upon hitting $\mathcal{L}$,

$$
G^L(x,y) = \mathbb{E}_x\left[\sum_{n=0}^{\sigma_L - 1} 1_{\{y\}}(\xi_n)\right]\,, \quad x,y \in \mathbb{Z}^d\,. \tag{A.A.12}
$$

Then, setting $\Lambda = \mathcal{L}^c$ we may use Lemma A.2 to obtain

Lemma A.3. *The fields ϕ_x and $\tau_x \equiv \phi_x - \zeta_x$ are independent. Moreover, for any $x \in \mathbb{Z}^d$,*

$$
\zeta_x = \sum_{z \in L\mathbb{Z}^d} \mathbb{P}_x\left(\xi_{\sigma_L} = z\right)\phi_z\,,
$$

and τ is the centered Gaussian field with covariance

$$
P_\alpha(\tau_x\tau_y) = G^L(x,y)\,. \tag{A.A.13}
$$

The following Lemma can be proven as in [6], Proposition A.12.

Lemma A.4. *Assume the random walk satisfies hypothesis H1, cf. chapter 5. Let*

$$
L = L(N) = \log N\,.
$$

Then

$$
\lim_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N}\left[\sqrt{G(x,x)} - \sqrt{G^L(x,x)}\right] = 0\,, \quad \text{and}
$$

$$
\lim_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N}\left[G(x,x) - G^L(x,x)\right] = 0\,.
$$

Moreover, there exist constants $a, b > 0$ such that for any $L > 0$, any $x, y \in \mathbb{Z}^d$ with $x \neq y$,

$$
G^L(x,y) \leqslant aL^b(\log|x-y|)^{d+2+\alpha}|x-y|^{-d-\alpha}\,. \tag{A.A.14}
$$

Appendix B

Hypercontractivity estimates

We start with an application of Nelson's famous result on the hypercontractivity of Gaussian measures. Let P^ε be the centered Gaussian field with spectral density f^ε given in (3.3.7) and covariance G^ε as in (3.3.8). Also let $\beta > 0$ be as in the decay assumption (3.3.1). We denote by $\|f\|_p$, $1 \leqslant p < \infty$ the L^p norm of f w.r.t. P^ε,

$$\|f\|_p = P^\varepsilon(|f|^p)^{1/p} \,.$$

Lemma B.1. *There exist constants $\alpha = \alpha(\beta) > 0$ and $l_0 > 0$ such that for all $l > l_0$ and for any pair of local functions $f, g \in L^1(P^\varepsilon)$ with $d(S_f, S_g) > l$,*

$$\|f\,g\|_1 \leqslant \|f\|_{1+\tau(l)}\,\|g\|_{1+\tau(l)}\,, \tag{B.B.1}$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function satisfying $\tau(l) \leqslant l^{-\alpha}$.

Proof. Given two configurations $\xi, \eta \in \Omega$ of finite support, we define

$$\langle \xi, \eta \rangle_\varepsilon = \sum_{x,y\in\mathbb{Z}^d} \xi_x\, G^\varepsilon(x,y)\, \eta_y \,,$$

and let $\|\xi\|_\varepsilon^2 = \langle \xi, \xi \rangle_\varepsilon$. Following [18] we define, for $\Lambda_1, \Lambda_2 \in \mathbb{F}$

$$\begin{aligned}\tau(\Lambda_1,\Lambda_2) = \sup\big\{ \langle \xi, \eta \rangle_\varepsilon\,,\ \ &\|\xi\|_\varepsilon = \|\eta\|_\varepsilon = 1\,,\\ &\xi = 0 \ \text{ on } \Lambda_1^c\,,\ \ \eta = 0 \ \text{ on } \Lambda_2^c \big\}\,,\end{aligned}$$

and set, for any $l > 0$

$$\tau(l) = \sup\,\big\{\, \tau(\Lambda_1,\Lambda_2)\,;\ \ \Lambda_1,\Lambda_2 \in \mathbb{F},\ \ d(\Lambda_1,\Lambda_2) > l \,\big\}\,. \tag{B.B.2}$$

We may use now Nelson's estimate (cf. e.g. Theorem 5.11 and Lemma 5.12 in [18]), to write

$$\|f\,g\|_1 \leqslant \|f\|_{1+\tau(l)}\,\|g\|_{1+\tau(l)}\,,$$

for any pair of local functions $f, g \in L^1(P^\varepsilon)$ with $d(S_f, S_g) > l$.
Let us check the decay of $\tau(l)$, as $l \to \infty$. Choose $\Lambda_1, \Lambda_2 \in \mathbb{F}$ and $\xi, \eta \in \Omega$ such that $\xi = 0$ on Λ_1^c and $\eta = 0$ on Λ_2^c, and $\|\xi\|_\varepsilon = \|\eta\|_\varepsilon = 1$. We write

$$\begin{aligned} |\langle \xi, \eta\rangle_\varepsilon| &\leqslant 2 \sum_{x,y\in\mathbb{Z}^d} (\xi_x^2 + \eta_y^2)\, |G^\varepsilon(x,y)| \\ &\leqslant 4 \sup_{x\in\Lambda_1} \sum_{y\in\Lambda_2} |G^\varepsilon(x,y)| + 4 \sup_{y\in\Lambda_2} \sum_{x\in\Lambda_1} |G^\varepsilon(x,y)|\,, \end{aligned} \tag{B.B.3}$$

having used the fact that, for any $\xi \in \Omega$ of finite support

$$\sum_{x\in\mathbb{Z}^d} \xi_x^2 \leqslant 2\, \|\xi\|_\varepsilon^2\,,$$

which is easily checked using e.g. the Fourier transform (1.1.6). It is then sufficient to show

$$\sum_{|x|>l} |G^\varepsilon(0,x)| \leqslant l^{-\alpha}\,, \tag{B.B.4}$$

for some $\alpha > 0$ and for all l sufficiently large[1]. We first observe that $G^\varepsilon(0,x)$ is indeed absolutely summable, cf. (3.3.15). We can then write

$$\sum_{|x|>l} |G^\varepsilon(0,x)| \leqslant \sum_{|x|>l} p(0,x) + \sum_{n=2}^{\infty} (1+\varepsilon)^{-n} \sum_{|x|>l} p^n(0,x)\,. \tag{B.B.5}$$

Let β be the constant appearing in (3.3.1). It is easy to check that the first term above can be bounded by $l^{-\beta}$ as soon as l is sufficiently large. Let $\mathbb{P}_0$ denote the probability measure for the random walk $\{\xi_0, \xi_1, \xi_2, \dots\}$ with transition p and starting point $\xi_0 = 0$. For each $n \geqslant 2$, we have

$$\sum_{|x|\geqslant l} p^n(0,x) = \mathbb{P}_0\big(|\xi_n| \geqslant l\big) \leqslant n\, \mathbb{P}_0\big(|\xi_1| \geqslant l/n\big) \leqslant c_1\, n^{1+\beta}\, l^{-\beta}\,,$$

where $c_1 = c_1(d)$ is a finite constant (independent of n and l). Therefore there exist constants $c_2 < \infty$ and $c_3(\varepsilon) > 0$, such that for any integer M,

$$\begin{aligned} \sum_{n=2}^{\infty} (1+\varepsilon)^{-n} \sum_{|x|>l} p^n(0,x) &\leqslant \sum_{n=2}^{M} \sum_{|x|>l} p^n(0,x) + \\ + \sum_{n=M+1}^{\infty} (1+\varepsilon)^{-n} \sum_{|x|>l} p^n(0,x) &\leqslant c_2\, M^{2+\beta}\, l^{-\beta} + \exp(-c_3(\varepsilon) M)\,. \end{aligned} \tag{B.B.6}$$

[1] As discussed in Remark 3.2, the bound (B.B.4) is rather crude.

We choose now $M = M(l) = c_3(\varepsilon)^{-1}\beta \log l$ in (B.B.6). From (B.B.5) we see that this implies (B.B.4) for any $0 < \alpha < \beta$ and l sufficiently large. $\square$

Let $f_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, be a finite collection of local functions. Given a number $l > 0$, we say that $f_1, f_2, \ldots, f_n$ are *l-separated* if their supports $S_1, S_2, \ldots, S_n$ are such that

$$d(S_i, S_j) > l\,, \quad \text{for all} \quad i \neq j\,.$$

Lemma B.2. *There exists a constant $l_0 > 0$ and a non-increasing function*

$$\rho : [0, \infty) \to [1, \infty)\,,$$

with $\rho(l) \to 1$, as $l \to \infty$, such that the following holds for arbitrary integers $n \in \mathbb{Z}_+$, $l > l_0$ and $L > l$. Suppose $f_1, f_2, \ldots, f_n$ are l-separated local functions in $L^1(P^\varepsilon)$. Suppose there exist sites $x_i \in (L+l)\mathbb{Z}^d$, $i = 1, \ldots, n$ with

$$S_i \subset B_i \;\equiv\; x_i \,+\, [0, L+l]^d \cap \mathbb{Z}^d\,, \quad i = 1, \ldots, n\,.$$

Then

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \prod_{i=1}^{n} \|f_i\|_{\rho(l)} \tag{B.B.7}$$

Proof. Let us consider the case $d = 2$. The general case requires an obvious generalization of the argument. Let $l > 0$, $L > l$ and let $f_1, f_2, \ldots, f_n$ and $B_1, \ldots, B_n$ be as above. We group the cubes B_i, $i = 1, \ldots, n$ into four families A,B,C and D, depending on whether the coordinates of the corresponding x_i are (even,even),(odd,even),(odd,odd) or (even,odd) respectively. Define

$$f_A \;\equiv\; \prod_{i:\, S_i \in A} f_i\,,$$

We define f_B, f_C and f_D in the same way. In this way f_A, f_B, f_C and f_D are l-separated functions, and Lemma B.1 yields

$$\begin{aligned}
\Big\| \prod_{i=1}^{n} f_i \Big\|_1 &\leqslant \|f_A\|_{1+\tau(l)}\, \|f_B f_C f_D\|_{1+\tau(l)} \\
&\leqslant \|f_A\|_{1+\tau(l)}\, \|f_B\|_{(1+\tau(l))^2}\, \|f_C\|_{(1+\tau(l))^3}\, \|f_D\|_{(1+\tau(l))^3}\,.
\end{aligned}$$

We can use the fact that $\|f\|_p \leqslant \|f\|_{p'}$, for $1 < p \leqslant p'$, to see that, for some constant $\gamma \in (0, \infty)$ (independent of n, L and l)

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \|f_A\|_{1+\gamma\tau(l)}\, \|f_B\|_{1+\gamma\tau(l)}\, \|f_C\|_{1+\gamma\tau(l)}\, \|f_D\|_{1+\gamma\tau(l)}\,. \tag{B.B.8}$$

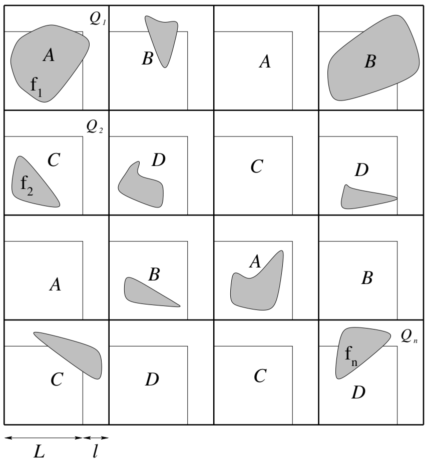

FIGURE B.1. Geometric construction in the proof of Lemma B.2 .

Since $L > l$, we see that, within each family A,B,C,D, the functions are now at least $2l$-separated. Therefore we can repeat the argument by dividing the family A into four sub-families AA,AB,AC and AD. Within each sub-family the functions are now $4l$-separated. Implementing this idea, and iterating (B.B.8), we have

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \prod_{i=1}^{n} \|f_i\|_{\rho(l)}$$

where we have defined

$$\rho(l) \; \equiv \; \prod_{k=1}^{\infty} (1 + \gamma\tau(2^k l)) \, .$$

Using the bound $\tau(l) \leqslant l^{-\alpha}$, cf. Lemma B.1, it is straightforward to check that $\rho(l)$ satisfies the desired properties. ☐

B.1 Logarithmic moment generating function

The hypothesis of Lemma B.2 are tailored in such a way that they apply to the following situation (see [23], Lemma 5.4.13, for the one-dimensional case). Suppose a local bounded function f is given, whose support S_f is contained in the cube of side $q \in \mathbb{Z}_+$

$$Q = [0, q-1]^d \cap \mathbb{Z}^d \, .$$

We show that the limit

$$\Lambda_Q(f) = \lim_{N\to\infty} N^{-d} \log P^{\varepsilon}\Big(\exp \sum_{x\in\Lambda_N} f \circ \theta_x \Big) \tag{B.B.9}$$

exists. To this end we may assume $f \geqslant 0$, otherwise replace f with $f + \|f\|_\infty$. By shift invariance of P^ε, we may replace Λ_N in (3.3.26) by the cube

$$B_N = [0, N-1]^d \cap \mathbb{Z}^d \, .$$

We also let $B_N(x)$ denote the translate $B_N + x$, $x \in \mathbb{Z}^d$. Let $M \in \mathbb{Z}_+$, $M < N$ and divide B_N into translates of the cube B_M. If $m = [N/M]$, the integer part, then $B_N = B_{mM} \cup R_{N,M}$, with $|R_{N,M}| \leqslant c_M N^{d-1}$, for some M-dependent constant c_M. We now enlarge the cubes B_M contained in B_{mM} by changing their side from M to $M+q+l$, for some $l > 0$. We choose m^d sites $x_i \in (M+q+l)\mathbb{Z}^d$, in such a way that,

$$B_{mM} \subset \bigcap_{i=1}^{m^d} B_{M+q+l}(x_i) \, .$$

We may estimate

$$\sum_{x \in B_N} f \circ \theta_x \leqslant \sum_{i=1}^{m^d} \sum_{x \in B_{M+q+l}(x_i)} f \circ \theta_x + c_M N^{d-1} \|f\|_\infty \, .$$

Moreover, there is a finite constant $c_{q,l}$, depending on q and l, such that for any $i = 1, \ldots, m^d$ and for any $M \in \mathbb{Z}_+$

$$| B_{M+q+l}(x_i) \setminus B_M(x_i) | \leqslant c_{q,l} M^{d-1} \, ,$$

and therefore

$$\sum_{x \in B_N} f \circ \theta_x \leqslant (c_{q,l} m^d M^{d-1} + c_M N^{d-1}) \|f\|_\infty + \\ + \sum_{i=1}^{m^d} \sum_{x \in B_M(x_i)} f \circ \theta_x \tag{B.B.10}$$

By construction, the supports S_i of the functions

$$f_i \equiv \sum_{x \in B_M(x_i)} f \circ \theta_x$$

are contained in $B_{M+q}(x_i)$, so that $d(S_i, S_j) > l$, for all $i \neq j$ and, setting $L = M+q$, we see that Lemma B.2 applies. If l is large enough we can then use the estimate (B.B.7) to obtain

$$P^\varepsilon\Big(\prod_{i=1}^{m^d} \exp f_i\Big) \leqslant \prod_{i=1}^{m^d} \Big[P^\varepsilon(\exp \rho(l) f_i)\Big]^{1/\rho(l)} \tag{B.B.11}$$

where $\rho(l) \to 1$, as $l \to \infty$. Also, notice that for each $i = 1, \dots, m^d$,

$$\frac{1}{\rho(l)} \log P^{\varepsilon}\big(\exp \rho(l) f_i\big) \leqslant \log P^{\varepsilon}\big(\exp f_i\big) + (\rho(l)-1)\, M^d \, \|f\|_\infty .$$

Let us define

$$\Gamma_N(f) = \log P^{\varepsilon}\Big(\exp \sum_{x \in B_N} f \circ \theta_x\Big).$$

Using shift invariance of P^{ε}, from (B.B.11) and (B.B.10) we have

$$\begin{aligned}\Gamma_N(f) \leqslant\; & m^d \Gamma_M(f) + \\ & + \big(m^d M^d (\rho(l)-1) + c_{q,l}\, m^d M^{d-1} + c_M N^{d-1}\big)\|f\|_\infty.\end{aligned}$$

For fixed M and l, this implies

$$\limsup_{N\to\infty} \frac{\Gamma_N(f)}{N^d} \leqslant \frac{\Gamma_M(f)}{M^d} + \Big[(\rho(l)-1) + \frac{c_{q,l}}{M}\Big]\|f\|_\infty$$

By letting first $M \to \infty$, and then $l \to \infty$, we have

$$\limsup_{N\to\infty} \frac{\Gamma_N(f)}{N^d} \leqslant \liminf_{M\to\infty} \frac{\Gamma_M(f)}{M^d}.$$

This proves the existence of the limit in (3.3.26).

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