UNIFORM POINCARÉ INEQUALITIES FOR UNBOUNDED CONSERVATIVE SPIN SYSTEMS: THE NON–INTERACTING CASE

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Abstract. We prove a uniform Poincaré inequality for non–interacting unbounded spin systems with a conservation law, when the single–site potential is a bounded perturbation of a convex function with polynomial growth at infinity. The result is then applied to Ginzburg-Landau processes to show diffusive scaling of the associated spectral gap.

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1. Introduction and main result

Consider a probability measure μ on $\mathbb{R}$ of the form

$$\mu(\mathrm{d}\eta) = \frac{e^{-V(\eta)}}{Z}\,\mathrm{d}\eta\,, \tag{1.1}$$

with $Z = \int e^{-V(\eta)}\mathrm{d}\eta$. Denote by μ_N the N-fold product measure obtained by tensorization of μ on $\mathbb{R}^N$, $N \in \mathbb{N}$. The canonical Gibbs measure with density $\rho \in \mathbb{R}$ is defined by conditioning μ_N on the $N-1$ dimensional hyperplane $\sum_{i=1}^N \eta_i = \rho N$, i.e.

$$\nu_{N,\rho} = \mu_N\Big(\,\cdot\,\Big|\,\sum_{i=1}^N \eta_i = \rho N\Big)\,. \tag{1.2}$$

We are going to give sufficient conditions on the potential V in order that the canonical measures $\nu_{N,\rho}$ satisfy a Poincaré inequality, uniformly in ρ and N.

For any probability measure ν we write $\nu(F) = \int F\mathrm{d}\nu$ for the mean of a function F and $\mathrm{Var}_\nu(F)$ for the variance $\nu(F^2) - \nu(F)^2$. For any smooth function F on $\mathbb{R}^N$ we write $\partial_i F$ for the partial gradient along the i-th coordinate. We say that a measure ν on $\mathbb{R}^N$ satisfies a Poincaré inequality if there exists a finite constant γ such that

$$\mathrm{Var}_\nu(F) \leqslant \gamma \sum_{i=1}^N \nu\big[(\partial_i F)^2\big]$$

holds for every smooth, real function F. Specializing to the canonical Gibbs measures (1.2) we define the quadratic form

$$\mathcal{E}_{N,\rho}(F) = \sum_{i=1}^N \nu_{N,\rho}\big[(\partial_i F)^2\big]\,.$$

For every $N \in \mathbb{N}$ and $\rho \in \mathbb{R}$ the Poincaré constant is given by

$$\gamma(N,\rho) = \sup_F \frac{\mathrm{Var}_{\nu_{N,\rho}}(F)}{\mathcal{E}_{N,\rho}(F)} \tag{1.3}$$

where the supremum is carried over all smooth, non–constant, real functions F on $\mathbb{R}^N$. We say that a uniform Poincaré inequality holds whenever

$$\sup_{N\in\mathbb{N}}\sup_{\rho\in\mathbb{R}}\gamma(N,\rho)\;<\;\infty\,. \tag{1.4}$$

The main result of this paper states that such an estimate holds when V is of the form $V=\varphi+\psi$ with ψ a smooth bounded function and φ a uniformly convex function satisfying some mild growth condition at infinity. In order to describe the latter we define the class Φ of functions $\varphi\in\mathcal{C}^2(\mathbb{R},\mathbb{R})$ with second derivative φ'' obeying the following conditions:

- *Uniform convexity:* There exists $\delta>0$ such that $\varphi''\geqslant\delta$.
- *Polynomial growth at infinity:* There exist constants $\beta_-,\beta_+\in[0,\infty)$ and a constant $C\in[1,\infty)$ such that

$$\frac{1}{C}\leqslant\liminf_{x\to\infty}\frac{\varphi''(\pm x)}{x^{\beta_\pm}}\leqslant\limsup_{x\to\infty}\frac{\varphi''(\pm x)}{x^{\beta_\pm}}\leqslant C\,. \tag{1.5}$$

Clearly, any uniformly convex polynomial belongs to Φ. The perturbation will be taken from the class Ψ, defined as the set of functions $\psi\in\mathcal{C}^2(\mathbb{R},\mathbb{R})$ such that $|\psi|_\infty<\infty$, $|\psi'|_\infty<\infty$ and $|\psi''|_\infty<\infty$.

Theorem 1.1. *Assume V is of the form $V=\varphi+\psi$ with $\varphi\in\Phi$ and $\psi\in\Psi$. Then the measures $\nu_{N,\rho}$ satisfy a uniform Poincaré inequality.*

The proof of Theorem 1.1 will be given in the next three sections. It relies on a powerful idea recently introduced by Carlen, Carvalho and Loss in [11, 12]. A similar technique was then used also in [10] to study the relaxation to equilibrium for a conservative lattice gas dynamics. The argument of [11, 12], see also [9], essentially shows that in view of the permutation symmetry of the measures (1.2) one can reduce the problem to the analysis of a one–dimensional process. The latter will be studied by means of a local limit theorem expansion. The technical condition (1.5) on the growth at infinity is of help in establishing uniform estimates in the local central limit theorem (see Lemma 2.5). It is also used to estimate the tails of the transition probabilities of the above mentioned one–dimensional process (see Lemma 3.4).

Poincaré inequalities for conservative systems are usually studied on the level of the corresponding Ginzburg-Landau or Kawasaki dynamics, [2, 3, 7, 21]. This is an ergodic diffusion process on the hyperplane $\sum_{i=1}^N\eta_i=\rho N$, with $\nu_{N,\rho}$ as reversible invariant measure, and Dirichlet form of the type

$$\mathcal{D}_{N,\rho}(F)=\sum_{i=1}^{N-1}\nu_{N,\rho}\big[(\partial_{i+1}F-\partial_iF)^2\big]\,.$$

In this context the Poincaré inequality becomes a statement about the gap in the spectrum of the associated self adjoint Markov generator, or equivalently about the rate of convergence to equilibrium in the $L^2(\nu_{N,\rho})$–norm. In section 4 we shall see that an immediate corollary of Theorem 1.1 is an estimate of the form

$$\mathrm{Var}_{\nu_{N,\rho}}(F)\leqslant C\,N^2\,\mathcal{D}_{N,\rho}(F) \tag{1.6}$$

for all smooth functions F with a constant C independent of ρ and N. This says that the spectral gap scales diffusively with the size of the system, uniformly in the density. Such estimates are usually a key step in establishing hydrodynamical limits, see [16]. The question of the generality under which estimate (1.6) holds was already raised in [22]. It was pointed out that when V is a uniformly convex function then (1.6) holds. Indeed, in this case a general argument based on the Bakry–Emery criterium applies, see [8, 13]. More directly, when there is no perturbation ($\psi = 0$), Theorem 1.1 (without the additional requirement (1.5)) becomes an immediate consequence of the Brascamp–Lieb inequality [6]. On the other hand the extension to bounded perturbations of a uniformly convex function proved to be rather challenging. We refer the reader to [1, 5, 15, 20, 23] and references therein to get an idea of the difficulties one has to face when leaving the purely convex setting. Recently it was shown in [17] that (1.6) holds when V is of the form $V(x) = a\,x^2 + \psi(x)$, $a > 0$ and ψ a bounded function. The authors prove the statement (1.6) by adapting the martingale method originally introduced in [21]. They also prove that the stronger logarithmic Sobolev inequality holds. The recent paper [13] gives further development along the same lines by slightly improving the hypothesis on the potential V. These results seem to rely strongly on the fact that V is essentially quadratic. Our approach is substantially simpler and covers a wider class of potentials. On the other hand it is based on the permutation symmetry (exchangeability) of the canonical measure and it might be difficult to adapt to truly interacting non-product cases.

We conclude this introduction with some comments on three questions posed by the referee about possible extensions of Theorem 1.1.

1. One can try to generalize Theorem 1.1 to the case of $\mathbb{R}^d$–valued variables, $d > 1$. Here the convex part of the potential is a function $\varphi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ with uniformly positive hessian matrix. The technical difficulty is in the derivation of the uniform expansion in Theorem 2.1 in the presence of a nontrivial covariance structure. Note, however, that some form of the Bobkov's estimate we use there is still available in the multidimensional case [4]. We have not worked out the details but we see no serious obstacle to the extension of the result to this case.

2. In contrast to the technical assumption (1.5), the uniform convexity assumption $\varphi'' \geqslant \delta$ seems to be necessary. If, for instance, the potential V is of the form $V(x) = |x|^{1+\alpha}$, $\alpha \in [0, 1)$, we cannot expect a uniform Poincaré inequality. Indeed, in this case direct computations show that the variance σ_ρ^2 appearing in (2.3) below diverges (when $|\rho| \to \infty$) as $|\rho|^{1-\alpha}$ for $\alpha > 0$, and as ρ^2 for $\alpha = 0$. An interesting question here is whether the ratio $\gamma(N, \rho)/\sigma_\rho^2$ remains bounded uniformly.

3. A very interesting problem is to prove that the assumptions of Theorem 1.1 are sufficient for a uniform logarithmic Sobolev inequality. As already mentioned, this has been shown to hold when φ is quadratic [17, 13]. While the results of section 2 and 3 below can be useful to attack this question, the argument of Theorem 4.1 relies entirely on Hilbert space techniques. An inspection of the reasoning in section 4 reveals that the log–Sobolev counterpart of our main estimate on the operator $\mathcal{P}$, cf. (4.6), can be formulated as a suitable approximate subadditivity property for entropies. To establish such an entropy version of the Carlen–Carvalho–Loss approach remains a challenging problem.

The rest of the paper goes as follows. In section 2 we prove a uniform local central limit theorem expansion. This is used in section 3 to study the one–dimensional process which plays a key role in the iterative proof of Theorem 1.1. The latter is given in section 4. In section 5 we discuss the application to spectral gap estimates for Ginzburg-Landau processes.

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2. Basic tools

We assume throughout that V is a potential satisfying the hypothesis of Theorem 1.1. Namely, $V(x) = \varphi(x) + \psi(x)$, $\varphi \in \Phi$ and $\psi \in \Psi$, where Φ is the class of uniformly convex $\mathcal{C}^2$–functions satisfying (1.5) and Ψ is the class of bounded $\mathcal{C}^2$–functions with bounded first and second derivatives. Given $\rho \in \mathbb{R}$ define the probability density

$$h^\rho(x) = \frac{e^{-V(x+\rho)-\lambda(\rho)x}}{Z_\rho}\,, \tag{2.1}$$

with $Z_\rho = \int e^{-V(x+\rho)-\lambda(\rho)x}\mathrm{d}x$. The parameter $\lambda = \lambda(\rho) \in \mathbb{R}$, the so-called chemical potential, is uniquely determined by ρ through the condition

$$\int x\, h^\rho(x)\,\mathrm{d}x = 0\,. \tag{2.2}$$

We write σ_ρ^2 for the variance

$$\sigma_\rho^2 = \int x^2\, h^\rho(x)\,\mathrm{d}x\,. \tag{2.3}$$

Unless otherwise specified all integrals here and below are understood to range over the real line. We call μ_ρ the probability measure with density $h^\rho(\cdot - \rho)$. If $\mu_{N,\rho}$ denotes the product $\mu_\rho \otimes \cdots \otimes \mu_\rho$ (N times), $N \in \mathbb{N}$, then the canonical measure $\nu_{N,\rho}$ can be equivalently obtained as in (1.2) with μ_N replaced by $\mu_{N,\rho}$. It will be useful to work directly with the density h^ρ, which brings the original measure "back to the origin". Note that, when the potential V is quadratic h^ρ is just a fixed gaussian density, independently of ρ.

2.1. Uniform local central limit theorem.

Let π_i be the canonical projection of $\mathbb{R}^N$ onto $\mathbb{R}$ given by $\pi_i\eta = \eta_i$, $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$. We call $\nu^1_{N,\rho}$ the one–site marginal of $\nu_{N,\rho}$, i.e. $\nu^1_{N,\rho} = \nu_{N,\rho} \circ \pi_1^{-1}$ is the distribution of η_1 under $\nu_{N,\rho}$. By permutation symmetry all one–site marginals coincide. The density $g_{N,\rho}$ of $\nu^1_{N,\rho}$ can be written in the form

$$g_{N,\rho}(x) = \frac{h^\rho(x-\rho)G^\rho_{N-1}(x-\rho)}{G^\rho_N(0)}\,, \tag{2.4}$$

with

$$G^\rho_N(x) = \int \mathrm{d}\eta_1 \cdots \int \mathrm{d}\eta_N \Big(\prod_{i=1}^N h^\rho(\eta_i)\Big)\,\delta\Big(\sum_{i=1}^N \eta_i + x\Big)\,. \tag{2.5}$$

Here we are using Dirac's notation

$$f(0) = \int \mathrm{d}x\, f(x)\,\delta(x)\,.$$

Note that if we consider independent random variables η_i with common distribution defined by the density h^ρ, then the normalized sum

$$\frac{1}{\sigma_\rho\sqrt{N}}\sum_{i=1}^N \eta_i$$

has a density given by

$$F_N^\rho(z) = \sigma_\rho\sqrt{N} G_N^\rho(-z\sigma_\rho\sqrt{N})\,.$$

We shall use the classical local central limit theorem expansion for the density F_N^ρ. Introduce the centered moments $m_{k,\rho}$:

$$m_{k,\rho} = \int x^k\, h^\rho(x)\,\mathrm{d}x\,, \qquad k = 1, 2, \ldots$$

so that in particular $m_{1,\rho} = 0$, $m_{2,\rho} = \sigma_\rho^2$.

Theorem 2.1. *Uniformly in $z \in \mathbb{R}$ and $\rho \in \mathbb{R}$:*

$$F_N^\rho(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\Big(1 + \frac{P_3(z)}{\sqrt{N}} + \frac{P_4(z)}{N}\Big) + O\big(N^{-\frac{3}{2}}\big) \tag{2.6}$$

where P_3, P_4 are the polynomials

$$P_3(z) = \frac{m_{3,\rho}}{6\sigma_\rho^3}(z^3 - 3z)\,, \qquad P_4(z) = \frac{m_{3,\rho}^2}{72\sigma_\rho^6}(z^3-3z) + \frac{m_{4,\rho} - 3\sigma_\rho^4}{24\sigma_\rho^4}(z^4 - 6z^2 + 3)\,. \tag{2.7}$$

For every fixed $\rho \in \mathbb{R}$ the above expansion holds uniformly in $z \in \mathbb{R}$ as soon as the fifth moment $m_{5,\rho}$ exists, see e.g. [14], chap. XVI, Theorem 2. What is important for us is that (2.6) holds uniformly in $z \in \mathbb{R}$ and $\rho \in \mathbb{R}$. This in turn follows by the standard proof provided one has a uniform bound on normalized moments and a uniform control of the characteristic functions. In particular, Theorem 2.1 is a consequence of the following properties:

- *Bounds on normalized moments:* for every n there exists $C_n < \infty$ such that

$$\sup_{\rho\in\mathbb{R}} \left|\frac{m_{n,\rho}}{\sigma_\rho^n}\right| \leqslant C_n\,. \tag{2.8}$$

- *Bounds on characteristic functions:* Let $v_\rho(\zeta) = \int e^{i\zeta x} h^\rho(x)\mathrm{d}x$ and set $\bar v_\rho(\zeta) = v_\rho(\zeta/\sigma_\rho)$. Then

$$\exists C < \infty\ : \qquad \sup_{\rho\in\mathbb{R}} |\bar v_\rho(\zeta)| \leqslant \frac{C}{\zeta^2}\,, \tag{2.9}$$

$$\forall \epsilon > 0\,,\ \exists c_\epsilon < 1\ : \qquad \sup_{\rho\in\mathbb{R}}\ \sup_{|\zeta|\,\geqslant\,\epsilon} |\bar v_\rho(\zeta)| \leqslant c_\epsilon \tag{2.10}$$

We prove the above bounds in Lemma 2.2 and Lemma 2.5 below.

Lemma 2.2. *Assume $V = \varphi + \psi$ with $\varphi \in \Phi$, $|\psi|_\infty < \infty$. Then there exists $k = k(|\psi|_\infty) < \infty$ such that for every $n \geqslant 2$*

$$\sup_{\rho\in\mathbb{R}} \left|\frac{m_{2n,\rho}}{\sigma_\rho^{2n}}\right| \leqslant \prod_{\ell=2}^n (1 + k\ell^2)\,. \tag{2.11}$$

In particular, (2.8) holds.

Proof. We first establish a bound for the Poincaré constant in terms of the variance σ_ρ^2. We denote by μ_ρ the probability measure with density $(\mathcal{Z}_\rho)^{-1}e^{-V(x)-\lambda(\rho)x}$ and by $\tilde\mu_\rho$ the probability measure with density $(\tilde{\mathcal{Z}}_\rho)^{-1}e^{-\varphi(x)-\lambda(\rho)x}$. Clearly $\mathcal{Z}_\rho = e^{-\rho\lambda(\rho)}Z_\rho$ (cf. (2.1)) and

$$\tilde{\mathcal{Z}}_\rho = \int e^{-\varphi(x)-\lambda(\rho)x}\mathrm{d}x \in \left[e^{-|\psi|_\infty}\mathcal{Z}_\rho, e^{|\psi|_\infty}\mathcal{Z}_\rho\right].$$

Let γ_ρ and $\tilde\gamma_\rho$ be the Poincaré constants associated to μ_ρ and $\tilde\mu_\rho$ respectively. Let also $\tilde\sigma_\rho^2$ denote the variance of $\tilde\mu_\rho$, $\tilde\sigma_\rho^2 = \tilde\mu_\rho(x^2) - \tilde\mu_\rho(x)^2$. We shall use here a result derived by Bobkov in [4], which says that since the density of $\tilde\mu_\rho$ is log–concave one has the bound

$$\tilde\gamma_\rho \leqslant 12\,\tilde\sigma_\rho^2\,. \tag{2.12}$$

It is not difficult to establish a similar bound for μ_ρ. Namely, for any smooth function f such that $\mu_\rho(f) = 0$ we write

$$\begin{aligned}\mu_\rho(f^2) &\leqslant \mu_\rho\big[(f-\tilde\mu_\rho(f))^2\big] \leqslant e^{2|\psi|_\infty}\tilde\mu_\rho\big[(f-\tilde\mu_\rho(f))^2\big]\\ &\leqslant e^{2|\psi|_\infty}\,\tilde\gamma_\rho\,\tilde\mu_\rho\big[(f')^2\big] \leqslant e^{4|\psi|_\infty}\,\tilde\gamma_\rho\,\mu_\rho\big[(f')^2\big]\,,\end{aligned}$$

so that

$$\gamma_\rho \leqslant e^{4|\psi|_\infty}\tilde\gamma_\rho\,.$$

On the other hand

$$\tilde\sigma_\rho^2 \leqslant \tilde\mu_\rho\big[(x-\rho)^2\big] \leqslant e^{2|\psi|_\infty}\mu_\rho\big[(x-\rho)^2\big] = e^{2|\psi|_\infty}\sigma_\rho^2\,.$$

From (2.12) we obtain

$$\gamma_\rho \leqslant 12\,e^{6|\psi|_\infty}\,\sigma_\rho^2\,. \tag{2.13}$$

Once we have such an estimate the proof of (2.11) is immediate. For every $n\in\mathbb{N}$ we have

$$\begin{aligned}m_{2n,\rho} &= \mathrm{Var}_{\mu_\rho}\big[(x-\rho)^n\big] + m_{n,\rho}^2\\ &\leqslant n^2\,\gamma_\rho\,\mu_\rho\big[(x-\rho)^{2(n-1)}\big] + m_{n,\rho}^2 = n^2\,\gamma_\rho\,m_{2(n-1),\rho} + m_{n,\rho}^2\,.\end{aligned}$$

Setting $k = 12\,e^{6|\psi|_\infty}$, from (2.13) we have

$$m_{2n,\rho} \leqslant k\,n^2\sigma_\rho^2\,m_{2(n-1),\rho} \,+\, m_{n,\rho}^2\,.$$

This implies the claim for $n = 2$ and one obtains the rest through induction using $m_{n+1,\rho}^2 \leqslant \sigma_\rho^2 m_{2n,\rho}$. This last inequality also shows that to prove (2.8) we can restrict to even powers. ☐

Remark 2.3. Another important application of Bobkov's bound (2.12) is the following exponential tail estimate. It is well known (see e.g. [19]) that Poincaré inequality implies exponential integrability. In our setting, if γ_ρ denotes the Poincaré constant of μ_ρ and $\xi_\rho(x) = x-\rho$, then

$$\sup_{\rho\in\mathbb{R}}\,\mu_\rho\Big[\exp\frac{|\xi_\rho|}{\sqrt{\gamma_\rho}}\Big] < \infty\,. \tag{2.14}$$

The above estimate is easily obtained from the following argument: set $u(t) = \mu_\rho[\exp t\xi_\rho]$ and use Poincaré inequality to write $u(t) - u(t/2)^2 \leqslant \gamma_\rho(t/2)^2u(t)$. For $t < 2/\sqrt{\gamma_\rho}$ this gives

$$u(t) \leqslant (1-\gamma_\rho(t/2)^2)^{-1}u(t/2)^2\,.$$

Iterating this inequality and using $u(t/s)^s \to 1$, $s \to \infty$, we obtain

$$u(t) \leqslant \prod_{k=1}^{\infty} (1 - \gamma_\rho (t/2^k)^2)^{-2^{k-1}},$$

which is finite as soon as $t < 2/\sqrt{\gamma_\rho}$. Setting $t = 1/\sqrt{\gamma_\rho}$ and repeating the argument for $\mu_\rho[\exp -t\xi_\rho]$ we arrive at (2.14).

On the other hand, by (2.13) one has $\gamma_\rho \leqslant k\sigma_\rho^2$ for some uniform constant $k < \infty$. Using Markov's inequality we deduce that there exists $C < \infty$ such that for every $T \in (0, \infty)$, $\rho \in \mathbb{R}$ one has the tail estimate

$$\mu_\rho\big[|\xi_\rho| \geqslant \sigma_\rho T\big] \leqslant C\, e^{-\frac{T}{C}}\,. \tag{2.15}$$

We shall need some control on σ_ρ^2 as a function of ρ. The following Lemma relies on the assumption (1.5).

Lemma 2.4. *Assume $V = \varphi + \psi$ with $\varphi \in \Phi$ and $|\psi|_\infty < \infty$. Then there exists $k < \infty$ such that for every $\rho \in \mathbb{R}$ one has*

$$\frac{1}{k\,\varphi''(\rho)} \leqslant \sigma_\rho^2 \leqslant \frac{k}{\varphi''(\rho)}\,. \tag{2.16}$$

Proof. Since the bounded perturbation ψ only affects constants (depending only on $|\psi|_\infty$) in (2.16) we may assume that the potential is convex from start. Thus for the rest of this proof we set $V = \varphi$, $\varphi \in \Phi$. To obtain the upper bound we use the Brascamp–Lieb inequality ([6])

$$\sigma_\rho^2 \leqslant \mu_\rho\big[(\varphi'')^{-1}\big]\,. \tag{2.17}$$

In particular, since $\varphi'' \geqslant \delta$, we have $\sigma_\rho^2 \leqslant \frac{1}{\delta}$. We then write, for any $\Delta > 0$

$$\sigma_\rho^2 \leqslant \mu_\rho\big[(\varphi'')^{-1}\,;\, |\xi_\rho| \leqslant \Delta\big] \;+\; \mu_\rho\big[(\varphi'')^{-1}\,;\, |\xi_\rho| > \Delta\big] \tag{2.18}$$

so that

$$\sigma_\rho^2 \leqslant \varphi''(\rho)^{-1} \sup_{\alpha:\,|\alpha| \leqslant \Delta} \left|\frac{\varphi''(\rho)}{\varphi''(\rho+\alpha)}\right| + \frac{1}{\delta}\,\mu_\rho\big[|\xi_\rho| > \Delta\big]\,. \tag{2.19}$$

Choose $\Delta = B\log(2 + |\rho|)$ with $B > 0$ to be fixed later. From (1.5) we infer that

$$\sup_{\rho\in\mathbb{R}} \sup_{\alpha:\,|\alpha| \leqslant \Delta} \left|\frac{\varphi''(\rho)}{\varphi''(\rho+\alpha)}\right| < \infty\,.$$

Moreover, by (2.15) and the bound $\sigma_\rho^2 \leqslant \delta^{-1}$ we see that the second term in (2.19) is bounded by $C(2+|\rho|)^{-\frac{B}{C}}$ for some uniform $C < \infty$. Collecting all this and choosing B sufficiently large we have the sought upper bound $\sigma_\rho^2 \leqslant k\,\varphi''(\rho)^{-1}$.

To find the lower bound we use the inequality

$$\sigma_\rho^2 \geqslant \mu_\rho[\varphi'']^{-1}\,. \tag{2.20}$$

To prove (2.20) note that for any $f \in L^2(\mu_\rho)$ one has $\sigma_\rho^2 = \mu_\rho[\xi_\rho^2] \geqslant 2\mu_\rho[f\xi_\rho] - \mu_\rho[f^2]$. Choose now $f(x) = \beta\frac{\mathrm{d}}{\mathrm{d}x}V_\rho(x)$ where $V_\rho(x) = \varphi(x) + \lambda(\rho)x = -\log h^\rho(x-\rho) + \text{const.}$ and $\beta \in \mathbb{R}$. Integration by parts shows that $\mu_\rho[f\xi_\rho] = \beta$ and $\mu_\rho[f^2] = \beta^2\mu_\rho[\varphi'']$. We have obtained

$$\sigma_\rho^2 \geqslant 2\beta - \beta^2\mu_\rho[\varphi'']\,, \quad \beta \in \mathbb{R}.$$

Optimizing over β gives (2.20). We can now estimate

$$\mu_\rho[\varphi''] \leqslant \varphi''(\rho) \sup_{\alpha:\,|\alpha|\,\leqslant\,\Delta} \left|\frac{\varphi''(\rho+\alpha)}{\varphi''(\rho)}\right| + \mu_\rho\big[\varphi''\,;\,|\xi_\rho|>\Delta\big]$$

Choosing $\Delta = B\log(2+|\rho|)$ the first term is bounded by $k\,\varphi''(\rho)$ as above. The second term can be bounded uniformly in ρ by taking B sufficiently large. Namely, we use (1.5) to write $\varphi''(x) \leqslant \beta(1+|x|)^\beta$ for some given constant $\beta<\infty$ and estimate $|x| \leqslant |\xi_\rho|(1+|\rho|/|\xi_\rho|)$. By Lemma 2.2 $\mu_\rho[\xi_\rho^{2\beta}]$ is bounded uniformly in ρ for every β and therefore using also (2.15) we can find a constant $C<\infty$ such that

$$\begin{aligned}\mu_\rho\big[\varphi''\,;\,|\xi_\rho|>\Delta\big] &\leqslant \mu_\rho\big[(\varphi'')^2\,;\,|\xi_\rho|>\Delta\big]^{\frac12}\,\mu_\rho\big[|\xi_\rho|>\Delta\big]^{\frac12}\\ &\leqslant C(1+|\rho|)^\beta(2+|\rho|)^{-\frac{B}{C}}\,.\end{aligned}$$

Taking B large we have obtained the desired bound $\mu_\rho[\varphi''] \leqslant k\varphi''(\rho)$. □

Lemma 2.5. *Assume $V=\varphi+\psi$ with $\varphi\in\Phi$, $\psi\in\Psi$. Then (2.9) and (2.10) hold.*

Proof. Let $\bar h_\rho(x) = \sigma_\rho h_\rho(\sigma_\rho x)$ denote the density of the normalized variable ξ_ρ/σ_ρ. Observe that

$$\bar v_\rho(\zeta) = \int e^{i\zeta x}\bar h_\rho(x)\mathrm{d}x\,.$$

Writing $\bar v_\rho(\zeta) = |\bar v_\rho(\zeta)|\,e^{i\theta_\rho(\zeta)}$ for some real function $\theta_\rho(\zeta)$ we have

$$|\bar v_\rho(\zeta)| = \int \cos(\zeta x-\theta_\rho(\zeta))\,\bar h_\rho(x)\mathrm{d}x\,. \tag{2.21}$$

A double integration by parts shows that

$$|\bar v_\rho(\zeta)| \leqslant \frac{1}{\zeta^2}\int|\bar h''_\rho(x)|\mathrm{d}x\,,$$

where $\bar h''_\rho(x)$ denotes the second derivative of the density $\bar h_\rho$. We compute

$$\int|\bar h''_\rho(x)|\mathrm{d}x = \sigma_\rho^2\int|h''_\rho(x)|\mathrm{d}x = \sigma_\rho^2\int|V'_\rho(x+\rho)^2 - V''(x+\rho)|h_\rho(x)\mathrm{d}x\,,$$

where V'_ρ and $V''_\rho = V''$ denote the first and second derivative of the potential $V_\rho(x) = V(x)+\lambda(\rho)x$. Integration by parts yields $\int V'_\rho(x+\rho)^2h_\rho(x)\mathrm{d}x = \mu_\rho[(V'_\rho)^2] = \mu_\rho[V''] = \mu_\rho[\varphi'']+\mu_\rho[\psi'']$. Using $|\psi''|_\infty<\infty$ and the bounds in Lemma 2.4 we thus conclude that $\sigma_\rho^2\int|h''_\rho(x)|\mathrm{d}x$ is uniformly bounded and therefore $|\bar v_\rho(\zeta)|\leqslant C/\zeta^2$ as claimed in (2.9).

We turn to the proof of the estimate (2.10). In view of the uniform bound $|\bar v_\rho(\zeta)| = O(\zeta^{-2})$ proven above we only need to check that for any given constants $\epsilon>0$ and $C<\infty$ we have some $c_\epsilon<1$ such that

$$\sup_{\rho\in\mathbb{R}}\ \sup_{\epsilon\,\leqslant\,|\zeta|\,\leqslant\,C}|\bar v_\rho(\zeta)| \leqslant c_\epsilon\,. \tag{2.22}$$

To prove (2.22) we rely on Lemma 5.5 in [18]. This lemma tells us that if for each ρ we can find an interval $I_\rho\subset\mathbb{R}$ such that $|I_\rho|\geqslant 10\pi\sigma_\rho/\epsilon$ and

$$\inf_{\rho\in\mathbb{R}}\int_{I_\rho}h_\rho(x)\mathrm{d}x>0\,, \tag{2.23}$$

$$\sup_{\rho\in\mathbb{R}}\ \sup_{x,y\in I_\rho}h_\rho(x)/h_\rho(y)<\infty\,, \tag{2.24}$$

then the integral in (2.21) is bounded uniformly by some $c_\epsilon<1$.

We choose $I_\rho = \{x : |x| \leqslant T\sigma_\rho\}$ for some $T > 0$ to be fixed below. For the first property we require $T \geqslant 5\pi/\epsilon$. The property (2.23) is guaranteed by (2.15):

$$\int_{I_\rho} h_\rho(x)\mathrm{d}x \geqslant 1 - Ce^{-\frac{T}{C}} > 0\,,$$

provided T is large enough. It remains to check (2.24). Set

$$u_\rho(x,y) = \varphi(y) + \lambda(\rho)y - \varphi(x) - \lambda(\rho)x\,,$$

and write $h_\rho(x-\rho)/h_\rho(y-\rho) = \exp\left[u_\rho(x,y) + \psi(y) - \psi(x)\right]$. Since ψ is bounded it suffices to show that $u_\rho(x,y)$ is uniformly bounded for $x, y \in I_\rho + \rho$. Since φ is convex, the function $g_\rho(x) := \varphi(x) + \lambda(\rho)x$ has a unique minimum ρ^*, solution of $\varphi'(x) = -\lambda(\rho)$. We first claim that $|\rho - \rho^*| \leqslant 2T\sigma_\rho$ when T is sufficiently large, uniformly in ρ. To see this, suppose $\rho < \rho^* - 2T\sigma_\rho$ (a similar argument applies in the case $\rho > \rho^* + 2T\sigma_\rho$). In this case g_ρ is strictly decreasing in the interval $[\rho - T\sigma_\rho, \rho^*]$. Therefore, letting (as in the proof of Lemma 2.2) $\tilde\mu_\rho$ denote the probability measure with density $(\tilde{\mathcal{Z}}_\rho)^{-1}e^{-g_\rho}$,

$$\tilde\mu_\rho[|x-\rho| \leqslant T\sigma_\rho] \leqslant 2\tilde\mu_\rho[x \in (\rho, \rho + T\sigma_\rho)] \leqslant 2\tilde\mu_\rho[|x - \rho^*| \leqslant T\sigma_\rho]\,. \tag{2.25}$$

On the other hand we know that $\tilde\mu_\rho[|x-\rho| \leqslant T\sigma_\rho] \geqslant 1 - \varepsilon$, for every $\varepsilon > 0$, whenever $T \geqslant T_\varepsilon$ with some $T_\varepsilon < \infty$ uniformly in ρ. The latter estimate follows from the uniform bound $\tilde\mu_\rho[|x-\rho| > T\sigma_\rho] \leqslant k\mu_\rho[|x-\rho| > T\sigma_\rho]$ (cf. the proof of Lemma 2.2) and (2.15). Clearly, this is in contradiction with (2.25) when ε is sufficiently small and $\rho < \rho^* - 2T\sigma_\rho$. Therefore $|\rho - \rho^*| \leqslant 2T\sigma_\rho$, as claimed.

Using this, we see that $x \in I_\rho + \rho$ implies $|x - \rho^*| \leqslant 3T\sigma_\rho$. Therefore an expansion of $u_\rho(x,y)$ up to second order around the point (ρ^*, ρ^*) allows to estimate

$$|u(x,y)| \leqslant 9\,T^2\,\sigma_\rho^2 \sup_{|z| \leqslant 3T\sigma_\rho} \varphi''(\rho^* + z)\,, \qquad x, y \in I_\rho + \rho$$

As in Lemma 2.4, $\sigma_\rho^2\varphi''(\rho^*+z) \leqslant C\sigma_\rho^2\varphi''(\rho) \leqslant C'$ uniformly in $|z| \leqslant 3T\sigma_\rho$, $|\rho-\rho^*| \leqslant 2T\sigma_\rho$, $\rho \in \mathbb{R}$. This implies $\sup_{x,y\in I_\rho} h_\rho(x)/h_\rho(y) < \infty$ and the proof of the lemma is completed. ∎

3. The operator $\mathcal{K}$

Here we introduce the relevant one–dimensional process and prove a key spectral estimate, see Theorem 3.1 below. Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R}, \nu^1_{N,\rho})$ and use the symbol $\langle\cdot,\cdot\rangle$ for the corresponding scalar product $\langle f, g\rangle = \nu_{N,\rho}[(\bar f \circ \pi_1)(g \circ \pi_1)]$, with $\bar f$ denoting the complex conjugate function. Write also $\langle f\rangle$ for the mean of a function $f \in \mathcal{H}$ w.r.t. $\nu^1_{N,\rho}$. We write $\mathcal{H}_0$ for the subspace of $f \in \mathcal{H}$ such that $\langle f\rangle = 0$. We define the stochastic self–adjoint operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ by the sesquilinear form:

$$\langle f, \mathcal{K}g\rangle = \nu_{N,\rho}\big[(\bar f \circ \pi_1)(g \circ \pi_2)\big]\,, \qquad f, g \in \mathcal{H}\,. \tag{3.1}$$

Let ξ_ρ be the linear function $\xi_\rho(x) = x - \rho$. A simple computation shows that

$$\mathcal{K}\xi_\rho = -\frac{1}{N-1}\,\xi_\rho \tag{3.2}$$

for every $\rho \in \mathbb{R}$. Thus the spectrum of $\mathcal{K}$ always contains the eigenvalues $-(N-1)^{-1}$ and 1. We prove below that the rest of the spectrum is confined around zero within a neighborhood of radius $O(N^{-\frac{3}{2}})$.

Theorem 3.1. *There exists $C < \infty$ independent of ρ and N such that for every $f \in \mathcal{H}_0$ satisfying $\langle f, \xi_\rho \rangle = 0$ one has*

$$\big|\langle f, \mathcal{K} f\rangle\big| \leqslant C\, N^{-\frac{3}{2}}\, \langle f, f\rangle\,. \tag{3.3}$$

The rest of this section deals with the proof of Theorem 3.1. The strategy is essentially the same as in [10] where this type of result has been established for a discrete lattice gas model. Adaptation to our setting, however, requires some non–trivial modifications.

Denote by $\tilde g_{N,\rho}(x,y)$ the density of the joint distribution of (η_1, η_2) under $\nu_{N,\rho}$:

$$\tilde g_{N,\rho}(x,y) = \frac{h^\rho(x-\rho) h^\rho(y-\rho) G^\rho_{N-2}((x-\rho)+(y-\rho))}{G^\rho_N(0)}\,. \tag{3.4}$$

When f is real and $\langle f\rangle = 0$ we write

$$\langle f, \mathcal{K} f\rangle = \int\!\!\int \mathrm{d}x\, \mathrm{d}y\, g_{N,\rho}(x) g_{N,\rho}(y) Q_{N,\rho}(x,y) f(x) f(y) \tag{3.5}$$

where we introduced the kernel

$$Q_{N,\rho}(x,y) = \frac{\tilde g_{N,\rho}(x,y) - g_{N,\rho}(x) g_{N,\rho}(y)}{g_{N,\rho}(x) g_{N,\rho}(y)}\,. \tag{3.6}$$

Define the set

$$\mathcal{B}_\rho = \{(x,y) \in \mathbb{R}^2 :\ \ |x-\rho| + |y-\rho| \leqslant B\, \sigma_\rho \log N\,\} \tag{3.7}$$

where B is a constant to be fixed later on. The following expansion is the key step in the proof of Theorem 3.1.

Lemma 3.2. *For every $B < \infty$, there exists $C < \infty$ such that*

$$\sup_{\rho\in\mathbb{R}}\ \sup_{(x,y)\in\mathcal{B}_\rho} \left| Q_{N,\rho}(x,y) + \frac{\xi_\rho(x)\,\xi_\rho(y)}{\sigma_\rho^2 N} \right| \leqslant C\, N^{-\frac{3}{2}} \tag{3.8}$$

Proof. In order to simplify notations we shall write

$$\bar x = \xi_\rho(x) = x - \rho\,, \qquad \bar y = \xi_\rho(y) = y - \rho\,.$$

Using (2.4) and (3.4) we rewrite

$$Q_{N,\rho}(x,y) = \frac{G^\rho_{N-2}(\bar x + \bar y) G^\rho_N(0) - G^\rho_{N-1}(\bar x) G^\rho_{N-1}(\bar y)}{G^\rho_{N-1}(\bar x) G^\rho_{N-1}(\bar y)}\,. \tag{3.9}$$

We now use the expansion of Theorem 2.1. With the change of variable

$$\sigma_\rho \sqrt{N} G^\rho_N(\bar x) = F^\rho_N(-\bar x/\sigma_\rho\sqrt{N}) \tag{3.10}$$

we see that the only terms in $\sigma_\rho\sqrt{N} G^\rho_N(\bar x)$ which are not negligible w.r.t. $O(N^{-\frac{3}{2}})$ in the range $|\bar x| \leqslant B\,\sigma_\rho \log N$ are given by the constant term in P_4 and the linear terms in P_3 and P_4. This implies

$$\sup_{\rho\in\mathbb{R}}\ \sup_{|\bar x|\,\leqslant\, B\,\sigma_\rho \log N} \left| \sigma_\rho\sqrt{N} G^\rho_N(\bar x) - \frac{e^{-\frac{\bar x^2}{2\sigma_\rho^2 N}}}{\sqrt{2\pi}} \Big(1 + \frac{\alpha + \beta \bar x}{N}\Big) \right| \leqslant C\, N^{-\frac{3}{2}} \tag{3.11}$$

$$\alpha := \frac{m_{4,\rho} - 3\sigma_\rho^4}{8\sigma_\rho^4}\,, \qquad \beta := \frac{m_{3,\rho}}{2\sigma_\rho^4} + \frac{m_{3,\rho}^2}{24\sigma_\rho^7\sqrt{N}}\,.$$

Note that by Lemma 2.2 α is uniformly bounded and $\beta\bar{x}$ is bounded by $C\log N$ in the range $|\bar{x}| \leqslant B\,\sigma_\rho \log N$ for some uniform $C < \infty$.

We introduce the following convention. We call $\varepsilon(N)$ anything which vanishes at least as $O(N^{-\frac{3}{2}})$ uniformly in $(x,y) \in \mathcal{B}_\rho$. Thus the result (3.11) will be used in the form

$$\sigma_\rho\sqrt{N}G^\rho_N(\bar{x}) = \frac{e^{-\frac{\bar{x}^2}{2\sigma_\rho^2 N}}}{\sqrt{2\pi}}\Big(1 + \frac{\alpha+\beta\bar{x}}{N}\Big) + \varepsilon(N)\,. \tag{3.12}$$

Use now (3.12) to write

$$\begin{aligned}
&2\pi\sigma_\rho^2(N-1)G^\rho_{N-1}(\bar{x})G^\rho_{N-1}(\bar{y}) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma_\rho^2(N-1)}}\Big(1+\frac{\alpha+\beta\bar{x}}{N-1}\Big)\Big(1+\frac{\alpha+\beta\bar{y}}{N-1}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma_\rho^2 N}}\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N)\,.
\end{aligned}$$

Furthermore, writing $q(N) = (N-1)/\sqrt{N(N-2)} = 1 + O(N^{-2})$, one has

$$\begin{aligned}
&2\pi\sigma_\rho^2(N-1)G^\rho_{N-2}(\bar{x}+\bar{y})G^\rho_N(0) \\
&\qquad = q(N)\,e^{-\frac{(\bar{x}+\bar{y})^2}{2\sigma_\rho^2(N-2)}}\Big(1+\frac{\alpha+\beta(\bar{x}+\bar{y})}{N-2}\Big)\Big(1+\frac{\alpha}{N}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{(\bar{x}+\bar{y})^2}{2\sigma_\rho^2 N}}\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma_\rho^2 N}}\Big(1-\frac{\bar{x}\bar{y}}{\sigma_\rho^2 N}\Big)\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N)\,.
\end{aligned}$$

Inserting in (3.9) we have obtained

$$Q_{N,\rho}(x,y) = -\frac{\bar{x}\bar{y}}{\sigma_\rho^2 N} + \varepsilon(N)\,.$$

$\square$

Remark 3.3. As a simple consequence of the expansion (2.6) and the change of variable (3.10) we have

$$\frac{G^\rho_{N-1}(x-\rho)}{G^\rho_N(0)} = e^{-\frac{(x-\rho)^2}{2\sigma_\rho^2(N-1)}} + O\big(N^{-\frac{1}{2}}\big) \tag{3.13}$$

uniformly in $x \in \mathbb{R}$ and $\rho \in \mathbb{R}$. In particular by (2.4) we have the uniform bound

$$g_{N,\rho}(x) \leqslant C\,h^\rho(x-\rho)\,. \tag{3.14}$$

Next we need to control the atypical region $\mathcal{B}_\rho^c$. We shall use $\mathbf{1}_{\mathcal{B}_\rho}$ and $\mathbf{1}_{\mathcal{B}_\rho^c}$ to denote the indicator function of the set $\mathcal{B}_\rho$ (defined in (3.7)) and its complement, respectively.

Lemma 3.4. *There exist constants $C, B < \infty$ such that uniformly in ρ and N*

$$\int\!\!\int \mathrm{d}x\,\mathrm{d}y\,\tilde{g}_{N,\rho}(x,y)\,|f(x)|\,|f(y)|\,\mathbf{1}_{\mathcal{B}_\rho^c}(x,y) \leqslant C\,N^{-\frac{3}{2}}\,\langle f,f\rangle\,. \tag{3.15}$$

Proof. We first consider the set

$$\mathcal{Q}_\rho = \{(x,y)\in\mathbb{R}^2 : \ |x-\rho|+|y-\rho| \leqslant \ k\log N\} \tag{3.16}$$

where k is a finite constant to be fixed later. When σ_ρ is very small this set is much larger than $\mathcal{B}_\rho$. We first show that (3.15) holds with $\mathcal{B}_\rho^c$ replaced by $\mathcal{Q}_\rho^c$ for k sufficiently large (but independent of N,ρ):

$$\int\!\!\int \mathrm{d}x\mathrm{d}y\, \tilde g_{N,\rho}(x,y)|f(x)|\,|f(y)|\mathbf{1}_{\mathcal{Q}_\rho^c}(x,y) \leqslant C\,N^{-\frac{3}{2}}\,\langle f,f\rangle\,. \tag{3.17}$$

To do this first step we proceed as follows. For all $x\in\mathbb{R}$ we set

$$\rho_x = \rho + \frac{\rho - x}{N-1}\,, \tag{3.18}$$

and observe that if $\tilde g_{N,\rho}(x,y)$ is the density of the joint law of (η_1,η_2), then g_{N-1,ρ_x} is the density of the law of η_1 under the conditioning $\eta_2 = x$. In particular, $\tilde g_{N,\rho}(x,y) = g_{N,\rho}(x)g_{N-1,\rho_x}(y)$. Moreover a simple computation shows that as soon as $N\geqslant 3$ we have $\mathbf{1}_{\mathcal{Q}_\rho^c}(x,y)\leqslant \chi_{x,\frac{k}{4}\log N}(y)+\chi_{y,\frac{k}{4}\log N}(x)$ where $\chi_{x,T}$ denotes the characteristic function of the event $\{|\xi_{\rho_x}|>T\}$, $T\geqslant 0$. We have

$$\begin{aligned}
&\int\!\!\int \mathrm{d}x\mathrm{d}y\, \tilde g_{N,\rho}(x,y)|f(x)|\,|f(y)|\,\chi_{x,\frac{k}{4}\log N}(y)\\
&\quad\leqslant \langle f,f\rangle^{\frac12}\Big(\int \mathrm{d}x\, g_{N,\rho}(x)\Big[\int \mathrm{d}y\, g_{N-1,\rho_x}(y)|f(y)|\,\chi_{x,\frac{k}{4}\log N}(y)\Big]^2\Big)^{\frac12}\\
&\quad\leqslant \langle f,f\rangle\Big(\sup_{x\in\mathbb{R}}\int \mathrm{d}y\, g_{N-1,\rho_x}(y)\,\chi_{x,\frac{k}{4}\log N}(y)\Big)^{\frac12}
\end{aligned}$$

Now by (3.14) we estimate $g_{N-1,\rho_x}(y)\leqslant Ch^{\rho_x}(y-\rho_x)$ and from the exponential tail bound (2.15) we obtain

$$\int \mathrm{d}y\, g_{N-1,\rho_x}(y)\,\chi_{x,\frac{k}{4}\log N}(y)\leqslant C\,\mu_{\rho_x}\big[|\xi_{\rho_x}|>\frac{k}{4}\,\log N\big]\leqslant C'\,N^{-\frac{k}{C'\,\sigma_{\rho_x}}} \tag{3.19}$$

for some constant $C'<\infty$. Since σ_{ρ_x} is bounded from above uniformly, see (2.17), it follows that there exist $C,k_0<\infty$ independent of ρ and N such that

$$\int\!\!\int \mathrm{d}x\mathrm{d}y\, \tilde g_{N,\rho}(x,y)|f(x)|\,|f(y)|\,\chi_{x,\frac{k}{4}\log N}(y)\ \leqslant\ C\,N^{-\frac{3}{2}}\langle f,f\rangle$$

holds as soon as $k\geqslant k_0$. Repeating the argument with x and y interchanged yields (3.17).

We turn to the original claim (3.15). From the previous estimate (3.17) we may replace $\mathbf{1}_{\mathcal{B}_\rho^c}$ by $\mathbf{1}_{\mathcal{Q}_\rho}\mathbf{1}_{\mathcal{B}_\rho^c}$ in (3.15). With the notations introduced above we write

$$\begin{aligned}
\mathbf{1}_{\mathcal{Q}_\rho}\mathbf{1}_{\mathcal{B}_\rho^c}(x,y) &\leqslant \mathbf{1}_{\{|\xi_\rho|\,\leqslant\, k\log N\}}(y)\,\chi_{y,\frac{\sigma_\rho B}{4}\log N}(x)\\
&\quad+\mathbf{1}_{\{|\xi_\rho|\,\leqslant\, k\log N\}}(x)\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\,.
\end{aligned} \tag{3.20}$$

Let us estimate one of the two terms coming from the decomposition (3.20).

$$\begin{aligned}
&\int\int \mathrm{d}x\mathrm{d}y\, \tilde g_{N,\rho}(x,y)|f(x)|\,|f(y)|\,\mathbf{1}_{\{|\xi_\rho|\,\leqslant\, k\log N\}}(x)\,\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\\
&\leqslant \langle f,f\rangle^{\frac12}\Big(\int \mathrm{d}x\, g_{N,\rho}(x)\,\mathbf{1}_{\{|\xi_\rho|\,\leqslant\, k\log N\}}(x)\Big[\int \mathrm{d}y\, g_{N-1,\rho_x}(y)|f(y)|\,\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\Big]^2\Big)^{\frac12}\\
&\leqslant \langle f,f\rangle\Big(\sup_{\substack{x\in\mathbb{R}:\\ |x-\rho|\,\leqslant\, k\log N}}\int \mathrm{d}y\, g_{N-1,\rho_x}(y)\,\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\Big)^{\frac12}
\end{aligned}$$

As in (3.19) we know that there exists $C<\infty$ such that for every $x\in\mathbb{R}^d$

$$\int \mathrm{d}y\, g_{N-1,\rho_x}(y)\,\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\;\leqslant\; C\,N^{-\frac{\sigma_\rho B}{C\,\sigma_{\rho_x}}}\,.$$

The point here is that we may restrict to x satisfying $|x-\rho|\leqslant k\log N$ and for such x Lemma 2.4 tells us that $\sigma_{\rho_x}/\sigma_\rho$ is bounded uniformly in ρ. More precisely, by (2.16) and the assumption (1.5) there exists $C<\infty$ and $\epsilon_0>0$ such that

$$\sup_{\rho\in\mathbb{R}}\ \sup_{\substack{\alpha\in\mathbb{R}:\\ |\alpha|\,\leqslant\,\epsilon_0}}\left|\frac{\sigma^2_{\rho+\alpha}}{\sigma^2_\rho}\right|\leqslant C\,. \tag{3.21}$$

When x satisfies $|x-\rho|\leqslant k\log N$ then $|\rho-\rho_x|\leqslant k\,\frac{\log N}{N-1}$ and taking N sufficiently large we can use (3.21) to arrive at

$$\sup_{\rho\in\mathbb{R}}\ \sup_{\substack{x\in\mathbb{R}:\\ |x-\rho|\,\leqslant\, k\log N}}\int \mathrm{d}y\, g_{N-1,\rho_x}(y)\,\chi_{x,\frac{\sigma_\rho B}{4}\log N}(y)\;\leqslant\; C'\,N^{-\frac{B}{C'}}\;\leqslant\; C'\,N^{-\frac32},$$

with some constant $C'<\infty$ and B sufficiently large. Repeating the argument with x and y interchanged we arrive at (3.15). This completes the proof of the lemma. □

We are now able to finish the proof of Theorem 3.1. Let us go back to (3.5) and split the integral there as

$$\begin{aligned}
\langle f,\mathcal{K}f\rangle &= \int\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\\
&\quad+\int\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}^c_\rho}(x,y)\,.
\end{aligned}\tag{3.22}$$

The second term here can be estimated from above by the sum

$$\begin{aligned}
&\int\int \mathrm{d}x\,\mathrm{d}y\, \tilde g_{N,\rho}(x,y)\,|f(x)|\,|f(y)|\,\mathbf{1}_{\mathcal{B}^c_\rho}(x,y)\\
&+\int\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,|f(x)|\,|f(y)|\,\mathbf{1}_{\mathcal{B}^c_\rho}(x,y)\,.
\end{aligned}$$

By Lemma 3.4 we control the first part in the above sum. The second part is simply estimated with Schwarz' inequality by

$$\langle f,f\rangle\Big(\int\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\mathbf{1}_{\mathcal{B}^c_\rho}(x,y)\Big)^{\frac12}\leqslant C\,N^{-\frac32}\,\langle f,f\rangle$$

where the last estimate follows from (3.14) and (2.15) provided B is sufficiently large.

The first term in (3.22) can be written as

$$-\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma_\rho^2 N}\, f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)$$
$$+\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\Big[Q_{N,\rho}(x,y)+\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma_\rho^2 N}\Big]\, f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)$$

A Scwharz inequality and Lemma 3.2 imply that the second term above is bounded by $C\,N^{-\frac{3}{2}}\,\langle f,f\rangle$. Since by assumption $\langle f,\xi_\rho\rangle=0$ we rewrite the first term above as

$$\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma_\rho^2 N}\, f(x)f(y)\mathbf{1}_{\mathcal{B}^c_\rho}(x,y)\,.$$

We estimate the absolute value of this expression by

$$\langle f,f\rangle\,\frac{1}{\sigma_\rho^2 N}\,\int \mathrm{d}x\, g_{N,\rho}(x)\xi_\rho(x)^2\,\mathbf{1}_{\{|\xi_\rho|\,\geqslant\,\frac{B}{2}\,\sigma_\rho\log N\}}(x)\,.$$

This last integral can be estimated using (3.14), (2.15) and the bound of Lemma 2.2:

$$\int \mathrm{d}x\, g_{N,\rho}(x)\xi_\rho(x)^2\,\mathbf{1}_{\{|\xi_\rho|\,\geqslant\,\frac{B}{2}\,\sigma_\rho\log N\}}(x)$$
$$\leqslant C\,\sqrt{m_{4,\rho}}\,\Big(\int \mathrm{d}x\, h^\rho(x-\rho)\,\mathbf{1}_{\{|\xi_\rho|\,\geqslant\,\frac{B}{2}\,\sigma_\rho\log N\}}(x)\Big)^{\frac{1}{2}}\leqslant C\,\sigma_\rho^2 N^{-\frac{1}{2}}\,.$$

We have obtained

$$\Big|\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\Big|\leqslant C\,N^{-\frac{3}{2}}\,\langle f,f\rangle\,.$$

This finishes the proof of Theorem 3.1.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the recursive inequality presented in the theorem below. Recall the definition (1.3) of the Poincaré constant $\gamma(N,\rho)$, and set

$$\gamma(N)=\sup_{\rho\in\mathbb{R}}\gamma(N,\rho)\,. \tag{4.1}$$

Theorem 4.1. *There exist constants $C<\infty$ and $N_0\in\mathbb{N}$ such that for every $N>N_0$*

$$\gamma(N)\leqslant\big[1+CN^{-\frac{3}{2}}\big]\,\gamma(N-1)\,. \tag{4.2}$$

Proof. Take an arbitrary real smooth function F on $\mathbb{R}^N$. For simplicity we drop all subscripts and simply write $\mathrm{Var}(F)$ for $\mathrm{Var}_{\nu_{N,\rho}}(F)$ and $\mathcal{E}(F)$ for $\mathcal{E}_{N,\rho}(F)$. Let $\mathcal{F}_k$ denote the $\sigma-$algebra generated by the one–site variables η_k, $k=1,\dots,,N$. $\mathrm{Var}(F\,|\,\mathcal{F}_k)$ denotes the $\mathcal{F}_k$-measurable random variable $\nu_{N,\rho}(F^2\,|\,\mathcal{F}_k)-\nu_{N,\rho}(F\,|\,\mathcal{F}_k)^2$. We use the notation

$$\mathcal{E}^{(k)}(F)=\sum_{i:\,i\neq k}\nu_{N,\rho}\big((\partial_i F)^2\big)\,.$$

Note that

$$\sum_{k=1}^N\mathcal{E}^{(k)}(F)=(N-1)\mathcal{E}(F)\,.$$

For any F one has the decomposition

$$\mathrm{Var}(F) = \frac{1}{N}\sum_{k=1}^{N} \nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big) + \frac{1}{N}\sum_{k=1}^{N} \mathrm{Var}\big(\nu(F\,|\,\mathcal{F}_k)\big)\,. \tag{4.3}$$

Observe that $\mathrm{Var}(F\,|\,\mathcal{F}_k)(\eta) = \mathrm{Var}_{\nu_{N-1,\rho_{\eta_k}}}(F)$, cf. (3.18). By definition (4.1), for each k we then have

$$\nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big) \leqslant \gamma(N-1)\mathcal{E}^{(k)}(F)\,.$$

Summing over k gives

$$\frac{1}{N}\sum_{k=1}^{N} \nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big) \leqslant \frac{N-1}{N}\,\gamma(N-1)\,\mathcal{E}(F)\,. \tag{4.4}$$

We turn to estimate the second term in (4.3). Here comes the idea of [11]. Namely assume without loss of generality that $\nu(F) = 0$ and write the quadratic form

$$\frac{1}{N}\sum_{k=1}^{N} \mathrm{Var}\big(\nu(F\,|\,\mathcal{F}_k)\big) = \nu\big(F\mathcal{P}F\big)$$

where the stochastic operator $\mathcal{P} : L^2(\nu) \to L^2(\nu)$ is defined by

$$\mathcal{P}F = \frac{1}{N}\sum_{k=1}^{N} \nu(F\,|\,\mathcal{F}_k)\,.$$

In this way (4.3) and (4.4) give

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) \leqslant \frac{N-1}{N}\,\gamma(N-1)\,\mathcal{E}(F)\,. \tag{4.5}$$

We need a spectral gap estimate for the generator $\mathbf{1}-\mathcal{P}$. We are going to prove

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) \geqslant \frac{N-1}{N}\big[1 - CN^{-\frac{3}{2}}\big]\,\nu\big(F^2\big)\,, \tag{4.6}$$

for all real $F \in L^2(\nu)$ such that $\nu(F) = 0$ with a uniform constant $C < \infty$ independent of the density ρ. Together with (4.5) this will complete the proof of the theorem.

Recalling the notation introduced in the previous section we define the closed subspace Γ of $L^2(\nu)$ consisting of sums of mean–zero functions of a single variable:

$$\Gamma = \Big\{ F \in L^2(\nu) :\ F = \sum_{k=1}^{N} f_k \circ \pi_k\,,\ \ f_1,\dots,f_N \in \mathcal{H}_0\,,\ \Big\} \tag{4.7}$$

Since $\mathcal{P}F \in \Gamma$ for every $F \in L^2(\nu)$ with $\nu(F) = 0$, we may restrict to $F \in \Gamma$ to prove (4.6). For $F \in \Gamma$, $F = \sum_k f_k \circ \pi_k$, we define $\Phi_F = \sum_k f_k$, a function in $\mathcal{H}_0$. Taking any real $F \in \Gamma$, a simple computation shows that

$$\nu\big(F^2\big) = \langle \Phi_F, \mathcal{K}\Phi_F\rangle + \sum_k \langle f_k, (\mathbf{1}-\mathcal{K}) f_k\rangle\,, \tag{4.8}$$

where $\mathcal{K}$ is the operator defined in (3.1). Similarly, for every k one computes

$$\nu\big(F\,\nu(F\,|\,\mathcal{F}_k)\big) = 2\langle \Phi_F, \mathcal{K}(\mathbf{1}-\mathcal{K}) f_k\rangle + \langle f_k, (\mathbf{1}-\mathcal{K})^2 f_k\rangle + \langle \Phi_F, \mathcal{K}^2\Phi_F\rangle\,.$$

Averaging over k and rearranging terms we then have

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) = \frac{N-2}{N}\,\langle \Phi_F, \mathcal{K}(\mathbf{1}-\mathcal{K})\Phi_F\rangle + \frac{1}{N}\sum_k \langle f_k, (\mathbf{1}-\mathcal{K})[(N-1)\mathbf{1}+\mathcal{K}]f_k\rangle\,. \tag{4.9}$$

Consider now the subspace $\mathcal{S}\subset\Gamma$ of symmetric functions:

$$\mathcal{S} = \Big\{F\in L^2(\nu):\; F=\sum_{k=1}^N f\circ\pi_k\,,\;\; f\in\mathcal{H}_0\Big\}\,. \tag{4.10}$$

Since $\mathcal{S}$ is invariant for $\mathcal{P}$, i.e. $\mathcal{P}\mathcal{S}\subset\mathcal{S}$ we may consider separately the cases $F\in\mathcal{S}$ and $F\in\mathcal{S}^\perp$, with $\mathcal{S}^\perp$ denoting the orthogonal complement in Γ. When $F\in\mathcal{S}$ we have $\Phi_F = Nf$ and rearranging terms in (4.8) and (4.9) we obtain

$$\nu\big(F^2\big) = N(N-1)\,\langle f, [\mathcal{K}+\frac{\mathbf{1}}{N-1}]f\rangle \tag{4.11}$$

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) = (N-1)^2\langle f, [\mathbf{1}-\mathcal{K}][\mathcal{K}+\frac{\mathbf{1}}{N-1}]f\rangle \tag{4.12}$$

By Theorem 3.1 we see that $\mathcal{K}+\frac{\mathbf{1}}{N-1}$ is non-negative on the whole subspace $\mathcal{H}_0$, for all N sufficiently large. Moreover by (3.2) and (4.11) we see that $\nu(F^2)=0$ when f is a multiple of ξ_ρ. We may then restrict to the case $\langle f,\xi_\rho\rangle = 0$. Writing $\tilde f = [\mathcal{K}+\frac{\mathbf{1}}{N-1}]^{\frac12}f$ and observing that $\langle\tilde f\rangle = 0$ and $\langle \tilde f,\xi_\rho\rangle=0$, Theorem 3.1 yields the estimate

$$\begin{aligned}\nu\big(F(\mathbf{1}-\mathcal{P})F\big) &\geqslant (N-1)^2\big[1-CN^{-\frac32}\big]\,\langle\tilde f,\tilde f\rangle\\ &= \frac{N-1}{N}\big[1-CN^{-\frac32}\big]\,\nu\big(F^2\big)\,,\qquad F\in\mathcal{S}\,.\end{aligned} \tag{4.13}$$

We turn to study the case $F\in\mathcal{S}^\perp$. Let us first observe that in the definition (4.7) of Γ one can assume without loss of generality that $\sum_k\langle f_k,\xi_\rho\rangle = 0$. Indeed if $c = (N\langle\xi_\rho,\xi_\rho\rangle)^{-1}\sum_k\langle f_k,\xi_\rho\rangle$ and $g_k = f_k - c\xi_\rho$, we have $\sum_k g_k\circ\pi_k = \sum_k f_k\circ\pi_k$ in $L^2(\nu)$ since by the conservation law $\sum_k\xi_\rho\circ\pi_k = 0$. Therefore $\langle\Phi_F,\xi_\rho\rangle = 0$ may be assumed from the start. Now, for every $G\in\mathcal{S}$, $G=\sum_k g\circ\pi_k$, with $g\in\mathcal{H}_0$ one has

$$\nu(FG) = (N-1)\,\langle\Phi_F, [\mathcal{K}+\frac{\mathbf{1}}{N-1}]g\rangle\,.$$

Thus $F\in\mathcal{S}^\perp$ implies that $[\mathcal{K}+\frac{\mathbf{1}}{N-1}]\Phi_F$ is a constant in $\mathcal{H}$. Since $\langle\Phi_F\rangle = 0$ and $\langle\Phi_F,\xi_\rho\rangle = 0$, Theorem 3.1 implies $\Phi_F = 0$. Writing $\hat f_k = (\mathbf{1}-\mathcal{K})^{\frac12}f_k$, then (4.8) and (4.9) imply

$$\nu\big(F^2\big) = \sum_k\langle\hat f_k,\hat f_k\rangle \tag{4.14}$$

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) = \frac{1}{N}\sum_k\langle\hat f_k, [(N-1)\mathbf{1}+\mathcal{K}]\hat f_k\rangle \tag{4.15}$$

Since $\langle\hat f_k\rangle = 0$ for all k we may use Theorem 3.1 to estimate

$$\langle\hat f_k,\mathcal{K}\hat f_k\rangle \geqslant -\frac{1}{N-1}\langle\hat f_k,\hat f_k\rangle\,.$$

From (4.14) and (4.15) we obtain
$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) \geqslant \frac{N-2}{N-1}\nu\big(F^2\big) = \frac{N-1}{N}\big[1-\frac{1}{(N-1)^2}\big]\nu\big(F^2\big)\,.$$
This ends the proof of the claim (4.6). □

Once we have Theorem 4.1 the conclusion of Theorem 1.1 is straightforward. Indeed,
$$\prod_{N=N_0+1}^{\infty} \big[1+CN^{-\frac{3}{2}})\big] \leqslant C' \tag{4.16}$$
for some uniform constant C' and Theorem 4.1 yields
$$\gamma(N) \leqslant C'\,\gamma(N_0)\,, \qquad N \geqslant N_0+1\,.$$
The uniform Poincaré inequality of Theorem 1.1 then follows from the fact that $\gamma(N_0)$ is indeed finite.

Lemma 4.2. *For every $N_0 \geqslant 2$*
$$\sup_{\rho\in\mathbb{R}} \gamma(N_0,\rho) < \infty\,. \tag{4.17}$$

Proof. Let $\tilde\nu_{N,\rho}$ denote the canonical measure obtained in (1.2) where the potential V is replaced by its convex component φ. Let also $\tilde\gamma(N,\rho)$ denote the corresponding Poincaré constant. Since $\varphi'' \geqslant \delta > 0$ one can use the Brascamp–Lieb inequality [6] to prove $\tilde\gamma(N,\rho) \leqslant \delta^{-1}$, uniformly in N and ρ, see also [8]. A standard argument (as in the proof of Lemma 2.2) on the other hand gives $\gamma(N,\rho) \leqslant e^{4N|\psi|_\infty}\tilde\gamma(N,\rho)$, for every $N\in\mathbb{N}$ and $\rho\in\mathbb{R}$. This gives, uniformly in ρ
$$\gamma(N,\rho) \leqslant \delta^{-1}e^{4N|\psi|_\infty}\,.$$
□

5. Ginzburg-Landau processes

We consider the discrete lattice $\mathbb{Z}^d$, with $d \geqslant 1$ an integer. Given a finite subset $\Lambda\subset\mathbb{Z}^d$, we denote by Λ^* the set of oriented bonds b contained in Λ, i.e. the couples $b=(x,y)$, $x,y\in\Lambda$ with $x=y+e$, e a unit vector in $\mathbb{Z}^d$. Denoting $\Lambda_L=\{1,2,\dots,L\}^d$, the $L-$hypercube in $\mathbb{Z}^d$, we define the product measure $\mu_{\Lambda_L,\rho}$ as the usual grand canonical measure $\mu_{N,\rho}$ with $N=L^d$. Then $\nu_{\Lambda_L,\rho}$ stands for the probability measure obtained from $\mu_{\Lambda_L,\rho}$ by conditioning on $L^{-d}\sum_{x\in\Lambda_L}\eta_x=\rho$. The Ginzburg-Landau dynamics is defined by the Dirichlet form
$$\mathcal{D}_{L,\rho}(F) = \frac{1}{2}\sum_{b\in\Lambda_L^*}\nu_{\Lambda_L,\rho}\big[(\nabla_b F)^2\big] \tag{5.1}$$
where we used the notation
$$\nabla_b F = \partial_y F - \partial_x F\,, \quad b=(x,y)\,.$$
The inverse of the spectral gap associated to $\mathcal{D}_{L,\rho}$ is given by
$$\chi(L,\rho) = \sup_F \frac{\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F)}{\mathcal{D}_{L,\rho}(F)} \tag{5.2}$$
with the supremum ranging over all real smooth functions on $\mathbb{R}^{\Lambda_L}$. As already observed in [8] we have a simple upper bound on $\chi(L,\rho)$ in terms of $\gamma(N,\rho)$ with $N=L^d$.

Lemma 5.1. *There exists a constant C only depending on d such that*

$$\chi(L,\rho) \leqslant C\, L^2\, \gamma(L^d,\rho) \tag{5.3}$$

Proof. We first make some observations about paths in Λ_L. We denote $\mathcal{C}_{xy}(L)$ the set of all paths γ_{xy} connecting sites $x,y\in\Lambda_L$, which use only bonds in Λ_L^*. The length of a path, denoted $|\gamma_{xy}|$ is the number of bonds composing it. Given $x,y\in\Lambda_L$ we need a rule to select a single path γ_{xy} from $\mathcal{C}_{xy}(L)$. We may choose γ_{xy} as follows. Fix $x,y\in\Lambda_L$ and define points $x^{(i)}$, $i=0,\dots,d$, such that $x^{(0)}=x$, $x^{(d)}=y$, and when $i=1,\dots,d-1$

$$x^{(i)}_j = \begin{cases} y_j & j=1,\dots,i \\ x_j & j=i+1,\dots,d \end{cases}$$

Call $\gamma^{(i)}$, $i=1,\dots,d$, the straight line parallel to the i-th axis joining sites $x^{(i-1)}$ and $x^{(i)}$. The path γ_{xy} is given by $\gamma^{(1)}\cup\dots\cup\gamma^{(d)}$. It is not difficult to prove the following properties: there exists a finite constant k only depending on d such that

- for every $x,y\in\Lambda_L$, $|\gamma_{xy}|\leqslant kL$, and
- for every $b\in\Lambda_L^*$, $\sum_{x,y\in\Lambda_L} 1_{\{b\in\gamma_{xy}\}}\leqslant kL^{d+1}$

When we write γ_{xy} below we always assume that this path has been chosen according to the above rule.

Given $\eta\in\mathbb{R}^{\Lambda_L}$, $y\in\Lambda_L$ we write $\eta^{(y)}$ for the configuration

$$\eta^{(y)}_x = \begin{cases} \eta_x & x\neq y \\ \rho L^d - \sum_{z\neq y}\eta_z & x=y \end{cases}.$$

For $F:\mathbb{R}^{\Lambda_L}\to\mathbb{R}$, we denote $F_y(\eta)$ the function $\eta\to F(\eta^{(y)})$. Clearly, for any $y\in\Lambda_L$ we have

$$\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F) = \mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F_y)\,. \tag{5.4}$$

It is then sufficient to show

$$L^{-d}\sum_{y\in\Lambda_L}\sum_{x\in\Lambda_L}\nu_{\Lambda_L,\rho}\big[(\partial_x F_y)^2\big] \leqslant C\, L^2\, \mathcal{D}_{L,\rho}(F)\,. \tag{5.5}$$

For any $x,y\in\Lambda_L$ we have

$$\partial_x F_y = \big(\partial_x F\big)_y - \big(\partial_y F\big)_y$$

and therefore

$$\nu_{\Lambda_L,\rho}\big[(\partial_x F_y)^2\big] = \nu_{\Lambda_L,\rho}\big[(\partial_x F-\partial_y F)^2\big]\,.$$

We write

$$\partial_x F - \partial_y F = \sum_{b\in\gamma_{yx}}\nabla_b F\,.$$

Since $|\gamma_{yx}|\leqslant kL$, Schwarz' inequality gives

$$\big(\partial_x F-\partial_y F\big)^2 \leqslant kL\sum_{b\in\Lambda_L^*}\big(\nabla_b F\big)^2 1_{\{b\in\gamma_{yx}\}}\,. \tag{5.6}$$

From the second property of our paths we see that (5.5) with $C=k^2$ follows from (5.6) when summing over x,y and dividing by L^d. ∎

Corollary 5.2. *Assume* $V = \varphi + \psi$ *with* $\varphi \in \Phi$ *and* $\psi \in \Psi$*. Then there exists* $C < \infty$ *such that for every* $\rho \in \mathbb{R}$ *and* $L \in \mathbb{N}$

$$\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F) \leqslant C\, L^2\, \mathcal{D}_{L,\rho}(F) \tag{5.7}$$

holds for every smooth function F *on* $\mathbb{R}^{\Lambda_L}$.

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