A NOTE ON WETTING TRANSITION FOR GRADIENT FIELDS

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Abstract. We prove existence of a wetting transition for two classes of gradient fields which include: 1) The Continuous SOS model in any dimension and 2) The Massless Gaussian model in dimension 2. Combined with a recent result proving the absence of such a transition for Gaussian models above 2 dimensions [5], this shows in particular that absolute-value and quadratic interactions can give rise to completely different behavior.

Keywords: Gradient models, entropic repulsion, pinning, wetting transition.

1. Introduction

In several recent papers [3, 9, 2, 1], the question has been raised whether the 2 dimensional massless Gaussian model exhibits a wetting transition. It is well-known that the Gaussian field in 2D is delocalized, with a logarithmically divergent variance, but that the introduction of an arbitrarily weak self-potential favoring height 0 is enough to localize it, in the sense that the variance remains finite [10, 2]; this result has recently been extended to a class of non Gaussian models in a stronger form, showing in particular existence of exponential moments for the heights [9] and exponential decay of covariances [11]. In higher dimensions, the field is already localized without pinning potential, but the introduction of such a potential turns the algebraic decay of the covariances into an exponential one. On the other hand, a Gaussian field with a positivity constraint ("surface above a hard-wall") exhibits entropic repulsion: The average height diverges like $\log N$ in 2D [8] and $\sqrt{\log N}$ in higher dimensions [4] (N being the linear size of the box). When both a positivity constraint and a pinning potential are present ("surface above an attractive hard wall"), there is a competition between these two effects. If there exists a (non zero) critical value for the strength of the pinning potential above which the interface is localized, but below which it is repelled by the wall, we say that the model exhibits a wetting transition. That such a transition occurs in a wide class of 1D model is well-known, see e.g. [6, 12, 5]. It was recently shown in [5] that the Gaussian model in dimensions 3 or higher does not display a wetting transition: The interface is always localized. The physically important case of the 2D model (describing a 2D interface in a 3D medium) remained however open.

In the present note, we prove that the 2D Gaussian field does exhibit a wetting transition; in fact the proof applies to any strictly convex interaction, see below. We also prove that the continuous SOS model has such a transition in *any* dimension, thus showing that the choice of the interaction can greatly affect the physics of the system. Our proofs are based on a variant of a beautiful and simple argument of Chalker [7], who proved the existence of a wetting transition in the discrete SOS model in dimension 2.

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2. Results

We consider a class of gradient models with single spin-space $\mathbb{R}^+$, i.e. modeling surfaces above a hard wall. Let Λ_N be the cube of side N centered at the origin, and $\Psi : \mathbb{R} \to \mathbb{R}$ an even function to be specified later; we consider the following Hamiltonian

$$H_N^{0,a,b}(\phi) = H_{0,N}^0(\phi) + V_N^{a,b}(\phi)\,,$$

where

$$H_{0,N}^0(\phi) = \sum_{\langle x,y\rangle \subset \Lambda_N} \Psi(\phi_x - \phi_y) \;+ \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda_N,\, y\notin\Lambda_N}} \Psi(\phi_x)\,,$$

$$V_N^{a,b}(\phi) = -b \sum_{x\in\Lambda_N} 1_{\{\phi_x \leqslant a\}}\,, \quad a,b > 0\,,$$

($\langle x,y\rangle$ denotes a pair of nearest neighbour sites). The corresponding Gibbs measure (on $(\mathbb{R}^+)^{\Lambda_N}$) is then given by

$$\mu_N^{0,+,a,b}(\,\mathrm{d}\phi) = \frac{e^{-H_N^{0,a,b}(\phi)}}{Z_N^{0,+,a,b}} \prod_{x\in\Lambda_N} \mathrm{d}\phi_x\,.$$

As in the pure pinning problem (i.e. without a wall) [9], the relevant parameter is $\varepsilon(a,b) = ae^b$ and not both a and b separately. As usual, we introduce the *δ-pinning* limit, which is the model described by the measure

$$\mu_N^{0,+,\varepsilon}(\,\mathrm{d}\phi) = \lim_{\substack{a\to 0\\ \varepsilon(a,b)=\varepsilon}} \mu_N^{0,+,a,b}(\,\mathrm{d}\phi) = \frac{e^{-H_{0,N}^0(\phi)}}{Z_N^{0,+,\varepsilon}} \prod_{x\in\Lambda_N} (\,\mathrm{d}\phi_x + \varepsilon\delta_0(\,\mathrm{d}\phi_x))\,.$$

Note that $\mu_N^{0,+,\varepsilon}$ can be written more explicitly as

$$\mu_N^{0,+,\varepsilon}(\,\mathrm{d}\phi) = \sum_{A\subset\Lambda_N} \varepsilon^{|A|} \frac{Z_{\Lambda_N\setminus A}^{0,+}}{Z_N^{0,+,\varepsilon}} \mu_{\Lambda_N\setminus A}^{0,+}(\,\mathrm{d}\phi)\,,$$

where

$$Z_{\Lambda_N\setminus A}^{0,+} = \int e^{-H_{0,N}^0(\phi)} \prod_{x\in\Lambda_N\setminus A} \mathrm{d}\phi_x \prod_{y\in A} \delta_0(\,\mathrm{d}\phi_y)\,,$$

and

$$\mu_{\Lambda_N\setminus A}^{0,+}(\,\mathrm{d}\phi) = (Z_{\Lambda_N\setminus A}^{0,+})^{-1}\, e^{-H_{0,N}^0(\phi)} \prod_{x\in\Lambda_N\setminus A} \mathrm{d}\phi_x \prod_{y\in A} \delta_0(\,\mathrm{d}\phi_y)$$

is the Gibbs measure on $\Lambda_N \setminus A$ with 0 boundary condition outside.

Remark: Here and everywhere else in this note, the integrals are restricted to the positive real axis, so we do not write this condition explicitly.

A quantity of interest is the density of *pinned sites*, i.e. of those sites where the interface feels the effect of the pinning potential; it is defined as

$$\rho_N(a,b) = |\Lambda_N|^{-1} \mu_N^{0,+,a,b}(\nu_N(\phi))\,,$$

where $\nu_N(\phi) = \sum_{x\in\Lambda_N} 1_{\{\phi_x \leqslant a\}}$. We also write $\rho(a,b) = \lim_{N\to\infty} \rho_N(a,b)$. The corresponding quantities in the δ-pinning limit are denoted $\rho_N(\varepsilon)$, $\rho(\varepsilon)$ (measuring the density of sites exactly at height 0). ρ will play the role of an order parameter for the wetting transition.

Our results can then be stated as follows.

Theorem 2.1. *Let $\Psi(x) = |x|$, $d \geqslant 1$. Suppose that $ae^b < (2d)^{-1}$, then there exist two constants $C_1(a,b,d)$ and $C_2(a,b,d)$ such that, for any $M > C_1\, N^{d-1}$,*

$$\mu_N^{0,+,a,b}(\nu_N(\phi) > M) \leqslant e^{-C_2\, M}\,.$$

In particular, $\rho(a,b) = 0$. The corresponding results also hold in the case of δ-pinning, provided $\varepsilon < (2d)^{-1}$.

Theorem 2.2. *Let $\Psi(x) = \frac{1}{2}x^2$, $d = 2$. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ there exist two constants C_3 and C_4 depending on ε such that, for any $M > C_3\, N^{d-1}$,*

$$\mu_N^{0,+,\varepsilon}(\nu_N(\phi) > M) \leqslant e^{-C_4\, M}\,.$$

In particular, $\rho(\varepsilon) = 0$. The same also holds in the case of the square-well potential provided ae^b is small enough.

We recall that it is not hard to prove that $\rho > 0$ when the pinning is strong. For example, in the δ-pinning case, we can proceed in the following way. Since

$$|\Lambda_N|^{-1} \log \frac{Z_N^{0,+,\varepsilon}}{Z_N^{0,+,0}} = \int_0^\varepsilon \frac{1}{\hat{\varepsilon}}\, \rho_N(\hat{\varepsilon})\ \mathrm{d}\hat{\varepsilon}\,,$$

the result follows from $Z_N^{0,+,\varepsilon} \geqslant \varepsilon^{|\Lambda_N|}$ and the existence of a constant C such that $Z_N^{0,+,0} \leqslant C^{|\Lambda_N|}$. To prove the latter inequality one can consider a shortest self-avoiding path ω on $\mathbb{Z}^d$ starting at some site of $\partial\Lambda_N$ and containing all the sites of Λ_N, and use $H_{0,N}^0(\phi) \geqslant \sum_{n=1}^{|\omega|-1} \Psi(\phi_{\omega_n} - \phi_{\omega_{n+1}})$.

The above results imply the existence of a wetting transition in these models[1]. Together with the result of [5] that in the Gaussian model in $d \geqslant 3$ there is *no* wetting transition, the first part of Theorem 2.2 shows a radical difference of behavior between the Gaussian and the SOS interactions.

Remark: 1. It is not difficult to see, looking at the proofs, that our theorems remain true if we replace the SOS interaction $\Psi(x) = |x|$ with any globally Lipschitz function (not necessarily symmetric), and the Gaussian interaction $\Psi(x) = \frac{1}{2}x^2$ with any even, convex Ψ such that $1/c \geqslant \Psi''(x) \geqslant c$ for some $c > 0$ and all x.

2. Even though the present work provides a proof of the wetting transition in the models considered, several important issues remain completely open. In particular, it would be most desirable to have a pathwise description of the field in both the localized and repelled regimes, i.e. a proof that $\rho > 0$ implies finiteness of the mean height of any fixed spin in the thermodynamic limit (if possible with estimate on the tail and exponential decay of correlations), and a proof that $\rho = 0$ implies that the mean height of any fixed spin diverges (if possible with estimates on the rate). Another question of physical interest would be to determine how the field delocalizes as the pinning strength decreases to its critical value.

[1] In the δ-pinning case, it can easily be seen that ρ is monotonous in ε, so that there is a single critical value.

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3. Proof of Theorem 2.1

We first consider the square-well potential. Let $M > 0$; following [7], we introduce the set $\mathcal{B}_M = \{\phi : \nu_N(\phi) \geqslant M\}$, and the set

$$\mathcal{C}_M = \{\phi : \sum_{x \in \Lambda_N} 1_{\{\phi_x \leqslant 2a\}} \geqslant M\}.$$

Since $\mathcal{B}_M \subset \mathcal{C}_M$, the first claim immediately follows from the estimate on conditional probabilities

$$\mu_N^{0,+,a,b}(\mathcal{B}_M | \mathcal{C}_M) \leqslant e^{-C_2 M}, \quad M > C_1 N^{d-1}. \tag{3.1}$$

Moreover, $\rho(a,b) = 0$ will follow from this and the obvious bound

$$\rho_N \leqslant \frac{M}{N^d} + \mu_N^{0,+,a,b}(\mathcal{B}_M),$$

by choosing M such that $N^d \gg M \gg N^{d-1}$.

We turn to the proof of (3.1). We define a map $\mathfrak{T}$ from $\mathcal{C}_M$ onto $\mathcal{B}_M$ by

$$(\mathfrak{T}\phi)_x = \begin{cases} \phi_x & \text{if } \phi_x \leqslant a, \\ \phi_x - a & \text{otherwise.} \end{cases}$$

If we write $e^{-V_N^{a,b}} = 1_{\{\phi_x > a\}} + e^b 1_{\{\phi_x \leqslant a\}}$ and expand the corresponding products, we have

$$\mu_N^{0,+,a,b}(\mathcal{C}_M) = \int \frac{e^{-H_N^{0,a,b}(\phi)}}{Z_N^{0,+,a,b}} 1_{\{\phi \in \mathcal{C}_M\}} \prod_{x \in \Lambda_N} \mathrm{d}\phi_x \tag{3.2}$$

$$= \sum_{\substack{A \subset \Lambda_N \\ |A| \geqslant M}} \sum_{B \subset A} e^{b|B|} \int \frac{e^{-H_{0,N}^0(\phi)}}{Z_N^{0,+,a,b}} \prod_{x \in B} 1_{\{\phi_x \leqslant a\}} \mathrm{d}\phi_x \prod_{y \in A \setminus B} 1_{\{a < \phi_y \leqslant 2a\}} \mathrm{d}\phi_y \prod_{z \in \Lambda_N \setminus A} 1_{\{2a < \phi_z\}} \mathrm{d}\phi_z.$$

Now, observe that

$$H_{0,N}^0(\phi) \leqslant H_{0,N}^0(\mathfrak{T}\phi) + da|\partial\Lambda_N| + 2da|B| \tag{3.3}$$

($\partial\Lambda_N$ being the set of $x \in \Lambda_N$ neighbouring a site $y \notin \Lambda_N$). After the change of variables $\widetilde{\phi}_x = (\mathfrak{T}\phi)_x$, we have

$$\mu_N^{0,+,a,b}(\mathcal{C}_M)$$

$$\geqslant e^{-da|\partial\Lambda_N|} \sum_{\substack{A \subset \Lambda_N \\ |A| \geqslant M}} \sum_{B \subset A} \left(e^{-2da} e^b\right)^{|B|} \int \frac{e^{-H_{0,N}^0(\widetilde{\phi})}}{Z_N^{0,+,a,b}} \prod_{x \in A} 1_{\{\widetilde{\phi}_x \leqslant a\}} \mathrm{d}\widetilde{\phi}_x \prod_{y \in \Lambda_N \setminus A} 1_{\{a < \widetilde{\phi}_y\}} \mathrm{d}\widetilde{\phi}_y$$

$$= e^{-da|\partial\Lambda_N|} \sum_{\substack{A \subset \Lambda_N \\ |A| \geqslant M}} e^{b|A|} \left(e^{-2da} + e^{-b}\right)^{|A|} \int \frac{e^{-H_{0,N}^0(\widetilde{\phi})}}{Z_N^{0,+,a,b}} \prod_{x \in A} 1_{\{\widetilde{\phi}_x \leqslant a\}} \mathrm{d}\widetilde{\phi}_x \prod_{y \in \Lambda_N \setminus A} 1_{\{a < \widetilde{\phi}_y\}} \mathrm{d}\widetilde{\phi}_y$$

$$\geqslant e^{-da|\partial\Lambda_N|} \left(e^{-2da} + e^{-b}\right)^M \mu_N^{0,+,a,b}(\mathcal{B}_M),$$

where we used $e^{-2da} + e^{-b} > 1$, which follows from $ae^b < (2d)^{-1}$. This proves (3.1).

Let us now consider the case of the δ-pinning potential. The proof is very similar. Let $M > 0$, and define $\mathcal{B}_M$ as in the previous case (but remember that now ν_N is the number of sites with height equal to 0). We also need a set analogous to the set $\mathcal{C}_M$ above: Let $\Delta \equiv (2d)^{-1} - \varepsilon$; we set

$$\mathcal{D}_M = \{\phi : \sum_{x\in\Lambda_N} 1_{\{\phi_x \leqslant \Delta\}} \geqslant M\}.$$

We are going to show that

$$\mu_N^{0,+,\varepsilon}(\mathcal{B}_M|\mathcal{D}_M) \leqslant e^{-C_4 M}, \quad M > C_3 N^{d-1}. \tag{3.4}$$

As above, (3.4) is sufficient to prove our claims.

To prove (3.4) define the map

$$(\mathfrak{S}\phi)_x = \begin{cases} \phi_x - \Delta & \text{if } \phi_x > \Delta \\ 0 & \text{otherwise} \end{cases}$$

from $\mathcal{D}_M$ onto $\mathcal{B}_M$. Note that $\mu_N^{0,+,\varepsilon}(\mathcal{D}_M)$ can be written

$$\sum_{\substack{A\subset\Lambda_N\\ |A|\geqslant M}} \sum_{B\subset A} \varepsilon^{|B|} \int \frac{e^{-H^0_{0,N}(\phi)}}{Z_N^{0,+,\varepsilon}} \prod_{x\in A\setminus B} 1_{\{\phi_x \leqslant \Delta\}} \,\mathrm{d}\phi_x \prod_{y\in\Lambda_N\setminus A} 1_{\{\Delta<\phi_y\}} \,\mathrm{d}\phi_y \prod_{z\in B} \delta_0(\,\mathrm{d}\phi_z). \tag{3.5}$$

We have

$$H^0_{0,N}(\phi) \leqslant H^0_{0,N}(\mathfrak{S}\phi) + \Delta d|\partial\Lambda_N| + \Delta 2d|A|,$$

and therefore, letting $\widetilde{\phi}_x = (\mathfrak{S}\phi)_x$ and integrating over the variables ϕ_x, $x \in A\setminus B$,

$$\begin{aligned}
\mu_N^{0,+,\varepsilon}(\mathcal{D}_M) &\geqslant e^{-\Delta d|\partial\Lambda_N|} \sum_{\substack{A\subset\Lambda_N\\ |A|\geqslant M}} \sum_{B\subset A} e^{-\Delta 2d|A|} \Delta^{|A|-|B|} \varepsilon^{|B|} \int \frac{e^{-H^0_{0,N}(\widetilde{\phi})}}{Z_N^{0,+,\varepsilon}} \prod_{x\in\Lambda_N\setminus A} \mathrm{d}\widetilde{\phi}_x \prod_{y\in A} \delta_0(\,\mathrm{d}\widetilde{\phi}_y) \\
&= e^{-\Delta d|\partial\Lambda_N|} \sum_{\substack{A\subset\Lambda_N\\ |A|\geqslant M}} \varepsilon^{|A|} e^{-\Delta 2d|A|} \left(1 + \frac{\Delta}{\varepsilon}\right)^{|A|} \frac{Z^{0,+}_{\Lambda_N\setminus A}}{Z_N^{0,+,\varepsilon}} \\
&\geqslant e^{-\Delta d|\partial\Lambda_N|} \big(e^{-\Delta 2d}(1 + \frac{\Delta}{\varepsilon})\big)^M \mu_N^{0,+,\varepsilon}(\mathcal{B}_M),
\end{aligned}$$

where we used $e^{-\Delta 2d}(1 + \frac{\Delta}{\varepsilon}) > 1$. This proves (3.4).

4. Proof of Theorem 2.2

We start with the δ-pinning potential. We proceed as in the corresponding proof of the previous section up to equation (3.5). To simplify notations, here we set $\Delta = 1$. Writing $W = A \cup {\Lambda_N}^c$, we have the estimate

$$H^0_{0,N}(\phi) \leqslant H^0_{0,N}(\mathfrak{S}\phi) + 2|\partial\Lambda_N| + 8|A| + 2 \sum_{x\in\partial W} \sum_{\substack{y\notin W\\ y\sim x}} (\mathfrak{S}\phi)_y. \tag{4.6}$$

Let us use the short-hand notation $\mathcal{X}(\phi) = 2\sum_{x\in\partial W}\sum_{y\notin W, y\sim x}\phi_y$. Inserting this estimate in (3.5) and changing variables to $\widetilde{\phi}_x = (\mathfrak{S}\phi)_x$, we obtain

$$\mu_N^{0,+,\varepsilon}(\mathcal{D}_M) \geqslant e^{-2|\partial\Lambda_N|} \sum_{\substack{A\subset\Lambda_N\\ |A|\geqslant M}} \varepsilon^{|A|}\,(e^{-8}(1+\frac{1}{\varepsilon}))^{|A|}\,\frac{Z^{0,+}_{\Lambda_N\setminus A}}{Z_N^{0,+,\varepsilon}}\,\mu^{0,+}_{\Lambda_N\setminus A}\left(e^{-\mathcal{X}(\phi)}\right).$$

Since Jensen's inequality implies that

$$\mu^{0,+}_{\Lambda_N\setminus A}\left(e^{-\mathcal{X}(\phi)}\right) \geqslant e^{-\mu^{0,+}_{\Lambda_N\setminus A}(\mathcal{X}(\phi))},$$

the conclusion will follow as before, once we prove that

$$\mu^{0,+}_{\Lambda_N\setminus A}(\mathcal{X}(\phi)) \leqslant c_1|\partial W| \leqslant c_1(|\partial\Lambda_N| + |A|),$$

with c_1 a finite independent constant. However, this follows immediately from the existence of the infinite-volume repulsed field pinned at the origin, which was established in [10] (Lemma 2.1). Notice that it is not the case in higher dimensions, and therefore the argument does not apply. In fact, as was proved in [5], there is *no* wetting transition in this case.

Let us finally discuss the square-well case. Again we adapt the proof given in the previous section. Consider the expression (3.2). Now the estimate (3.3) must be replaced by the analogous of (4.6) for the square well potential. In particular we are led to prove an upper bound for

$$\mu^{0,+}_{\Lambda_N}\left(\widetilde{\mathcal{X}}(\phi)\,\middle|\,\phi_x\leqslant a,\ \forall x\in A;\ \phi_y > a,\ \forall y\notin A\right),$$

where $\widetilde{\mathcal{X}}(\phi) = 2a\sum_{x\in\partial\widetilde{W}}\sum_{y\notin\widetilde{W}, y\sim x}\phi_y$, $\widetilde{W} = B\cup{\Lambda_N}^c$, for fixed $B\subset A\subset\Lambda_N$. However, FKG inequality implies that this expectation increases if one raises both the boundary conditions in ${\Lambda_N}^c$ and the conditioning in A to $\phi_x = a$, and modify the wall constraint to $\phi_y\geqslant a$ for all y. Thus, after a last change of variables, we get

$$\mu^{0,+}_{\Lambda_N}\left(\widetilde{\mathcal{X}}(\phi)\,\middle|\,\phi_x\leqslant a,\ \forall x\in A;\ \phi_y > a,\ \forall y\notin A\right) \leqslant \mu^{0,+}_{\Lambda_N\setminus A}\left(\widetilde{\mathcal{X}}(\phi+a)\right),$$

and therefore - using $|\partial\widetilde{W}| \leqslant |\partial\Lambda_N| + |B|$ - the conclusion (3.1) follows as above.

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