Lezione 10/05/19

(1) Esercizi [A] p. 161

- Trovare poli e residui di:

$\ast\ Res_{b_1} = \frac{1}{(1-1)!} \partial^{1-1} \left((z-b_1) f(z)\right)\Big|_{z=b_1} = \frac{1}{b_1 - b_2}$

(a) $\frac{1}{z^2+5z+6} = \frac{1}{(z+2)(z+3)}$

$b_1 = -2$, $b_2 = -3$ poli semplici $\rightarrow \left(\frac{1}{(z-b)} \cdot g(z)\right)$ con $g(z)$ analitica e $g(b) \neq 0$

$R_{b_1} = (b_1 - b_2)^{-1} \overset{*}{=} 1$

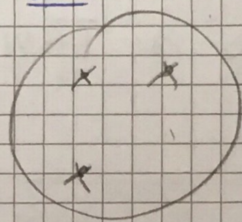
$R_{b_2} = (b_2 - b_1)^{-1} = -1$

OSS

Se f è una funzione razionale (meromorfa con numero finito di poli) e $|f| \leq \frac{C}{R^2}$ definitivamente

Allora $\sum Res\, f = 0$

dim $\quad \ast$ = polo b_j



$R \geq R_0 > |b_j|$, $\{b_j\}$ poli di f

per teo dei Residui

$\int_{|z|=R} f(z)\,dz := \sum_j Res_{b_j} f$

faccio una stima $\rightsquigarrow \left|\int_{|z|=R} f(z)\,dz\right| \leq \frac{C}{R^2} R \longrightarrow 0$

CVD

(b) $\frac{1}{(z^2-1)^2} = \frac{1}{(z+1)^2(z-1)^2}$ ha poli doppi

$b_1 = 1$ polo di ordine 2

$b_2 = -1$ polo di ordine 2

$\left(\frac{B_h}{(z-a)^h} + \dots + \frac{B_1}{z-a} + \dots\right)(z-a)^h$ voglio

$\parallel$

$B_h + \dots + B_1(z-a)^{h-1} + \dots$

$\rightarrow R_{b_1} = \partial (z-1)^2 \frac{1}{(z+1)^2(z-1)^2}\Big|_{z=1} = \partial \frac{1}{(z+1)^2}\Big|_{z=1} = -2\frac{1}{8} = -\frac{1}{4}$

$R_{b_2} \overset{OSS}{=} \frac{1}{4}$ (spesso è più saggio fare il calcolo)

(c) $f(z) = \frac{1}{\sin z}$ devo capire quali sono poli e come sono fatti

$\rightarrow$ studio gli zeri di $\sin z$. $\sin z = 0 \Leftarrow z = k\pi \quad k \in \mathbb{Z}$

$\left(e^z = e^{x+iy} = e^x e^{iy} \Rightarrow \Leftrightarrow e^{iy} = 0 \text{ mai}\right)$

$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} = 0$ sse $e^{iz} - e^{-iz} = 0$

$e^{2iz} = 1 = e^{2ix} e^{-2y} \Longrightarrow y = 0$ (prendendo il modulo)

$\rightsquigarrow e^{2ix} = 1 \Leftrightarrow \cos 2x = 1$, $\cos 2x = 0$ sse $x = k\pi$

gli zeri di $\sin z$ sono tutti e soli i $k\pi$ $k \in \mathbb{Z}$
e sono zeri semplici

$$\sin z \underset{\uparrow}{=} \sin(z - k\pi + k\pi) = \cos k\pi \cdot \sin(z - k\pi) = (-1)^k \sin(z - k\pi)$$

NOTA

teorema di addizione vale in $\mathbb{C}$ per il principio di identità (anche dalla def)

$\left(\forall f \text{ analitica in } \Omega \text{ (aperto connesso)} \right)$ se $z_k \in \Omega \longmapsto z_0 \in \Omega$ e $f(z_k) \equiv 0$ e z_k distinti $\Longrightarrow f \equiv 0$ su Ω

dim se $f \equiv 0$ su curva γ non banale $\longrightarrow f \equiv 0$

$f(z) := \sin(z+w) - (\sin z \cos w + \cos z \sin w)$ w fissato in $\mathbb{R}$

$f(z) = 0$ se $z \in \mathbb{R}$ (formula di addizione del seno in $\mathbb{R}$)

$\rightsquigarrow f \equiv 0$ su $\mathbb{C}$

Ripeto l'argomento su $F(w) := \sin(z+w) - (\sin z \cos w + \cos z \sin w)$ con z fissato in $\mathbb{R}$

CVD

$$\frac{1}{\sin z}(z - k\pi) = (-1)^k \frac{(z-k\pi)}{\sin(z-k\pi)} \longmapsto (-1)^k = R_{k\pi}$$

$$\left(\frac{\sin z}{z} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j+1)!} \quad \text{è intera e in } z=0 \text{ vale } 1. \right)$$

(2) Esercizio [A] p. 161

(a) $\displaystyle\int_0^{\pi} \frac{dx}{a + \sin^2 x}$ con $|a| > 1$

$$\text{pari} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{a + \sin^2 x} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{a + \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2} = \frac{1}{2i}\int_{|z|=1} \frac{1}{z} \frac{dz}{a - \frac{1}{4}\left(z - \frac{1}{z}\right)^2} =$$

$|z| = 1$ $e^{ix} = z$ $dz = ie^{ix}dx \rightsquigarrow dx = iz\, dz$ (su $|z|=1$ $z = e^{ix}$)

$$= -\frac{2}{i}\int_{|z|=1} \frac{z}{z^4 - (4a+2)z^2 + 1} dz = -\frac{2}{i} 2\pi i \sum_{|z|<1} \operatorname{Res} f = *$$

Cerco i poli di f:

$$z^4 - (4a+2)z^2 + 1 = 0 \iff z_{\pm}^2 = z_{\pm}^2 = 2a + 1 \pm \sqrt{2a - 1}$$

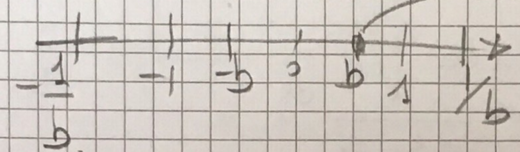
$$\int_0^{2\pi} R(\cos\theta \sin\theta)\, d\theta \qquad \left.\begin{array}{l} \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \\ \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right) \\ d\theta = \frac{dz}{iz} \end{array}\right\} |z|=1$$

$\rightsquigarrow z_{\pm} = \pm\sqrt{2a+1 \pm \sqrt{2a-1}}$ devo andare a pescare le soluzioni nel cerchio

se $a>1 \to$ (le soluzioni sono coniugate $\rightsquigarrow$ una interna e una ext)

se $|a|>1 \to$

$+\sqrt{2a+1-\sqrt{(2a-1)^2-1}}$



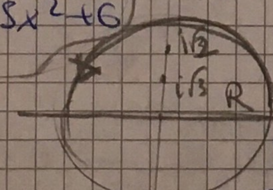
ho 4 poli semplici

$\stackrel{*}{=} -4\pi\left(\mathrm{Res}_b f + \mathrm{Res}_{-b} f\right)$ da finire

$$\mathrm{Res}_b f = \left.\frac{b}{(z+b)\left(z+\frac{1}{b}\right)\left(z-\frac{1}{b}\right)}\right|_{z=b} = \frac{1}{2}\frac{b^2}{(b^4-1)}$$

(b) $\displaystyle\int_0^{\infty} \frac{x^2}{x^4+5x^2+6}\, dx$

$$\underset{\text{pari}}{=} \frac{1}{2}\int_{-\infty}^{+\infty} \underbrace{\frac{x^2}{x^4+5x^2+6}}_{f(x)}\, dx$$



$\displaystyle\int_{|z|=R} f(z)\, dz = 0 = 2\pi i \sum_{\text{poli}} \mathrm{Res}_z f$

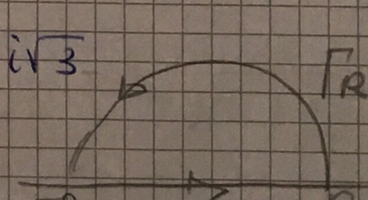
$R > R_0$

studio i poli

$x^4+5x^2+6=0$

$(x^2+2)(x^2+3)=0 \iff z=\pm i\sqrt{2} \quad z=\pm i\sqrt{3}$

$=(z-i\sqrt{2})(z+i\sqrt{2})(z+i\sqrt{3})(z-i\sqrt{3})$



prendo il semicerchio sup.

$$\int f(z)\, dz = 0 = 2\pi i \sum_{\text{poli } y>0} \mathrm{Res}_z f = \left(\mathrm{Res}_{i\sqrt{2}}(f) + \mathrm{Res}_{i\sqrt{3}}(f)\right) 2\pi i$$

$\displaystyle\int_0^{\infty} f(x)\, dx \xrightarrow{R\to\infty} \Gamma_R$

$$\rightsquigarrow \int_0^{\infty} f(x)\, dx = \pi i \left(R_{i\sqrt{2}} f + R_{i\sqrt{2}} f\right)$$

e $\quad R_{i\sqrt{2}} = -\dfrac{(-2)}{2i\sqrt{2}\cdot(\sqrt{2}-\sqrt{3})(\sqrt{2}+3)} = \dfrac{i}{\sqrt{2}}$

$R_{i\sqrt{3}} = \dfrac{(-3)\,i}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+2)\,2\sqrt{3}} = -\dfrac{\sqrt{3}}{2}\,i$

$\rightsquigarrow \displaystyle\int_0^{\infty} f(x)\,dx = \pi i\left(\frac{i}{\sqrt{2}} - \frac{\sqrt{3}}{2}i\right) = \pi\,\frac{\sqrt{3}-\sqrt{2}}{2}$

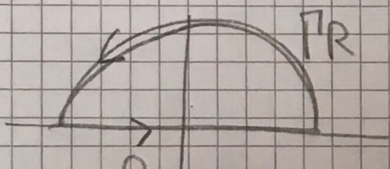
(e) $\displaystyle\int_0^{\infty} \frac{\cos x}{a^2+x^2}\,dx$

pari $\displaystyle = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos x}{a^2+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{ix}}{a^2+x^2}\,dx$

$f(x)\in L^1,\quad \displaystyle\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,e^{-itx}\,dx$

$f(x) = \dfrac{1}{a^2+x^2} \in L^1 \qquad \displaystyle\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{-ixt}}{x^2+\pi^2}\,dx$

$|f| \in R^* \subseteq L^1$



Γ_R

$t>0$

$xt=\xi$

$\displaystyle = \frac{1}{2\pi}\, t \int_{\mathbb{R}} \frac{e^{-i\xi}}{\xi^2+ta^2}\,d\xi$

si fa come prima

$\displaystyle = t\int_{\mathbb{R}} \frac{e^{ix}}{x^2+t^2a^2}\,dx$

$\Gamma_R = Re^{it} \quad t\in(0,\pi)$

$\displaystyle \overset{*}{=} i\int_0^{\pi} \frac{e^{iRe^{it}}}{a^2+R^2e^{2it}}\,Re^{it}\,dt \xrightarrow{R\to+\infty} 0$

$\left|e^{iRe^{it}}\right| \le e^{iR(\cos t + i\sin t)} = e^{-R\sin t} \le 1$

$\left|\dfrac{1}{a^2+R^2e^{it}}\right| = \dfrac{1}{R^2}\,\dfrac{1}{\left|e^{it}+\frac{a^2}{R^2}\right|} \le \dfrac{1}{R^2}\,\dfrac{1}{\left(1-O\left(\frac{1}{R^2}\right)\right)}$

$\rightsquigarrow \overset{*}{=} 2\pi i\,\mathrm{Res}_{ia} f = 2\pi i\,\dfrac{e^{-a}}{2ia} = \pi\,\dfrac{e^{-a}}{a}$