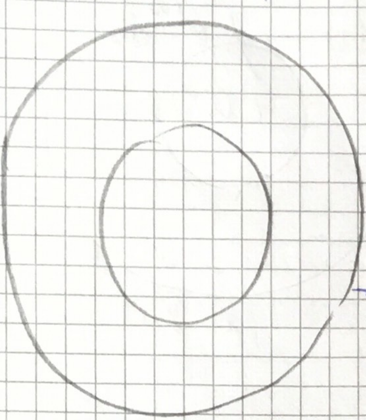
Lezione 15/05/19

dim di thm p.103



f analitica in $\Omega := \{r < |z-z_0| < R\}$

$r < \rho < |z-z_0| < \rho_1 < R$

$\Omega' = \{\rho < |z-z_0| < \rho_1\}$

dalla formula di Cauchy su Ω' $(C_{\rho_1} \sim C_\rho \text{ mod } \Omega)$

$$\to f(z) = \frac{1}{2\pi i}\int_{C_{\rho_1}-C_\rho} \frac{f(\xi)}{\xi - z}\, d\xi =$$

$$= \underbrace{\frac{1}{2\pi i}\int_{C_{\rho_1}} \frac{f(\xi)}{\xi - z}\, d\xi}_{f_1} - \underbrace{\frac{1}{2\pi i}\int_{C_{\rho}} \frac{f(\xi)}{\xi - z}\, dz}_{f_2} =$$

$$= f_1(z) + f_2(z) \quad \text{con } (*) \begin{cases} f_1 \text{ analitica in } |z-z_0| < R \\ f_2 \text{ analitica in } |z-z_0| < R \end{cases}$$

(la decomposizione) la rappresentazione di f come $f_1 + f_2$ con $(*)$ e $\lim_{z \to +\infty} f_2 = 0$ è UNICA:

Infatti: $a_n = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \quad \forall n \in \mathbb{Z}$

$$f_1 = \sum_{n \geq 0} a_n (z-z_0)^n \quad \text{e} \quad f_2 = \sum_{n < 0} a_n (z-z_0)^n$$

NB

a_n non dipende da $\rho \in (r, R)$ per il thm di Cauchy

$(C_\rho \sim C_{\rho'} \text{ mod } \Omega \ \forall \rho, \rho' \in (r, R)$

$\Leftrightarrow C_\rho - C_{\rho'} \sim 0 \text{ mod } \Omega$ e $\xi \longmapsto \frac{f(\xi)}{(\xi - z_0)^{n+1}}$ è analitica in Ω $\forall n \in \mathbb{Z}$

$$\to a_n \underset{\xi = z_0 + \rho e^{it}}{=} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{(\rho e^{it})^{n+1}} \rho e^{it} i\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho^n e^{int}}\, dt \quad (*')$$

$\leadsto f_1(z) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(\xi)}{\xi - z}\, d\xi$ f_1 è analitica in $|z-z_0| < R$

e $f_1 = \sum_{n \geq 0} a_n (z-z_0)^n$ con $a_n = \frac{f_1^{(n)}(z)}{n!} = (*')$

⇝ $f_2 = -\frac{1}{2\pi i}\int_{C_\rho} \frac{f(\xi)}{\xi - z}\, d\xi \underset{z = z_0 + \frac{1}{w}}{=} -\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{it})}{\rho e^{it} - \frac{1}{w}}\, i\rho e^{it}\, dt =$

$w = (z-z_0)^{-1}$

$|z-z_0|>\rho \iff |w| > \rho^{-1}$

$$= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+\rho e^{it})}{w\rho e^{it}-1}\, w\rho e^{it}\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it}) \frac{w\rho e^{it}}{1-w\rho e^{it}}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it}) \sum_{m=1}^{\infty} (w\rho e^{it})^m = \sum_{m=1}^{\infty} w^m \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it})(\rho e^{it})^m\, dt =$$

converge assolutamente ⇝ porto la somma fuori

$n=-m$

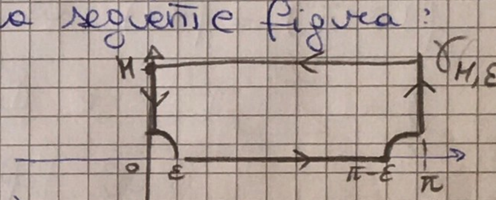
$$= \sum_{n<0} (z - z_0)^n \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\frac{f(z_0+\rho e^{it})}{\rho^n e^{int}}\, dt}_{a_n}$$

cvd.

Esercizi [Integrali]

(1) $\int_0^{\infty} \log(\sin\theta)\, d\theta$

Suggerimenti:
- $f(z) := \log(1-e^{2iz})$
- la seguente figura:

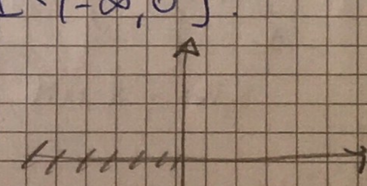


$1-e^{2iz} = -2ie^{iz}\sin z$

$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ ⇝ $f(z) = \log(-2ie^{iz}\sin z)$

È possibile definire il ramo analitico di $\log z$ su $\mathbb{C}\setminus(-\infty,0]$.
Ad esempio il ramo principale in cui $\log 1 = 0$

$\iff \log z = \log|z| + i\,\mathrm{Arg}(z)$ con $\mathrm{Arg}\, z \in (-\pi,\pi)$



⇝ voglio capire dove $1-e^{2iz}$ è negativo

$1-e^{2iz} \underset{z=x+iy}{=} 1-e^{2ix}e^{-2y} = 1-e^{-2y}(\cos 2x + i\sin 2x) \in \mathbb{R} \iff \sin 2x = 0$

$\iff x = \frac{k\pi}{2},\ k\in\mathbb{Z}$ ⇝ $\cos k\pi = (-1)^k$ → $1-e^{-2y}(-1)^k<0$ voglio k pari

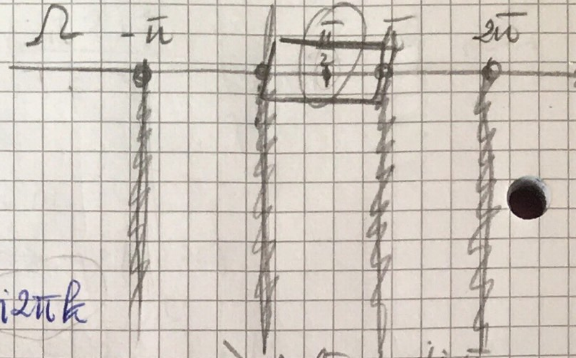
$\to 1-e^{-2y}(-1)^k<0 \iff \begin{cases} x = k\pi & k\in\mathbb{Z}\\ y<0 \end{cases}$

⟹ $f(z)$ è analitica in $\mathbb{C}\setminus\bigcup_{k\in\mathbb{Z}}\{k\pi+iy \mid y\geq 0\} =: \Omega$
(con $f(\frac{\pi}{2}) = \log 2$)

OSS

$$f(z) = \mathrm{Log}\left(2e^{-i\frac{\pi}{2}} e^{iz} \sin z\right) \overset{?}{=} \dots$$

è analitica in Ω. se mi metto in un intorno di $\frac{\pi}{2}$



$$\dots = \log 2 + \log e^{-i\frac{\pi}{2}} + \log e^{iz} + \log(\sin z) + i2\pi k$$

$$\overset{?}{=} \mathrm{Log}\,2 - i\frac{\pi}{2} + iz + \mathrm{Log}(\sin z) + i2\pi k$$

non può saltare di π se $x \in (0,\pi)$, $|y| < \varepsilon$ $(z = x+iy)$

stanno vicine nel dischetto

se trovo un k per cui sono uguali $\rightsquigarrow$ sono uguali dove mi interessa (p.d.I)

$\mathrm{Log}\left(2e^{-i\frac{\pi}{2}} e^{iz} \sin z\right)$ | $\mathrm{Log}\,2 - i\frac{\pi}{2} + iz + \mathrm{Log}\,\sin z + i2\pi k$

se $z = \frac{\pi}{2}$ vale $\mathrm{Log}\,2$ | $z = \frac{\pi}{2}$, $k = 0$ — ramo principale $\Rightarrow$ $k = 0$

$$\Rightarrow f(z) = \mathrm{Log}\,2 - i\frac{\pi}{2} + iz + \mathrm{Log}\,\sin z$$ ramo principale

$f = \mathrm{Log}$ in $\{x \in (0,\pi), |y| < \varepsilon\}$

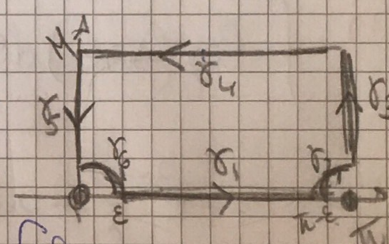
Allora: $f(z) = \mathrm{Log}(1 - e^{2iz})$

$f\left(\frac{\pi}{2}\right) = \mathrm{Log}\,2$

$$f(z) = \mathrm{Log}\,2 - i\frac{\pi}{2} + iz + \mathrm{Log}\,\sin z \quad \text{in } \{x \in (0,\pi), y \in \mathbb{R}\}$$

ora, usando il 2° suggerimento:

$f(z+\pi) = f(z) \rightsquigarrow f(z)$ è periodica di π.



$\gamma = \sum_1^6 \gamma_i$ e $\gamma_3 = \pi - \gamma_5$

periodicità

$$\rightsquigarrow \int_{\gamma_3} f = \int_{\pi-\gamma_5} f = \int_{-\gamma_5} f \rightarrow \int_{\gamma_3} f + \int_{\gamma_5} f = 0$$

$$\int_{\gamma_4} f = -\int_0^\pi \mathrm{Log}\left(1 - e^{-2M} e^{2it}\right) dt \xrightarrow[M \to +\infty]{} 0$$ uniformemente in $t \in (0, \pi/2)$

altrimenti non posso fare il limite dentro

$-\gamma_4(t) = t + iM$, $0 \le t \le \pi$

$$\left| -\int_{-\gamma_4} f \right| \le \int_0^\pi \left| \mathrm{Log}\left(1 - e^{-2M} e^{2it}\right) \right| dt$$ l'argomento è sempre limitato

$$\mathrm{Log}(1+z) \xrightarrow[z \to 0]{} 0 \qquad \forall \varepsilon > 0\ \exists \delta \text{ t.c. } |z| < \delta$$

$$\Rightarrow |\mathrm{Log}(1+z)| < \frac{\varepsilon}{\pi}$$

$\int_{\gamma_6} f\,dz = \dots$

$-\gamma_6 = \varepsilon e^{it}$, $t \in \left[0, \frac{\pi}{2}\right]$

$dz = \varepsilon i e^{it}$

$$\dots = -\int_0^{\pi/2} f(\varepsilon e^{it})\, \varepsilon i e^{it}\, dt \xrightarrow{\varepsilon \to 0} 0$$ esercizio

$$\mathrm{Log}\left(-2ie^{i\varepsilon} \sin \varepsilon\right) = \mathrm{Log}\left(2|e^{i\varepsilon}||\sin \varepsilon|\right) + i\,O(1)$$
$$= \mathrm{Log}\,\varepsilon\, O(1) + i\,O(1)$$

Lo stesso argomento vale per γ_2 ⇝ posso applicare ora il (thm dei Residui) thm di Cauchy
NO

⇝ $\displaystyle\int_{\gamma_{\varepsilon,M}} f(z)\,dz = 0$

$\downarrow \begin{array}{l}\varepsilon \to 0\\ M \to +\infty\end{array}$

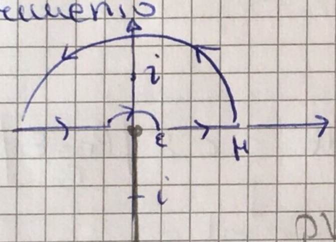
$$\int_0^{\pi} f(x)\,dx \overset{\text{ma è}}{=} \int_0^{\pi}\left(\log 2 - i\frac{\pi}{2} + ix + \log(\sin x)\right)dx =$$

$$= \pi\log 2 - i\frac{\pi^2}{2} + i\frac{\pi^2}{2} + \int_0^{\pi}\log(\sin x)\,dx$$

⇝ $\displaystyle\int_0^{\pi}\log(\sin x)\,dx = -\pi\log 2.$

(2) esercizio Calcolare $\displaystyle\int_0^{\infty}\underbrace{\frac{\log x}{x^2+1}}\,dx$

suggerimento



è analitica in $\mathbb{C}\setminus(i[-\infty,0])$
e $\log(1)=0$
il ramo principale
prendo $\operatorname{Arg} z \in \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right)$

(3) esercizio Calcolare $\displaystyle\int_{-\infty}^{+\infty}\frac{\cos\omega t}{(\operatorname{ch} t)^2}\,dt \qquad \omega\in\mathbb{R}.$

sugg:
se $\omega = 0 \longrightarrow \displaystyle\int_{-\infty}^{+\infty}\frac{1}{(\cosh t)^2} = 2\int_0^{\infty}(\operatorname{th} t)'\,dt = 2$ ($\operatorname{th}\infty = 1$)

se ω grande $\cos\omega t$ oscilla molto $\sim e^{-\frac{\omega\pi}{2}}\pi\omega$ (↑ si comporta come) $\omega\to\infty$