Lezione 17/05/19

Esercizi: Integrali con thm dei Residui

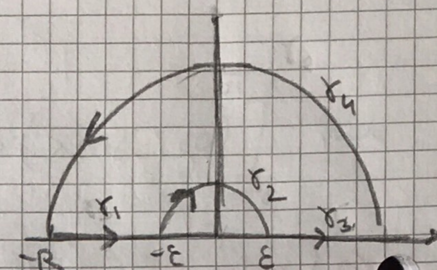
(1) $\displaystyle\int_0^{\infty} \frac{\log x}{1+x^2}\,dx$

$f(z) = \dfrac{\log z}{1+z^2}$ f ha un polo semplice in $z = \pm i$

prendo come ramo (principale) del logaritmo quello con $\operatorname{Im}(\log z) \in \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right)$

$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. $\gamma_4 = Re^{it}$ $t \in [0,\pi]$

$\gamma_2 = \varepsilon e^{it}$ $t \in [0,\pi]$



- sul cerchio γ_4 $\displaystyle\left|\int_{\{|z|=R,\ \operatorname{Arg} z \in (0,\pi)\}} f(z)\,dz\right| \leq \frac{\log R}{1+R^2} R \xrightarrow[R\to+\infty]{} 0$

- γ_2 sul cerchietto $\displaystyle\int_{\{z=\varepsilon e^{it},\ t\in(0,\pi)\}} \frac{(|\log \varepsilon| + \pi)\,\varepsilon}{1-\varepsilon^2}\,dt \underset{\varepsilon \leq \frac{1}{2}}{\leq} c\,\varepsilon\,|\log \varepsilon| \xrightarrow{\varepsilon \searrow 0} 0$

- $\displaystyle\int_{-\infty}^{0} \frac{\log|x| + i\pi}{1+x^2}\,dx + \int_0^{\infty} \frac{\log x}{1+x^2} = 2\int_0^{\infty} \frac{\log x}{1+x^2}\,dx + i\frac{\pi^2}{2}$

$\rightsquigarrow$ $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_i(f) = 2\pi i \frac{\pi}{4} = i\frac{\pi^2}{2}$

$\downarrow \varepsilon\to 0,\ R\to+\infty$

$\displaystyle i\frac{\pi^2}{2} + 2\int_0^{\infty} \frac{\log x}{1+x^2}\,dx \implies \int_0^{\infty} \frac{\log x}{1+x^2}\,dx = 0$

(2) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos\omega t}{(\operatorname{ch} t)^2}\,dt = \frac{1}{2}\int_0^{\infty} \frac{\cos\omega t}{(\operatorname{ch} t)^2}\,dt \quad \omega \in \mathbb{R}$

se $\omega = 0$ $\displaystyle\int_0^{\infty} \frac{1}{\operatorname{ch}^2 t}\,dt = 1$

se $\omega \neq 0$ $\displaystyle \varphi(\omega) := \int_{-\infty}^{\infty} \frac{\cos\omega t}{\operatorname{ch}^2 t}\,dt \quad \varphi \in C^{\infty}(\mathbb{R})$

NO.

infatti $\displaystyle\varphi^{(k)}(\omega) = \int_{-\infty}^{\infty} \frac{(\cos\omega t)^{(k)}}{\operatorname{ch}^2 t}\,dt$

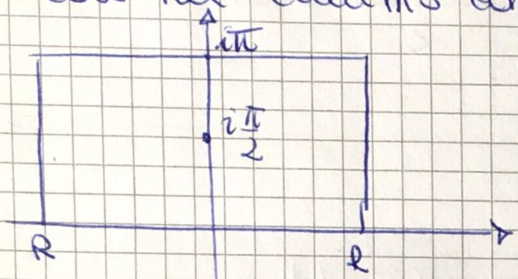
prendo $f(z) = \frac{\cos \omega t}{ch^2 t}$

$$ch\, z = \frac{e^{z} + e^{-z}}{2} = \frac{e^{iiz} + e^{-iiz}}{2} = \cos iz$$

NB $\boxed{ch\, z = \cos(iz)}$

$\rightsquigarrow ch(z) = 0 \iff iz = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z} \iff z = \frac{i\pi}{2} + ik\pi \quad k \in \mathbb{Z}$

così nel dominio considerato ho solo un polo doppio



calcolo il $\operatorname{Res}_{\frac{i\pi}{2}} f = \lim_{z \to \frac{i\pi}{2}} D\left[\left(z - \frac{i\pi}{2}\right)^2 f(z)\right] =$

cambio di variabile, traslo in $\frac{\pi i}{2}$

$$= \lim_{\xi \to 0} \partial_\xi \left(f\left(i\frac{\pi}{2} + \xi\right) \xi^2 \right)$$

$$\xi^2 f\left(\frac{i\pi}{2} + \xi\right) = \frac{\xi^2 \cos\left[\left(\frac{i\pi}{2} + \xi\right)\omega\right]}{ch^2\left(\frac{i\pi}{2} + \xi\right)} = \xi^2 \frac{\cos\left(\frac{i\omega\pi}{2}\right)\cos \omega\xi - \sin\left(\frac{i\omega\pi}{2}\right)\sin \omega\xi}{ch^2\left(i\frac{\pi}{2} + \xi\right)}$$

$ch^2\left(i\frac{\pi}{2} + \xi\right) = \cos^2\left(\frac{\pi}{2} - i\xi\right) = \sin^2 i\xi$

$$= \xi^2 \frac{1}{\sin^2(\xi i)} \left[\left(\cos i\omega\frac{\pi}{2}\right)\cos \omega\xi - \left(\sin i\omega\frac{\pi}{2}\right)\sin \omega\xi \right]$$

$\sin i\xi = \frac{e^{-\xi} - e^{+\xi}}{2i} = i\, sh\, \xi$

$$= -\frac{\xi^2}{sh^2 \xi} \left[\left(\cos i\omega\frac{\pi}{2}\right)\cos \omega\xi - \left(\sin i\omega\frac{\pi}{2}\right)\sin \omega\xi \right]$$

$$= \frac{1}{g(\xi)} [\,\ldots\,]$$

con $g(\xi) = \left(\frac{sh\, \xi}{\xi}\right)^2 = 1$ in $\xi = 0$

$\frac{sh\, \xi}{\xi} = 1 + \frac{\xi^2}{6} + O(\xi^5)$

$g(\xi) = 1 + \frac{\xi^2}{3} + O(\xi^5)$

$\rightsquigarrow g(0) = 1$, $g'(0) = 0$

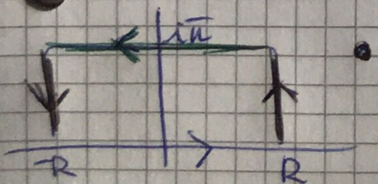
$\Rightarrow$ faccio solo le derivate di questo

$$\operatorname{Res}_{\frac{i\pi}{2}}(f) = -1\left(\ldots\right)'\Big|_0$$

$$= -\left(-\omega \sin\left(i\omega\frac{\pi}{2}\right)\right) = i\omega \sinh \omega\frac{\pi}{2}$$

$$\rightsquigarrow \int_{\gamma_R} f(z) = 2\pi i \operatorname{Res}_{\frac{i\pi}{2}}(f) = -2\pi\omega \sinh \frac{\pi\omega}{2}$$

Ora studio f su γ_R



$$\left| \int_{\{\pm R + it,\ t \in (0,\pi)\}} f(z)\,dz \right| = \left| \int_0^{\pi} \frac{|\cos(\omega R + it\omega)|}{|\cosh(\pm R + it)|^2} dt \right| \leq \ldots$$

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il numeratore è limitato

$$\left|\frac{e^{i\omega(\pm R+it)}+e^{-i\omega(\pm R+it)}}{2}\right| \leq ch(\omega t) \leq ch(\omega\pi)$$

il numeratore :

$$|ch(\pm R+it)| = \left|\frac{e^{\pm R}e^{it}+e^{\mp R}e^{it}}{2}\right| \geq \frac{e^{R}-e^{-R}}{2} = sh\,R$$

$$\dots \leq \frac{\pi\, ch\,\omega\pi}{sh^2 R} \xrightarrow{R\to\infty} 0$$

$$-\int_{-R}^{R}\frac{\cos\omega(i\pi+t)}{\cosh^2(i\pi+t)}dt \xrightarrow{R\to+\infty} \int_{-\infty}^{+\infty}\frac{\cos\omega(i\pi+t)}{ch^2(i\pi+t)}dt =$$

$t+i\pi$

$-R\leq t\leq R$

$$= -\int_{-\infty}^{+\infty}\frac{\cos(i\omega\pi)\cos(\omega t)-\cancel{\sin(i\pi\omega)\sin\omega t}}{(-\cosh t)^2}$$

dispari

$ch\, iz$
$=$
$\cos z$

$$= -\int_{-\infty}^{+\infty}\frac{\cos i\omega\pi\cos\omega t}{\cosh^2 t}dt$$

$$= -ch\,\pi\omega\int_{-\infty}^{+\infty}\frac{\cos\omega t}{ch^2 t}dt$$

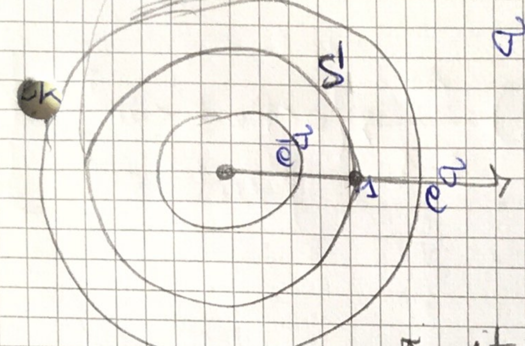
$$\longrightarrow \underbrace{\int_{\gamma_R} f}_{= -2\pi\omega\, sh\frac{\omega\pi}{2}} \longrightarrow (1-ch\,\omega\pi)\int_{-\infty}^{+\infty}\frac{\cos\omega t}{ch^2 t}dt$$

$$\longrightarrow \int_{-\infty}^{+\infty}\frac{\cos\omega t}{ch^2 t}dt = 2\pi\omega\frac{sh\frac{\pi\omega}{2}}{ch\,\omega t-1} = \frac{\pi\omega}{sh\left(\frac{\pi\omega}{2}\right)}$$

$$ch\,2x = 1+2sh^2x$$

$$\frac{sh\,x}{ch\,x2-1} = \frac{1}{2\,sh\,x}$$

Esercizio: Serie di Fourier



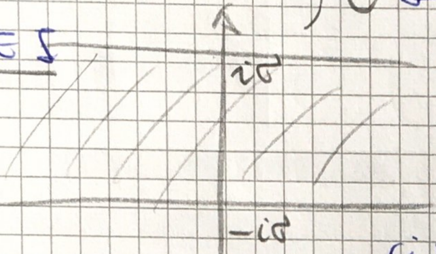
$\sigma > 0$

$A_\sigma = \{ z \in \mathbb{C} \mid e^{-\sigma} < |z| < e^{\sigma} \}$

$S_\sigma := \{ t \in \mathbb{C} : |\operatorname{Im} t| < \sigma \}$

e^{it} mappa conforme.

PARTE I



ES

Sia $f(z)$ analitica in A_σ e limitata. Posso definire $F(t) := f(e^{it})$. Dimostrare che

(i) F analitica in S_σ e anche limitata su S_σ

$z = e^{it} = e^{ix} e^{-y}$

(ii) $F(t + 2\pi) = F(t)$ $\quad F(e^{it+2\pi}) = F(e^{it})$

(iii) $F(t) = \sum_{k \in \mathbb{Z}} \hat{F}_k e^{itk}$ ($< \infty$ come la serie di Laurent)

con $\hat{F}_k = \frac{1}{2\pi i} \int_0^{2\pi} F(t) e^{-ikt} \, dt$ $(t \in S_\sigma)$

e $\hat{F}_k \leq M e^{-|k|\sigma}$; $M := \sup_{A_\sigma} |f| = \sup_{S_\sigma} |F| < \infty$

Quindi converge assolutamente in S_σ

~~Cenno di dim (iii)~~

$F(t) = f(e^{it}) = \sum_{k \in \mathbb{Z}} a_n e^{ikt}$ $\quad f(z) = \sum a_n z^n$

con $\hat{F}_k = a_k$ $\quad a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})}{e^{int}} \, dt$

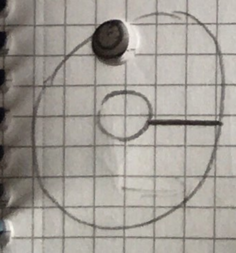
la decadenza è lasciata per es.

CVD

PARTE II

Sia F analitica e limitata in S_σ e periodica di periodo 2π

(iv) definisco $f(z)$ in questo modo, dove:



$$f(z) = \begin{cases} F\left(\frac{\log z}{i}\right) & z \in A_\sigma \setminus [0, +\infty) \\ F\left(\frac{\log_* z}{i}\right) & z \in A_\sigma \cap \mathbb{R}_+ \end{cases}$$

$\log z$ ramo analitico in $\mathbb{C} \setminus [0, +\infty)$

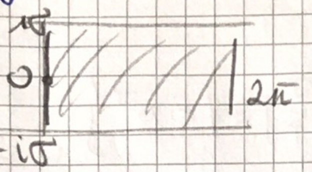
$\log(-1) = i\pi$; $\operatorname{Arg}(z) \in (0, 2\pi)$

$\log_* z$ ramo analitico in $\mathbb{C} \setminus (-\infty, 0]$

$\log_* 1 = 0$; $\operatorname{Arg} z \in (-\pi, \pi)$

Dimostrare che

(iv) $f(z)$ è analitica in A_σ



$$\frac{\operatorname{Log} z}{i} : A_\sigma \setminus [0,+\infty) \xrightarrow[1-1]{\Omega} \left\{ 0 < \operatorname{Re} t < 2\pi, \ |\operatorname{Im} t| < \sigma \right\}$$

(v) $F(t) = \sum_{k \in \mathbb{Z}} F_k e^{ikn}$