Lezione 27/03/19

Teorema di Cauchy

Proposizione

$f : \Omega$ regione di $\mathbb{C} \longmapsto \mathbb{C}$, f continua su Ω

$\oint_\gamma f\,dz = 0$, $\forall \gamma$ chiusa in $\Omega \iff \int_{\gamma(z_0,z_1)} f\,dz = F(z_0,z_1)$ dipende solo da z_0 e z_1 ($\forall\, \gamma(z_0,z_1)$ curve di estremi z_0 e z_1)

l'integrale è lo stesso

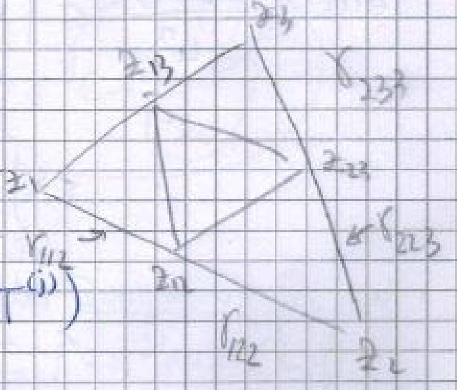
Teorema di Cauchy sui convessi

Ω regione convessa e f analitica su Ω.
Allora $\oint_\gamma f\,dz = 0 \quad \forall \gamma$ chiusa in Ω

NB f analitica su Ω convesso $\longmapsto f$ ha una primitiva

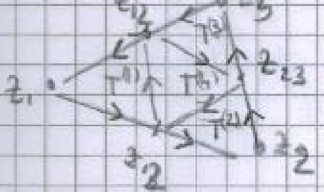
Lemma 1

f analitica su T triangolo chiuso (non degenere $\rightsquigarrow z_2 - z_1,\ z_3 - z_1 \to$ linearm indip) ($\exists\, \Omega$ regione tc $\Omega \supseteq T$ e f analitica su Ω)
allora $\int_{\partial T} f = 0$



dim

se T triangolo in Ω, $\phi(T) := \oint_{\partial T} f\,dz$



$z_{ij} = \dfrac{z_i + z_j}{2}$, $\quad \phi(T) = \sum_{1}^{4} \phi(T^{(j)})$

triangolo orientato

$z_{12} z_{23}$ viene percorso 2 volte in sensi opposti

$\partial T = \partial T^{(1)} + \partial T^{(2)} + \partial T^{(3)} + \partial T^{(4)} \quad (\partial T - \partial T^{(4)} = \partial T^{(1)} + \partial T^{(3)} + \partial T^{(2)})$

$\rightsquigarrow |\phi(T)| = \left|\sum_{1}^{4} \phi(T^{(j)})\right|$; $\exists\, j_0$ tc $|\phi(T^{(j_0)})| \geq \dfrac{|\phi(T)|}{4}$

$\alpha = \sum_{j=1}^{n} \alpha_j \quad \alpha, \alpha_j \in \mathbb{C} \longrightarrow \exists\, j_0$ tc $|\alpha_{j_0}| \geq \frac{1}{n}|\alpha|$

Iteriamo: $T_0 := T$ $\quad T_1 := T^{(j_0)} \dots \quad T^{(n)}$

e vale che (a) $T_n \subseteq T_{n-1}$ $\qquad$ (c) $\bigcap_{n\in\mathbb{N}} T_n = z_0 \in T$

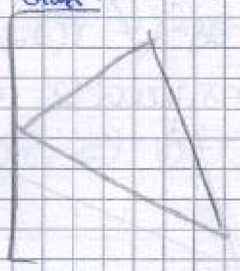
$$\text{(b)}\ |\Phi(T_1)| \geq \frac{|\Phi(T_0)|}{4}$$

$$\vdots$$

$$|\Phi(T_n)| \geq \frac{\Phi(T)}{4^n}$$

sia $L_n :=$ perimetro di T_n $\quad$ e $\quad L_n \overset{*}{=} \frac{L}{2^n}$, $\quad L = L_0$

dim *



$$\text{perimetro} = |z_{12} - z_{23}| + |z_{23} - z_{13}| + |z_{12} - z_{13}|$$

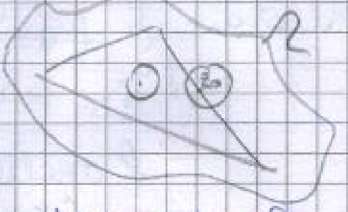
$$\left|\frac{z_1+z_2}{2} - \frac{z_2+z_3}{2}\right| + \dots = \frac{|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|}{2}$$

(c) $\bigcap_{n\in\mathbb{N}} T_n = z_0 \in T$ $\qquad \bigcap_{0}^{N} T_n = T_N$

$z, w \in T_N \longrightarrow |z - w| \leq L_N$

Quindi sia $z_n \in T_n$, $\{z_n\}$ è di Cauchy

$|z_n - z_{n-1}| \leq L_{n-1} \longrightarrow z_n$ converge $z_0 \in T$



Sia $\varepsilon > 0$, sia $\delta > 0$ tc $\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \frac{\varepsilon}{L^2}$ $\quad \forall |z - z_0| < \delta$

n tc $T_n \subseteq B_\delta(z_0)$

ora $\frac{|\Phi(T)|}{4^n} \overset{(a)}{\leq} |\Phi(T_n)| = \left|\int_{\partial T_n} f\, dz\right| = \left|\int_{\partial T_n} (f(z) - f(z_0)) + f'(z_0)(z - z_0)\right|$

se $g = G'$ analitica $\oint_\gamma G = 0$, qui $g = a + bz$

$$\leq \varepsilon \int_{\partial T_n} \frac{|z - z_0|}{L^2} |dz|$$

$$\leq \frac{\varepsilon}{L^2} L_n^2 = \frac{\varepsilon}{L^2}\left(\frac{L}{2^n}\right)^2 = \frac{\varepsilon}{4^n}$$

$\longrightarrow |\Phi(T)| \leq \varepsilon \longrightarrow \Phi(T) = 0$

CVD

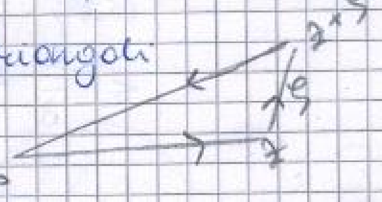
dim (thm di Cauchy sui convessi)
costruiamo una primitiva di f in Ω

Fissiamo $z_0 \in \Omega$, $z \in \Omega$, $F(z) := \int_{z_0}^{z} f = \int_{z_0}^{z} f(w)\,dw$

$$\int_{\sigma(z_0,z)} f \qquad \sigma(z_0,z) = \{\overbrace{z_0(1-t)+tz}^{z(t)} \mid t\in[0,1]\}$$

$$\sigma(z,z_0) = -\sigma(z_0,z)$$

$$\frac{F(z+\xi)-F(z)}{\xi} = \frac{\int_{z_0}^{z+\xi} f - \int_{z_0}^{z} f}{\xi} = \quad \rightarrow \text{Cauchy sui triangoli}$$



$$= \frac{1}{\xi}\int_{z}^{z+\xi} f = \frac{1}{\xi}\int_0^1 f(z+t\xi)\,\xi\,dt \xrightarrow{\xi\to 0} f(z)$$

$z(t) = z + t\xi \quad t\in[0,1]$

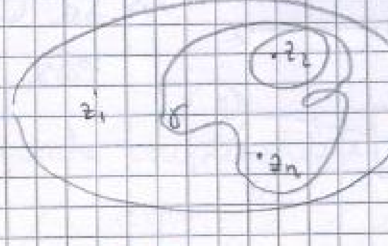
$z'(t) = \xi$

(f continua)

QED

def
$f: \Omega \setminus \{z_0\}$ $z_0 \in \Omega$ regione, f analitica, ha una SINGOLARITÀ ELIMINABILE in z_0 se $\lim_{z\to z_0} f(z)(z-z_0) = 0$

Teorema 2 [Estensione di Cauchy]
f analitica su $\Omega' := \Omega\setminus\{z_1, \dots z_n\}$, $z_j \in \Omega$ (Ω regione convessa)
e z_j sono singolarità eliminabili per f.
Allora $\oint_\gamma f = 0 \quad \forall \gamma$ chiusa in Ω'

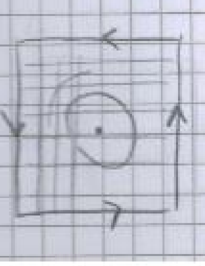


Lemma
f analitica su un quadrato $Q\setminus\{z_0\}$, z_0 centro di Q e z_0 sing eliminabile di f
$$\Longrightarrow \int_{\partial Q} f = 0$$

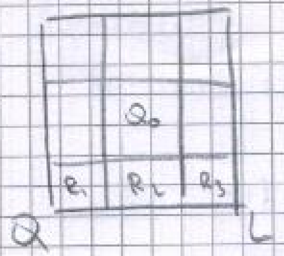
def
Un QUADRATO Q_L è:
$$Q_L := \begin{cases} 0 \le \operatorname{Im} z \le L \\ 0 \le \operatorname{Re} z \le L \end{cases}$$
$Q = z_0 + Q_L \quad z_0 \in \mathbb{C} \quad L>0$

dim

$\lim f(z)(z-z_0) = 0$
mi fisso ε, δ t.c. $z\in B_\delta$ $|f(z)(z-z_0)| < \frac{\varepsilon}{8}$



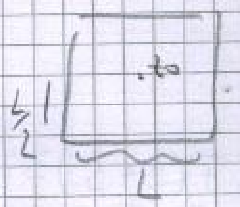
dividiamo Q in 9 rettangoli con al centro un quadrato $Q_0 \subseteq B_\delta$



$$\partial Q = \sum_j \partial R_j$$ le altre non contengono singol.

$$\int_{\partial Q} f = \sum_j \int_{\partial R_j} f \overset{TC}{=} \int_{\partial Q_0} f$$

$$\rightsquigarrow \left| \int_{\partial Q_0} f \, dz \right| \le \frac{\varepsilon}{8} \int_{\partial Q_0} \frac{|dz|}{|z - z_0|}$$ ma $|z - z_0| \geq \frac{L}{2}$



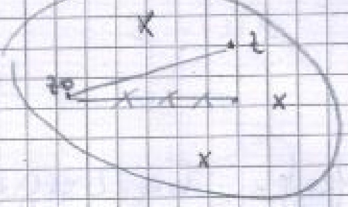
$$\le \frac{\varepsilon}{8} \, \frac{2}{L} \oint_{\partial Q_0} |dz| = \varepsilon$$

CVD

<u>dim</u> thm

costruisco una primitiva in Ω'

x = sing

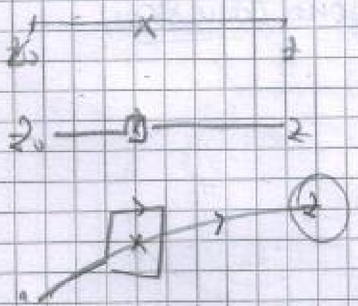


$z_0 \neq z_j \ \forall j$ e se $z \in \Omega'$

$$F(z) := \begin{cases} \displaystyle\int_{z_0}^{z} f & \text{se } z_j \notin \sigma(z_0, z) \ \forall j \\ \displaystyle\int_{\gamma(z_0,z)} f & \exists\, z_j : z_j \in \sigma(z_0, z) \end{cases}$$

una singolarità sia sul segmento

se $z_* \in \sigma(z_0, z)$



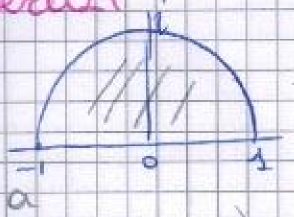
$\rightsquigarrow$ sia Q_0 un qualche quadrato di centro z_*, $Q_0 \subset \Omega$

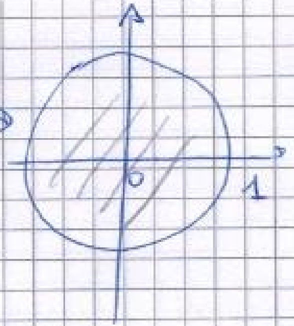
così elimino il problema

CVD

Esercizi

(1)





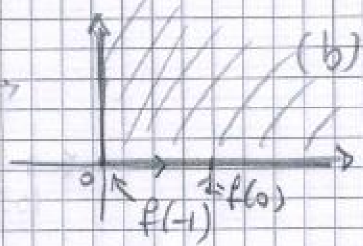
mando

$1 \longmapsto \infty$

$-1 \longmapsto 0$

$w = f(z) = \frac{-(z+1)}{z-1}$

$(f(0) = -1)$

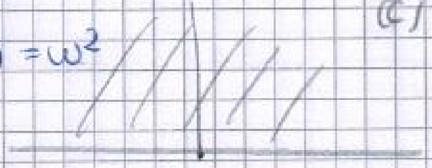


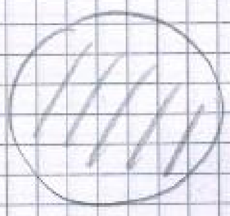
$(a) \xrightarrow{f} (b)$

$(b) \xrightarrow{g} (c)$

$g(w) = w^2$

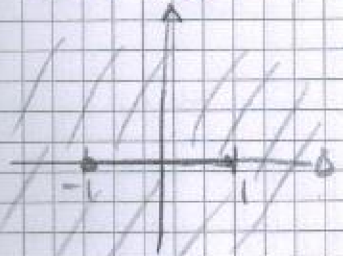
(c)

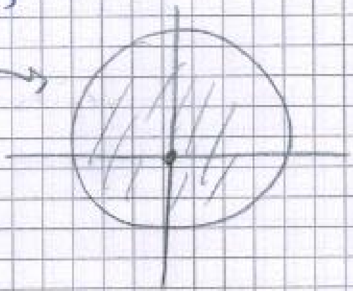




$\frac{z+1}{z-1}$

(2) $\mathbb{C} \setminus [-1,1] \longrightarrow D \setminus \{0\}$





una semiretta chiusa

$f(1)$ $f(0)$ $f(-1)$ $f(\infty) = 1$

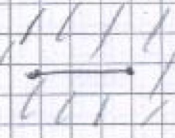
e manda complementare in

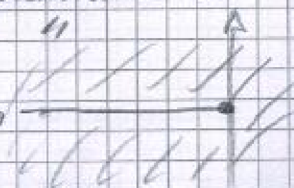
$f(z) = +\frac{z+1}{z-1}$

$1 \to \infty$

$-1 \to 0$

$0 \to -1$





$\sqrt{w}$ con $\sqrt{1} = 1$, $\sqrt{1} = \pm 1$ in $\mathbb{C}$