sum

(12) $$\lim_{m\to\infty}\sum_{-m}^{m}\frac{(-1)^n}{z-n}=\frac{1}{z}+\sum_{1}^{\infty}(-1)^n\frac{2z}{z^2-n^2}$$

which evidently represents a meromorphic function. It is very natural to separate the odd and even terms and write

$$\sum_{-(2k+1)}^{2k+1}\frac{(-1)^n}{z-n}=\sum_{n=-k}^{k}\frac{1}{z-2n}-\sum_{n=-k-1}^{k}\frac{1}{z-1-2n}.$$

By comparison with (11) we find that the limit is

$$\frac{\pi}{2}\cot\frac{\pi z}{2}-\frac{\pi}{2}\cot\frac{\pi(z-1)}{2}=\frac{\pi}{\sin\pi z},$$

and we have proved that

(13) $$\frac{\pi}{\sin\pi z}=\lim_{m\to\infty}\sum_{-m}^{m}(-1)^n\frac{1}{z-n}.$$

EXERCISES

1. Comparing coefficients in the Laurent developments of $\cot\pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_1^\infty\frac{1}{n^2},\quad\sum_1^\infty\frac{1}{n^4},\quad\sum_1^\infty\frac{1}{n^6}.$$

Give a complete justification of the steps that are needed.

2. Express

$$\sum_{-\infty}^{\infty}\frac{1}{z^2-n^2}$$

in closed form.

3. Use (13) to find the partial fraction development of $1/\cos\pi z$, and show that it leads to $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

4. What is the value of

$$\sum_{-\infty}^{\infty}\frac{1}{(z+n)^2+a^2}?$$

5. Using the same method as in Ex. 1, show that

$$\sum_1^\infty\frac{1}{n^{2k}}=2^{2k-1}\frac{B_k}{(2k)!}\pi^{2k}.$$

(See Sec. 1.3, Ex. 4, for the definition of B_k.)

2.2. Infinite Products. An infinite product of complex numbers

(14) $$p_1p_2\cdots p_n\cdots=\prod_{n=1}^{\infty}p_n$$

is evaluated by taking the limit of the partial products $P_n=p_1p_2\cdots p_n$. It is said to converge to the value $P=\lim_{n\to\infty}P_n$ if this limit exists and is different from zero. There are good reasons for excluding the value zero. For one thing, if the value $P=0$ were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors. On the other hand, in certain connections this convention is too radical. In fact, we wish to express a function as an infinite product, and this must be possible even if the function has zeros. For this reason we make the following agreement: The infinite product (14) is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero.

In a convergent product the general factor p_n tends to 1; this is clear by writing $p_n=P_n/P_{n-1}$, the zero factors being omitted. In view of this fact it is preferable to write all infinite products in the form

(15) $$\prod_{n=1}^{\infty}(1+a_n)$$

so that $a_n\to 0$ is a necessary condition for convergence.

If no factor is zero, it is natural to compare the product (15) with the infinite series

(16) $$\sum_{n=1}^{\infty}\log(1+a_n).$$

Since the a_n are complex we must agree on a definite branch of the logarithms, and we decide to choose the principal branch in each term. Denote the partial sums of (16) by S_n. Then $P_n=e^{S_n}$, and if $S_n\to S$ it follows that P_n tends to the limit $P=e^S$ which is $\neq 0$. In other words, the convergence of (16) is a sufficient condition for the convergence of (15).

In order to prove that the condition is also necessary, suppose that $P_n\to P\neq 0$. It is not true, in general, that the series (16), formed with the principal values, converges to the principal value of $\log P$; what we wish to show is that it converges to some value of $\log P$. For greater clarity we shall temporarily adopt the usage of denoting the principal value of the logarithm by Log and its imaginary part by Arg.

Because $P_n/P \to 1$ it is clear that $\operatorname{Log}(P_n/P) \to 0$ for $n \to \infty$. There exists an integer h_n such that $\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h_n \cdot 2\pi i$. We pass to the differences to obtain $(h_{n+1} - h_n)2\pi i = \operatorname{Log}(P_{n+1}/P) - \operatorname{Log}(P_n/P) - \operatorname{Log}(1 + a_n)$ and hence $(h_{n+1} - h_n)2\pi = \operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) - \operatorname{Arg}(1 + a_n)$. By definition, $|\operatorname{Arg}(1 + a_n)| \leqq \pi$, and we know that $\operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) \to 0$. For large n this is incompatible with the previous equation unless $h_{n+1} = h_n$. Hence h_n is ultimately equal to a fixed integer h, and it follows from $\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h \cdot 2\pi i$ that $S_n \to \operatorname{Log} P - h \cdot 2\pi i$. We have proved:

Theorem 5. *The infinite product* $\prod_1^\infty (1 + a_n)$ *with* $1 + a_n \neq 0$ *converges simultaneously with the series* $\sum_1^\infty \log(1 + a_n)$ *whose terms represent the values of the principal branch of the logarithm.*

The question of convergence of a product can thus be reduced to the more familiar question concerning the convergence of a series. It can be further reduced by observing that the series (16) converges absolutely at the same time as the simpler series $\Sigma |a_n|$. This is an immediate consequence of the fact that

$$\lim_{z \to 0} \frac{\log(1 + z)}{z} = 1.$$

If either the series (16) or $\sum_1^\infty |a_n|$ converges, we have $a_n \to 0$, and for a given $\varepsilon > 0$ the double inequality

$$(1 - \varepsilon)|a_n| < |\log(1 + a_n)| < (1 + \varepsilon)|a_n|$$

will hold for all sufficiently large n. It follows immediately that the two series are in fact simultaneously absolutely convergent.

An infinite product is said to be absolutely convergent if and only if the corresponding series (16) converges absolutely. With this terminology we can state our result in the following terms:

Theorem 6. *A necessary and sufficient condition for the absolute convergence of the product* $\prod_1^\infty (1 + a_n)$ *is the convergence of the series* $\sum_1^\infty |a_n|$.

In the last theorem the emphasis is on absolute convergence. By simple examples it can be shown that the convergence of $\sum_1^\infty a_n$ is neither sufficient nor necessary for the convergence of the product $\prod_1^\infty (1 + a_n)$.

It is clear what to understand by a uniformly convergent infinite product whose factors are functions of a variable. The presence of zeros may cause some slight difficulties which can usually be avoided by considering only sets on which at most a finite number of the factors can vanish. If these factors are omitted, it is sufficient to study the uniform convergence of the remaining product. Theorems 5 and 6 have obvious counterparts for uniform convergence. If we examine the proofs, we find that all estimates can be made uniform, and the conclusions lead to uniform convergence, at least on compact sets.

EXERCISES

1. Show that

$$\prod_{n=2}^\infty \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

2. Prove that for $|z| < 1$

$$(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots = \frac{1}{1 - z}.$$

3. Prove that

$$\prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

4. Prove that the value of an absolutely convergent product does not change if the factors are reordered.

5. Show that the function

$$\theta(z) = \prod_1^\infty (1 + h^{2n-1}e^z)(1 + h^{2n-1}e^{-z})$$

where $|h| < 1$ is analytic in the whole plane and satisfies the functional equation

$$\theta(z + 2 \log h) = h^{-1}e^{-z}\,\theta(z).$$

2.3. Canonical Products. A function which is analytic in the whole plane is said to be *entire*, or *integral*. The simplest entire functions which are not polynomials are e^z, $\sin z$, and $\cos z$.

If $g(z)$ is an entire function, then $f(z) = e^{g(z)}$ is entire and $\neq 0$. Conversely, if $f(z)$ is any entire function which is never zero, let us show that $f(z)$ is of the form $e^{g(z)}$. To this end we observe that the function $f'(z)/f(z)$, being analytic in the whole plane, is the derivative of an entire function $g(z)$. From this fact we infer, by computation, that $f(z)e^{-g(z)}$ has the derivative zero, and hence $f(z)$ is a constant multiple of $e^{g(z)}$; the constant can be absorbed in $g(z)$.

By this method we can also find the most general entire function with a finite number of zeros. Assume that $f(z)$ has m zeros at the origin (m may be zero), and denote the other zeros by $a_1, a_2, \ldots, a_N$, multiple zeros being repeated. It is then plain that we can write

$$f(z) = z^m e^{g(z)} \prod_1^N \left(1 - \frac{z}{a_n}\right).$$

If there are infinitely many zeros, we can try to obtain a similar representation by means of an infinite product. The obvious generalization would be

$$f(z) = z^m e^{g(z)} \prod_1^\infty \left(1 - \frac{z}{a_n}\right). \tag{17}$$

This representation is valid if the infinite product converges uniformly on every compact set. In fact, if this is so the product represents an entire function with zeros at the same points (except for the origin) and with the same multiplicities as $f(z)$. It follows that the quotient can be written in the form $z^m e^{g(z)}$.

The product in (17) converges absolutely if and only if $\sum_1^\infty 1/|a_n|$ is convergent, and in this case the convergence is also uniform in every closed disk $|z| \leq R$. It is only under this special condition that we can obtain a representation of the form (17).

In the general case convergence-producing factors must be introduced. We consider an arbitrary sequence of complex numbers $a_n \neq 0$ with $\lim_{n\to\infty} a_n = \infty$, and prove the existence of polynomials $p_n(z)$ such that

$$\prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{18}$$

converges to an entire function. The product converges together with the series with the general term

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z)$$

where the branch of the logarithm shall be chosen so that the imaginary part of $r_n(z)$ lies between $-\pi$ and π (inclusive).

For a given R we consider only the terms with $|a_n| > R$. In the disk $|z| \leq R$ the principal branch of $\log(1 - z/a_n)$ can be developed in a Taylor series

$$\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots.$$

We reverse the signs and choose $p_n(z)$ as a partial sum

$$p_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}.$$

Then $r_n(z)$ has the representation

$$r_n(z) = -\frac{1}{m_n + 1}\left(\frac{z}{a_n}\right)^{m_n+1} - \frac{1}{m_n + 2}\left(\frac{z}{a_n}\right)^{m_n+2} - \cdots$$

and we obtain easily the estimate

$$|r_n(z)| \leq \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}. \tag{19}$$

Suppose now that the series

$$\sum_{n=1}^\infty \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \tag{20}$$

converges. By the estimate (19) it follows first that $r_n(z) \to 0$, and hence $r_n(z)$ has an imaginary part between $-\pi$ and π as soon as n is sufficiently large. Moreover, the comparison shows that the series $\Sigma r_n(z)$ is absolutely and uniformly convergent for $|z| \leq R$, and thus the product (18) represents an analytic function in $|z| < R$. For the sake of the reasoning we had to exclude the values $|a_n| \leq R$, but it is clear that the uniform convergence of (18) is not affected when the corresponding factors are again taken into account.

It remains only to show that the series (20) can be made convergent for all R. But this is obvious, for if we take $m_n = n$ it is clear that (20) has a majorant geometric series with ratio < 1 for any fixed value of R.

Theorem 7. *There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{21}$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and $g(z)$ is an entire function.

This theorem is due to Weierstrass. It has the following important corollary:

Corollary. *Every function which is meromorphic in the whole plane is the quotient of two entire functions.*

In fact, if $F(z)$ is meromorphic in the whole plane, we can find an entire function $g(z)$ with the poles of $F(z)$ for zeros. The product $F(z)g(z)$ is then an entire function $f(z)$, and we obtain $F(z) = f(z)/g(z)$.

The representation (21) becomes considerably more interesting if it is possible to choose all the m_n equal to each other. The preceding proof has shown that the product

$$\prod_{1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h} \tag{22}$$

converges and represents an entire function provided that the series $\sum_{n=1}^{\infty}(R/|a_n|)^{h+1}/(h+1)$ converges for all R, that is to say provided that $\Sigma 1/|a_n|^{h+1} < \infty$. Assume that h is the smallest integer for which this series converges; the expression (22) is then called the *canonical product* associated with the sequence $\{a_n\}$, and h is the *genus* of the canonical product.

Whenever possible we use the canonical product in the representation (21), which is thereby uniquely determined. If in this representation $g(z)$ reduces to a polynomial, the function $f(z)$ is said to be of finite genus, and the genus of $f(z)$ is by definition equal to the degree of this polynomial or to the genus of the canonical product, whichever is the larger. For instance, an entire function of genus zero is of the form

$$Cz^m\prod_{1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

with $\Sigma 1/|a_n| < \infty$. The canonical representation of an entire function of genus 1 is either of the form

$$Cz^me^{\alpha z}\prod_{1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n}$$

with $\Sigma 1/|a_n|^2 < \infty$, $\Sigma 1/|a_n| = \infty$, or of the form

$$Cz^me^{\alpha z}\prod_{1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

with $\Sigma 1/|a_n| < \infty$, $\alpha \neq 0$.

As an application we consider the product representation of $\sin \pi z$. The zeros are the integers $z = \pm n$. Since $\Sigma 1/n$ diverges and $\Sigma 1/n^2$ converges, we must take $h = 1$ and obtain a representation of the form

$$\sin \pi z = ze^{g(z)}\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}.$$

In order to determine $g(z)$ we form the logarithmic derivatives on both sides. We find

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $z = n$. By comparison with the previous formula (10) we conclude that $g'(z) = 0$. Hence $g(z)$ is a constant, and since $\lim_{z\to 0} \sin \pi z/z = \pi$ we must have $e^{g(z)} = \pi$, and thus

$$\sin \pi z = \pi z\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}. \tag{23}$$

In this representation the factors corresponding to n and $-n$ can be bracketed together, and we obtain the simple form

$$\sin \pi z = \pi z\prod_{1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{24}$$

It follows from (23) that $\sin \pi z$ is an entire function of genus 1.

EXERCISES

1. Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

Hint: Let $g(z)$ be a function with simple zeros at the a_n. Show that

$$\sum_{1}^{\infty} g(z)\frac{e^{\gamma_n(z-a_n)}}{z-a_n}\cdot\frac{A_n}{g'(a_n)}$$

converges for some choice of the numbers γ_n.

2. Prove that

$$\sin \pi(z+\alpha) = e^{\pi z \cot \pi\alpha}\prod_{-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}$$

whenever α is not an integer. *Hint:* Denote the factor in front of the canonical product by $g(z)$ and determine $g'(z)/g(z)$.

3. What is the genus of $\cos \sqrt{z}$?

4. If $f(z)$ is of genus h, how large and how small can the genus of $f(z^2)$ be?

5. Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real. *Hint:* Consider $\operatorname{Im} f'(z)/f(z)$.

2.4. The Gamma Function. The function $\sin \pi z$ has all the integers for zeros, and it is the simplest function with this property. We shall now introduce functions which have only the positive or only the negative integers for zeros. The simplest function with, for instance, the negative integers for zeros is the corresponding canonical product

$$G(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{25}$$

It is evident that $G(-z)$ has then the positive integers for zeros, and by comparison with the product representation (23) of $\sin \pi z$ we find at once

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}. \tag{26}$$

Because of the manner in which $G(z)$ has been constructed, it is bound to have other simple properties. We observe that $G(z - 1)$ has the same zeros as $G(z)$, and in addition a zero at the origin. It is therefore clear that we can write

$$G(z - 1) = ze^{\gamma(z)}G(z),$$

where $\gamma(z)$ is an entire function. In order to determine $\gamma(z)$ we take the logarithmic derivatives on both sides. This gives the equation

$$\sum_{n=1}^{\infty} \left(\frac{1}{z - 1 + n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z + n} - \frac{1}{n}\right). \tag{27}$$

In the series to the left we can replace n by $n + 1$. By this change we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{z - 1 + n} - \frac{1}{n}\right) &= \frac{1}{z} - 1 + \sum_{1}^{\infty} \left(\frac{1}{z + n} - \frac{1}{n + 1}\right) \\ &= \frac{1}{z} - 1 + \sum_{n=1}^{\infty} \left(\frac{1}{z + n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + 1}\right).\end{aligned}$$

The last series has the sum 1, and hence equation (27) reduces to $\gamma'(z) = 0$.

Thus $\gamma(z)$ is a constant, which we denote by γ, and $G(z)$ has the reproductive property $G(z - 1) = e^{\gamma}zG(z)$. It is somewhat simpler to consider the function $H(z) = G(z)e^{\gamma z}$ which evidently satisfies the functional equation $H(z - 1) = zH(z)$.

The value of γ is easily determined. Taking $z = 1$ we have

$$1 = G(0) = e^{\gamma}G(1),$$

and hence

$$e^{-\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}.$$

Here the nth partial product can be written in the form

$$(n + 1)e^{-(1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n)},$$

and we obtain

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

The constant γ is called Euler's constant; its approximate value is .57722.

If $H(z)$ satisfies $H(z - 1) = zH(z)$, then $\Gamma(z) = 1/[zH(z)]$ satisfies $\Gamma(z - 1) = \Gamma(z)/(z - 1)$, or

$$\Gamma(z + 1) = z\Gamma(z). \tag{28}$$

This is found to be a more useful relation, and for this reason it has become customary to implement the restricted stock of elementary functions by inclusion of $\Gamma(z)$ under the name of *Euler's gamma function.*

Our definition leads to the explicit representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{29}$$

and the formula (26) takes the form

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}. \tag{30}$$

We observe that $\Gamma(z)$ is a meromorphic function with poles at $z = 0, -1, -2, \ldots$ but *without zeros.*

We have $\Gamma(1) = 1$, and by the functional equation we find $\Gamma(2) = 1$, $\Gamma(3) = 1 \cdot 2$, $\Gamma(4) = 1 \cdot 2 \cdot 3$ and generally $\Gamma(n) = (n - 1)!$. The Γ-function can thus be considered as a generalization of the factorial. From (30) we conclude that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Other properties are most easily found by considering the second