Funzioni meromorfe e fratti parziali

Mittag-Leffler

Teorema f meromorfa con poli in z_j e parti singolari $P_j\left(\frac{1}{z-z_j}\right)$ (P_j polinomio senza termine noto) allora $\exists$ polinomi p_j e una funzione intera $g(z)$ t.c.

$$f(z) = \sum_j \left(P_j\left(\frac{1}{z-z_j}\right) - p_j(z) \right) + g(z)$$

dove la serie è assol. conv. in $\mathbb{C} \setminus \{z_j\}$.

Lemma (Resto della serie di Taylor).

$\sum a_n (z-z_0)^n$ conv. in $|z-z_0|<R$. Allora, $\forall\, 0<r<R$, $\forall\, 0<t<1$ e $\forall\, |z-z_0| \le tr$, $|R_N| \le \sum_{n \ge N+1} |a_n||z-z_0|^n \le M \frac{t^{N+1}}{1-t}$, $M := \max_{|z-z_0|\le r} |f|$

Dim $a_n = \frac{f^{(n)}(z_0)}{n!} \underset{\text{Cauchy}}{=} \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta$

$\Rightarrow |a_n| \le \frac{M}{r^n}$. Quindi, se $|z-z_0| \le tr$,

$$\sum_{n \ge N+1} |a_n| |z-z_0|^n \le \sum_{n \ge N+1} \frac{M}{r^n} t^n r^n = M \sum_{n \ge N+1} t^n = M \frac{t^{N+1}}{1-t}. \square$$

Nota: se $t = \frac{1}{2}$, $|R_N| \le \frac{M}{2^N}$

Dim. (del teorema di Mittag-Leffler)

Possiamo assumere $j \geq 1$ e $z_j \neq 0$ $\forall j$..

(se z_0 è un polo, riordiniamo i poli con $z_0 = 0$, z_1, z_2, etc.

e poniamo $P_0 = 0$)

N.B. la convergenza della serie dipende dalla coda!

→ $P_j\left(\frac{1}{z-z_j}\right)$ è analitica in $|z| < |z_j| =: R_j$.

Prendiamo $p_j(z) =$ polinomio di Taylor di ordine N_j in 0

e $r_j = \frac{R_j}{2} = \frac{|z_j|}{2}$. Allora se $|z| \leq \frac{|z_j|}{4}$ $\left(t = \frac{1}{2}\right)$

si ha che

$$\left| P_j\left(\frac{1}{z-z_j}\right) - p_j(z) \right| \leq \frac{M_j}{2^{N_j}}, \quad M_j = \max_{|z| \leq \frac{|z_j|}{2}} \left|P_j\left(\frac{1}{z-z_j}\right)\right|$$

Scegliamo N_j t.c. $2^{N_j} \geq 2^j M_j \iff \frac{M_j}{2^{N_j}} \leq \frac{1}{2^j}$

Quindi $\forall$ $z \neq z_j$, sia $j_0 \mid$ $|z_j| \geq 4|z|$ $\forall j \geq j_0$

$$\sum_j \left| P_j\left(\frac{1}{z-z_j}\right) - p_j(z) \right| \leq \sum_{j < j_0} \max \left| P_j\left(\frac{1}{z-z_j}\right) - p_j(z) \right| + \sum_{j \geq j_0} \frac{1}{2^j}$$

$$= c(z)$$

Dunque, la funzione $F(z) := \sum_j \left(P_j\left(\frac{1}{z-z_j}\right) - p_j(z)\right)$

è una funzione meromorfa con poli in z_j e parte sing. $P_j\left(\frac{1}{z-z_j}\right)$

e $g(z) := f - F$ è intera (ha sing. eliminabili in z_j). $\square$

Sappiamo che

(1). $$\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

Dimostriamo anche:

(2) $$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \lim \sum_{-m}^{m} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

(3) $$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{1}^{\infty} (-1)^n \frac{2z}{z^2-n^2} = \lim_{m\to+\infty} \sum_{-m}^{m} (-1)^n \frac{1}{z-n}$$

<u>Dim (2)</u> $$\frac{\pi^2}{(\sin \pi z)^2} = D(-\pi \cot \pi z) \quad e \quad \frac{1}{(z-n)^2} = D\left(-\frac{1}{z-n}\right).$$

$$\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \sum_{n\neq 0} \frac{z}{(z-n)n},$$ che conv. ass. per $z \neq n$, $\forall n \neq 0$

e <u>unif. sui compatti</u> $D \mid D \cap \mathbb{Z} = \emptyset$

↑ <u>ES</u>

Per Weierstrass, $f(z) := \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) \in H(\Omega \setminus \mathbb{Z})$

e $f'(z) = -\sum_{n\neq 0} \frac{1}{(z-n)^2}$. Dunque,

$$D(\pi \cot \pi z) = -\frac{\pi^2}{(\sin \pi z)^2} \overset{(1)}{=} -\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = -\frac{1}{z^2} - \sum_{n\neq 0} \frac{1}{(z-n)^2}$$

$$= D\left(\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)\right)$$

Quindi

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) + c$$

Ora $\lim_{z\to 0}\left(\pi \cot \pi z - \frac{1}{z}\right) = \lim_{z\to 0}\left(\pi \frac{\cos \pi z}{\sin \pi z} - \frac{1}{z}\right) = 0$

Quindi $c = 0$.

Infine,

$$\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \lim_{m\to+\infty} \sum_{\substack{-m\le n\le m \\ n\neq 0}} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

$$= \frac{1}{z} + \lim_{m\to\infty} \sum_{1}^{m}\left(\frac{1}{z-n} + \cancel{\frac{1}{n}} + \frac{1}{z+n} - \cancel{\frac{1}{n}}\right)$$

$$= \frac{1}{z} + \lim_{m\to+\infty} \sum_{1}^{m} \frac{2z}{z^2-n^2} = \lim_{m\to\infty} \sum_{-m}^{m} \frac{1}{z-n}. \blacksquare$$

Dim (3) $\quad \lim_{m\to+\infty} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \lim_{m\to\infty} \sum_{-(2m+1)}^{2m+1} \frac{(-1)^n}{z-n}$

$$= \lim_{m\to\infty}\Big(\underbrace{\sum_{-k}^{k} \frac{1}{z-2n}}_{n \text{ pari}} - \underbrace{\sum_{-(k+1)}^{k} \frac{1}{z-1-2n}}_{n \text{ dispari}} \Big)$$

$$\overset{(2)}{=} \frac{\pi}{2}\cot\frac{\pi z}{2} - \frac{\pi}{2}\cot\frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z}$$

$$\cot \alpha - \cot\left(\alpha - \frac{\pi}{2}\right) = \frac{2}{\sin 2\alpha}$$

Rappresentazione in fratti parziali di $\frac{\pi}{\sin \pi z}$:

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{(-1)^n}{z-n} + \frac{(-1)^n}{n}\right)$$

$$P_n(\xi) = (-1)^n \xi \quad e \quad p_n(\xi) = \frac{(-1)^{n+1}}{n}, \quad g = 0.$$

Infatti,

$$\frac{\pi}{\sin \pi z} = \lim_{m} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \lim \sum_{\substack{-m \\ n\neq 0}}^{m} \frac{(-1)^n}{z-n}$$

$$= \frac{1}{z} + \lim \sum_{\substack{-m \\ n\neq 0}}^{m} \underbrace{\frac{(-1)^n}{z-n} + \frac{(-1)^n}{n}} - \frac{(-1)^n}{n} \quad \text{ma} \sum_{\substack{n=-m \\ n\neq 0}}^{m} \frac{(-1)^n}{n} = 0$$