## Other series expansion for meromorphic functions

We have proven that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \quad . \qquad (1)$$

Now,

$$\frac{\pi^2}{\sin^2\pi z} = D\left(-\pi \cot \pi z\right), \quad \text{and} \quad D\, \frac{1}{z-n} = -\frac{1}{(z-n)^2}$$

The function $\pi \cot \pi z$ has *simple* poles at the integers with singular part

$$\pi\, \frac{\cos \pi z}{\sin \pi z} \underset{\uparrow}{=} \pi\, \frac{\cos \pi(z-n)}{\sin \pi(z-n)} = \frac{1}{z-n} + h(z)$$

$\cot \pi z$ has period 1.

with $h(z)$ analytic near $n$ (and $h(z) \xrightarrow[z\to n]{} 0$).

Now, $\sum \frac{1}{z-n}$ does not converge but $\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)$ does

as $\sum_{n\neq 0} \frac{1}{z-n} + \frac{1}{n} = \sum_{n\neq 0} \frac{z}{(z-n)\,n}$ and defines

a meromorphic function on $\mathbb{C} \setminus (\mathbb{Z}\setminus\{0\})$ with simple poles at $z = n \neq 0$.

Then,

$$-D\left(\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)\right) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \overset{(1)}{=} \frac{\pi^2}{\sin^2 \pi z}$$

$$= -D\, \pi \cot \pi z$$

so that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) + c$$

but

$$\pi \cot \pi z - \frac{1}{z} \xrightarrow[z\to 0]{} 0$$

and so does $\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) \Rightarrow c = 0.$

We proved that

$$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) \overset{\text{Ex.}}{=} \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{z-n}} \qquad (2)$$

This last formula suggests that

(3) $$\boxed{\frac{\pi}{\sin \pi z} = \lim_{m\to+\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}}$$

Proof First notice that

$$\sum_{-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{n=1}^{m} (-1)^n \left(\frac{1}{z-n} - \frac{1}{z+n}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{m} (-1)^n \frac{2z}{z^2-n^2},$$

which converges uniformly on compact sets in $\mathbb{C} \setminus \mathbb{Z}$.

Now,

$$\lim_{m\to+\infty} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \lim \sum_{-(2m+1)}^{2m+1} \frac{(-1)^n}{z-n}$$

$$= \lim \Big( \underbrace{\sum_{j=-m}^{m}}_{n=2j} \frac{1}{z-2j} - \underbrace{\sum_{j=-m}^{m}}_{n=2j+1} \frac{1}{z-2j-1} \Big)$$

$$= \frac{1}{2} \lim \left( \sum_{-m}^{m} \frac{1}{\frac{z}{2}-j} - \sum_{j=-m}^{m} \frac{1}{\frac{z-1}{2}-j} \right)$$

$$\overset{(2)}{=} \frac{\pi}{2} \left( \cot \frac{\pi z}{2} - \cot \frac{\pi(z-1)}{2} \right) = \frac{\pi}{\sin \pi z} . \blacksquare$$

$$\cot \alpha - \cot\left(\alpha - \frac{\pi}{2}\right) = \frac{\cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{\cos\alpha} = \frac{1}{\sin\alpha\cos\alpha} = \frac{2}{\sin 2\alpha}$$

---

Theorem (Mittag-Leffler) $f$ meromorphic in $\mathbb{C}$ with poles in $z_n \to \infty$ and singular part at $z_n$, $P_n\left(\frac{1}{z-z_n}\right)$, $P_n$ polyn (with $P_n(0)=0$). Then, $\exists$ polynomials $p_n$ and an entire function $g$ s.t.

$$f(z) = \sum_n \left( P_n\left(\frac{1}{z-z_n}\right) - p_n(z) \right) + g$$

where the series converges absolutely in $\mathbb{C} \setminus \{z_n\}$ and uniformly on compact sets in $\mathbb{C} \setminus \{z_n\}$.