venerdì 11 dicembre 2020 08:10

$$\underline{\underline{\mathrm{Exp}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}}}$$ è assolutamente convergente $\underline{\forall\ x \in \mathbb{R}}$ ossia converge $\sum_{k=0}^{\infty} \frac{|x|^k}{k!}$ (CRIT DEL RAPPORTO)

Teorema $\mathrm{Exp}(x) = \exp(x) := e^x$ $\left(e = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n \right)$

Dim.

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{x^k}{n^k} = \underline{\sum_{k=0}^{n} a_{nk} \frac{x^k}{k!}} \qquad (1)$$

dove $a_{nk} = \dfrac{n!}{(n-k)!\ n^k}$ $\quad 0 \le k \le n$

$a_{n0} = 1$, $a_{n1} = 1$

$$0 < a_{nk} = \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < 1 \qquad 2 \le k \le n$$

$$\frac{\overbrace{n(n-1)\cdots(n-(k-1))}^{\#k}}{} \qquad \frac{n\cdot(n-1)\cdots(n-k+1)}{n^k}$$

$$= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-(k-1)}{n}$$

$$= 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\lim_{n\to+\infty} a_{nk} = 1$$

Sia $n > m \ge 1$

$$\left| \left(1+\frac{x}{n}\right)^n - \sum_{k=0}^{m} a_{nk} \frac{x^k}{k!} \right| \overset{(1)}{=} \left| \sum_{k=0}^{n} a_{nk} \frac{x^k}{k!} - \sum_{k=0}^{m} a_{nk} \frac{x^k}{k!} \right|$$

$$= \left| \sum_{k=m+1}^{n} a_{nk} \frac{x^k}{k!} \right| \le \sum_{k=m+1}^{n} a_{nk} \frac{|x|^k}{k!} \underset{\uparrow}{\le} \sum_{k=m+1}^{\infty} \frac{|x|^k}{k!}$$

$a_{nk}<1$ e che $\sum_0^\infty \frac{|x|^k}{k!}$ converge

Quindi.

$$\left| \underbrace{\left(1+\frac{x}{n}\right)^n - \sum_{k=0}^{m} \boxed{a_{nk}} \frac{x^k}{k!}}_{} \right| \le \sum_{k=m+1}^{\infty} \frac{|x|^k}{k!}$$

$\downarrow n\to+\infty$ $\downarrow$ A.d.L

$$\left| e^x - \sum_{k=0}^{m} \frac{x^k}{k!} \right| \le \boxed{\sum_{k=m+1}^{\infty} \frac{|x|^k}{k!}}$$

$\downarrow m\to+\infty$ $\quad\downarrow$ essendo la coda di una serie convergente

$$\left| e^x - \mathrm{Exp}(x) \right| \le 0$$

$$\Rightarrow \quad e^x = \mathrm{Exp}(x). \ ☒$$

Abbiamo usato $(1+x)^n \to e^x \quad \forall\ x \in \mathbb{R}$

$(1+\frac{\cdot}{n})$

segue dal limite notevole (d) di pag 118 $\boxed{\lim_{t\to+\infty}\left(1+\frac{x}{t}\right)^t = e^x}$, $\forall x\in\mathbb{R}$

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(2) $\boxed{e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}}$

(3) $\sinh x = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$

(4) $\cosh x = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$

Dim, ad esempio, la (3): $\sinh x := \frac{e^x - e^{-x}}{2} = \frac{1}{2}\cdot\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)$

$$= \frac{1}{2}\left(\cancel{\sum_{\substack{k=0\\ k\text{ pari}}}^{\infty}\frac{x^k}{k!}} + \sum_{\substack{k=0\\ k\text{ dispari}}}^{\infty}\frac{x^k}{k!} - \cancel{\sum_{\substack{k=0\\ k\text{ pari}}}^{\infty}\frac{(-x)^k}{k!}} - \underbrace{\sum_{\substack{k=0\\ k\text{ dispari}}}^{\infty}\frac{(-x)^k}{k!}}_{\frac{x^k}{k!}}\right)$$

$$= \sum_{\substack{k=0\\ k\text{ dispari}}}^{\infty}\frac{x^k}{k!} = \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}$$

k dispari $\|$ $k=1,3,5$ $\quad 2j+1$

$\left(-(-x)^k,\ k \text{ dispari},\ = x^k,\ \forall x\in\mathbb{R}\right)$

Dare stime delle code esponenziali $t_n(x) := \sum_{k=n}^{+\infty}\frac{x^k}{k!}$, $(t_0(x) = e^x)$, $n\in\mathbb{N}_0$

$t_1(x) = e^x - 1$

Prop 5.4 Sia $n\in\mathbb{N}_0$

(6) $t_n(|x|) \le \frac{|x|^n}{n!}e^{|x|}$, $\forall x\in\mathbb{R}$

(7) $t_n(x) < \frac{x^n}{n!}\frac{n+1}{n+1-x}$, $0<x<n+1$

(8) $t_n(x) < 2\frac{x^n}{n!}$, $0<x\le\frac{n+1}{2}$

(N.B. (7) $\Rightarrow$ (8), $\frac{n+1}{n+1-x} \le \frac{n+1}{n+1-\frac{n+1}{2}} = 2$)

$\left[\frac{a}{b-x} \le \frac{a}{b-y} \Leftrightarrow \cancel{b}-y - \cancel{b}-x,\ x<y;\ b>y>x,\ a>0\right]$

Dim (7).

$x>0$, $t_n(x) = \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \dots$

$$= \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x^2}{(n+2)(n+1)} + \frac{x^3}{(n+3)(n+2)(n+1)} + \dots + \frac{x^k}{\underbrace{(n+k)(n+k-1)\cdots(n+1)}_{k\text{ termini}}}\right)$$

$$< \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x^2}{(n+1)^2} + \frac{x^3}{(n+1)^3} + \dots + \frac{x^k}{(n+1)^k} + \dots\right)$$

$$= \frac{x^n}{}\sum^{\infty}\left(\frac{x}{}\right)^k$$

$$\frac{\;}{n!} \sum_{k=0} (n+1)$$

$\boxed{x<n+1}$

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$$

$1+t+t^2+\dots$

$$= \frac{x^n}{n!} \frac{1}{1-\frac{x}{n+1}}$$

$$= \frac{x^n}{n!} \frac{n+1}{n+1-x}$$

(6) $x=0$ è ok. Basta dim per $x>0$.

$$t_n(x) = \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x^2}{(n+2)(n+1)} + \frac{x^3}{(n+3)(n+2)(n+1)} + \dots + \frac{x^l}{(n+l)\cdots(n+1)}\right.$$

$$\le \frac{x^n}{n!}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^k}{\underbrace{k(k-1)\cdots 1}_{k!}} + \dots\right)$$

$$= \frac{x^n}{n!} e^x, \quad \forall x>0 \quad \Rightarrow \quad t_n(|x|) \le \frac{x^n}{n!} e^{|x|}$$

notare che per $n>0$ $\quad \underline{t_n(x) < \frac{x^n}{n!} e^x}$

La (7) $\quad t_n(x) < \frac{x^n}{n!} \frac{n+1}{n+1-x}, \quad 0<x<n+1$

$$\frac{1}{n!}\frac{n+1}{n} < \frac{1}{n-1} \Leftrightarrow \underset{\underset{n^2-1}{\shortparallel}}{(n+1)(n-1)} < n^2$$

$\Updownarrow$

se $\underline{0<x\le 1}$, $n\ge 2 \Rightarrow x<n+1$

$$(7) \Rightarrow t_n(x) < \frac{1}{n!}\frac{n+1}{n+1-x} \le \frac{1}{n!}\frac{n+1}{n+1-1} = \frac{1}{n!}\frac{n+1}{n} < \frac{1}{(n-1)!}\frac{1}{n-1}$$

(9) $\quad t_n(x) < \frac{1}{(n-1)!}\frac{1}{(n-1)}, \quad \forall \underline{0<x\le 1}$ e $n\ge 2$

Teorema $e \notin \mathbb{Q}$.

Dim supp., p. a., che $e \in \mathbb{Q}_+$, $e>2$ $\left(e = \lim e_n,\ e_n \nearrow \quad \left(1+\frac{1}{n}\right)^n < e \quad 2=e_2<e\right)$

$$\begin{cases} e = \frac{p}{q} \quad \text{con } p,q \in \mathbb{N} \text{ e primi tra loro} \\ \quad = \sum_{k=0}^{\infty} \frac{1}{k!} \end{cases}$$

$$\left(\sum_{k=0}^{n} \frac{1}{k!} \nearrow e\right)$$

$$0 < \underbrace{\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}} = \sum_{k=q+1}^{\infty} \frac{1}{k!} \overset{(9)}{<} \frac{1}{q!}\frac{1}{q}, \quad \underline{q+1\ge 2}$$

$$0 < \frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q!}\frac{1}{q}$$

moltiplico per $q!$

$$0 < \underbrace{\underbrace{p(q-1)!}_{} - \sum_{k=0}^{q} \underbrace{\frac{q!}{k!}}_{\in \mathbb{N}}}_{} < \boxed{\frac{1}{q}}$$

$\left(\frac{1}{q} \le 1, \quad q \ge 1\right)$

$$\underset{\substack{\uparrow\\ m\in\mathbb{Z}}}{0 < m < 1} \qquad \underline{\text{assurdo}}\ ,\ \boxtimes$$

SERIE TRIGONOMETRICHE

(PUNTO DI VISTA DIVERSO DA QUELLO DI EUCLIDE)

DEFINIZIONE $\forall\, x\in\mathbb{R}$ definiamo

$$\cos x := \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\ ,\qquad \sin x := \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

N.B. Sono serie assolutamente convergenti $\forall\ x\in\mathbb{R}$

$$|\cos x| = \left|\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\right| \leq \cosh x \underset{\uparrow}{\leq} e^{|x|}$$

(= $\cosh|x|$ essendo $\cosh x$ una funzione pari)

$$\frac{e^x+e^{-x}}{2}\leq\frac{2e^{|x|}}{2}$$

$$\underline{|\sin x|} \leq \underline{\sinh|x|}$$

$$\underline{|\cos x-1|} = \left|\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\right| \leq \sum_{k=1}^{\infty}\frac{x^{2k}}{(2k)!} \leq \underline{e^{|x|}-1}$$

$$\| $$

$$\underbrace{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots}\quad -1$$

$$\leq 1+\underline{\underline{|x|}}+\frac{|x|^2}{2}+\underline{\underline{\frac{|x|^3}{3!}}}+\frac{|x|^4}{4!}\dots\quad -1 \ = e^{|x|}-1$$

- $\lim_{x\to 0}\sin x = 0$ ✓ essendo e^x una funzione continua

  $\overset{\|}{\lim}\ \frac{e^{|x|}-e^{-|x|}}{2}$

- $\lim_{x\to 0}\cos x = 1$

  $\sin 0 = 0$ ovvio $\qquad \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$

  $\cos 0 = 1$ ovvio $\qquad \cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots$

- $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

ES* dimostrare $\underbrace{-0{,}417} < \cos 2 < \underbrace{-0.415}$

$$\left(\frac{2^{11}}{10!} = \frac{8}{14175}\right)$$

$$\frac{\overbrace{\cancel{2}\ \cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cancel{2}\cancel{2}\ \cancel{2}}^{11}}{\underset{5}{\cancel{10}}\cdot 9\ \underset{\cancel{4}}{\cancel{8}}\ 7\cdot\underset{3}{\cancel{6}}\ 5\ \underset{\cancel{2}}{\cancel{4}}\cdot 3\cdot\cancel{2}} = \frac{8}{\ }$$

$$\textcircled{9\cdot 7}\textcircled{5\cdot 3} = 63 \quad 15\cdot 15 = 63\cdot 225$$

$$\begin{array}{r} 225 \\ 63 \\ \hline 675 \\ 1350 \\ \hline 14175 \end{array}$$

Lemma $\forall$ $\boxed{|x| \le 1}$

$|\operatorname{sen} x - x| \le \frac{|x|^3}{3}$ ✓ $\qquad |\cos x - 1 + \frac{x^2}{2}| \le \frac{x^4}{12}$ ✓

$|\operatorname{sen} x| \le \frac{4}{3}|x|$ ✓ $\quad , \quad |1-\cos x| \le \frac{7}{12}x^2$ ✓

$\left|\frac{\operatorname{sen} x}{x} - 1\right| \le \frac{x^2}{3}$ $(x \ne 0)$ $\qquad \left|\frac{1-\cos x}{x^2} - \frac{1}{2}\right| \le \frac{x^2}{12}$ $(x \ne 0)$

NB

$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$

$\lim \frac{1-\cos x}{2} = \frac{1}{2}$

Ricordiamo

$t_n(|x|) \underset{\uparrow}{\le} 2\frac{|x|^n}{n!}$, $\forall |x| \le \frac{n+1}{2}$, $n \ge 1$

Dim

$$|\operatorname{sen} x - x| = \left|\sum_{k=1}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right| \le \sum_{k=1}^{\infty}\frac{|x|^{2k+1}}{(2k+1)!} \le \sum_{k=3}^{\infty}\frac{|x|^k}{k!} = t_3(|x|) \le \cancel{2}\frac{|x|^3}{3\cancel{2}}$$

$$\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| = \left|\sum_{k=2}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\right| \le \sum_{k=2}^{\infty}\frac{|x|^{2k}}{(2k)!} \le \sum_{k=4}^{\infty}\frac{|x|^k}{k!} \le \cancel{2}\frac{|x|^4}{4\cdot 3\cdot\cancel{2}} = \frac{|x|^4}{12}$$

$$|\operatorname{sen} x| = |\operatorname{sen} x - x + x| \le |\operatorname{sen} x - x| + |x| \le \frac{|x|^3}{3} + |x| \le \frac{|x|}{3} + |x| = \frac{4}{3}|x|$$

$$|1-\cos x| \le \left|\cos x - 1 + \frac{x^2}{2}\right| + \frac{|x|^2}{2} \le \frac{x^4}{12} + \frac{x^2}{2} \le x^2\ \frac{7}{12}$$

$\left|\frac{\operatorname{sen} x}{x} - 1\right| = \left|\frac{\operatorname{sen} x - x}{x}\right| \le \frac{x^2}{3}$ Analog. per l'ultima dis $\blacksquare$

Es $\lim \frac{1-\cos x}{x} = \lim \underset{\substack{\downarrow \\ \frac{1}{2}}}{\frac{1-\cos x}{x^2}} \cdot \underset{\substack{\downarrow \\ 0}}{x} = 0$ $\qquad \left(\overset{\text{sappiamo}}{\lim \frac{1-\cos x}{x^2} = \frac{1}{2}}\right)$

Es. Se $\lim_{x\to 0}\frac{f(x)}{x^n} = \alpha \in \mathbb{R} \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^k} = 0$ $\quad \forall\, 0 \le k < n$

$$\frac{f(x)}{x^k} = \underset{\substack{\downarrow \\ \alpha}}{\frac{f(x)}{x^n}}\ \underset{\substack{\downarrow \\ 0}}{x^{n-k}} \xrightarrow[x\to 0]{} 0$$

Teorema* $\cos(x+y) = \cos x \cos y - \sin x \sin y$.