lunedì 14 dicembre 2020 08:17

$$\cos x := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin x := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Teorema di addizione $\quad \cos(x+y) = \cos x \cos y - \sin x \sin y$

Dim (schema)

$$\cos(x+y) := \sum_{k=0}^{\infty} (-1)^k \frac{(x+y)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \sum_{j=0}^{2k} \binom{2k}{j} x^j y^{2k-j}$$

$$= \sum_{k=0}^{\infty} (-1)^k \underbrace{\sum_{j=0}^{2k} \frac{x^j}{j!} \frac{y^{2k-j}}{(2k-j)!}}_{h=0} = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{\substack{0 \le j \le 2k \\ j \text{ pari}}} \frac{x^j}{j!} \frac{y^{2k-j}}{(2k-j)!} + \sum_{\substack{0 \le j \le 2k \\ j \text{ dispari}}} \frac{x^j}{j!} \frac{y^{2k-j}}{(2k-j)!} \right)$$

$j = 2h,\ 0 \le h \le k$ $\qquad$ $j = 2h+1,\ 0 \le h \le k-1$

$$\overset{(*)}{=} \sum_{k=0}^{\infty} (-1)^k \sum_{h=0}^{k} \frac{x^{2h}}{(2h)!} \frac{y^{2(k-h)}}{(2(k-h))!} + \sum_{k=1}^{\infty} (-1)^k \sum_{h=0}^{k-1} \frac{x^{2h+1}}{(2h+1)!} \frac{y^{2(k-h)-1}}{(2(k-h)-1)!}$$

il termine $k=0$

$k = n+1$, $n = k-1$: $2(k-h)-1 = 2(n+1-h)-1 = 2(n-h)+1$

$$= \sum_{n=0}^{\infty} (-1)^n \sum_{h=0}^{n} \frac{x^{2h}}{(2h)!} \frac{y^{2(n-h)}}{(2(n-h))!} - \sum_{n=0}^{\infty} (-1)^n \sum_{h=0}^{n} \frac{x^{2h+1}}{(2h+1)!} \frac{y^{2(n-h)+1}}{(2(n-h)+1)!}$$

$k = n$ $\qquad$ $0 \le h \le n$

$$\left(\begin{gathered} k-1 = n,\ k = n+1 \\ (-1)^k = (-1)^{n+1} = -(-1)^n \end{gathered} \right)$$

(**) Per il lemma

$$\sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{x^{2h}}{(2h)!} \frac{y^{2(n-h)}}{(2(n-h))!} = \sum_{h=0}^{\infty} \sum_{n=h}^{\infty} \frac{x^{2h}}{(2h)!} \frac{y^{2(n-h)}}{(2(n-h))!} = \sum_{h=0}^{\infty} \frac{x^{2h}}{(2h)!} \sum_{n=h}^{\infty} \frac{y^{2(n-h)}}{(2(n-h))!} = \sum_{h=0}^{\infty} \frac{x^{2h}}{(2h)!} \sum_{m=0}^{\infty} \frac{y^{2m}}{(2m)!}$$

$$= \cosh x \cosh y < \infty$$

$$\overset{(**)}{=} \sum_{h=0}^{\infty} \sum_{n=h}^{\infty} (-1)^n \frac{x^{2h}}{(2h)!} \frac{y^{2(n-h)}}{(2(n-h))!} - \sum_{h=0}^{\infty} \sum_{n=h}^{\infty} (-1)^n \frac{x^{2h+1}}{(2h+1)!} \frac{y^{2(n-h)+1}}{(2(n-h)+1)!}$$

Posso scambiare l'ordine delle somme

$$= \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} \underbrace{(-1)^{m+h}}_{(-1)^h (-1)^m} \frac{x^{2h}}{(2h)!} \frac{y^{2m}}{(2m)!} - \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} \underbrace{(-1)^{m+h}}_{(-1)^h (-1)^m} \frac{x^{2h+1}}{(2h+1)!} \frac{y^{2m+1}}{(2m+1)!}$$

$m = n-h$

$$= \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{(2h)!} \underbrace{\left(\sum_{m=0}^{\infty} (-1)^m \frac{y^{2m}}{(2m)!} \right)}_{\cos y} - \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{2h+1} \underbrace{\left(\sum_{m=0}^{\infty} (-1)^m \frac{y^{2m+1}}{(2m+1)!} \right)}_{\sin y}$$

$$= \cos y \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{(2h)!} - \sin y \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{(2h+1)!}$$

$= \cos x \ldots - \sin x \sin x \ldots$ ∎

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum a_k + \sum b_k$$

se le serie $\sum a_k$ e $\sum b_k$ convergono

(*) è convergente:

$$\sum_{k=0}^{\infty} \sum_{\substack{0 \le j \le 2k \\ j \text{ pari}}} \frac{|x|^j |y|^{2k-j}}{j!\,(2k-j)!} < \infty \quad - j \text{ dispari}$$

$$\Leftarrow \sum_{k=0}^{\infty} \left(\sum_{j=0}^{2k} \frac{|x|^j}{j!} \frac{|y|^{2k-j}}{(2k-j)!} \right) < \infty \ \checkmark$$

(**) Lemma: siano $a_{nk} \ge 0$. Allora

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{nk}$$

(identità in $\overline{\mathbb{R}}^*$)

$= \cos y \sin x - \dots 0$

Esempio $\quad a_{nk} = \frac{1}{n^k}$ $\quad n,k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \zeta(k) < \infty \iff k \geq 2$$

$$\sum_{k=1}^{\infty} \frac{1}{n^k} = \sum_{k=1}^{\infty} \left(\frac{1}{n}\right)^k < \infty \iff n \geq 2$$

$$\| \qquad \frac{\frac{1}{n}}{1-\frac{1}{n}} = \frac{1}{n-1}$$

$$\sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{k=2}^{\infty} \zeta(k)$$

$$\|$$

$$+\infty = \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{n^k} = \sum_{n=1}^{\infty} \frac{\frac{1}{n^2}}{1-\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{1}{n(n-1)} = +\infty$$

$$\boxed{\sum_{k=m}^{\infty} x^k = \frac{x^m}{1-x}, \quad |x|<1}$$

$$\sum_{k=2}^{\infty} \zeta(k) = +\infty$$

$$\sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^k} = \boxed{\sum_{k=2}^{\infty} \left(\zeta(k) - 1\right) = 1}$$

$$\|$$

$$\sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{n^k} = \sum_{n=2}^{\infty} \frac{\frac{1}{n^2}}{1-\frac{1}{n}} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

In particolare $\zeta(k) - 1 \underset{k\to+\infty}{\to} 0$, $\quad \zeta(k) \underset{k\to+\infty}{\to} 1$

Sull'uguaglianza con (*): segue da

$$\sum_{k=0}^{\infty} \sum_{j=0}^{2k} \frac{|x|^j}{j!} \frac{|y|^{2k-j}}{(2k-j)!} = \sum_{j=0}^{\infty} \frac{|x|^j}{j!} \sum_{k \geq \frac{j}{2}} \frac{|y|^{2k-j}}{(2k-j)!}$$

!!

Lemma

$0 \leq j \leq 2k$

$$= \underbrace{\sum_{j=0}^{\infty} \frac{|x|^j}{j!}} \underbrace{\sum_{m=0}^{\infty} \frac{|y|^m}{m!}} \qquad 2k-j = m \geq 0$$

$$= e^{|x|+|y|} < \infty \quad \forall\, x,y$$

Osservazioni $\quad \boxed{\cos x \cos y - \sin x \sin y = \cos(x+y)} \quad \forall\, x,y \in \mathbb{R}$

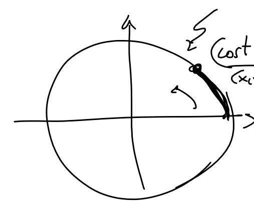
Se $y = -x$

(i) $\sin^2 x, \cos^2 x \leq \boxed{\cos^2 x + \sin^2 x = 1} \quad \forall\, x \in \mathbb{R}$.

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$(\cos t, \sin t) \in S^1$

(x,y)



(ii) $\Rightarrow$ $|\cos x|, |\sin x| \le 1$

Se $y = x$,

(iii) $\underline{\cos(2x) = \cos^2 x - \sin^2 x} = 1 - 2\sin^2 x$

(iv) $\lim_{z \to x} \cos z \underset{z = x+y}{=} \lim_{y \to 0} \cos(x+y) = \lim_{y\to 0} (\cos x \underbrace{\cos y}_{\downarrow\, 1} - \sin x \underbrace{\sin y}_{\downarrow\, 0}) = \cos x$

$\Rightarrow \cos \in C(\mathbb{R})$

<u>Definizione analitica di π.</u>

Sappiamo che $\cos 0 = 1$, $\cos \in C(\mathbb{R})$

<u>Es</u> $\underline{-0.417 < \cos 2 < -0.415}$

Dall'esercizio e dal teorema degli zeri per funzioni continue. $\cos \cdot [0,2]$ $\cos 0 = 1$, $\cos 2 < 0$

$\Rightarrow \exists!\ \beta > 0 \mid \cos x > 0 \quad \forall\ 0 \le x < \beta$ e $\cos\beta = 0$

<u>DEF.</u> $\underline{\pi := 2\beta} \in (3,4)$

<u>Oss</u> Sappiamo che $|\cos x - 1| \le \frac{7}{12}$, $\forall\ |x| \le 1$

$\Rightarrow \cos x - 1 \ge -\frac{7}{12} \Leftrightarrow \cos x \ge \frac{5}{12} \quad \forall\ |x| \le 1 \Rightarrow$ $\boxed{1 < \beta < 2}$

$2 < \pi < 4$

Dimostriamo (in maniera alternativa) che $\underline{\cos 2 < 0}$.

$$\cos 2 = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!} = \underset{k=0}{1} - \underset{k=1}{\frac{4}{2}} + \underset{k=2}{\frac{16}{24}} + \sum_{k=3}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!}$$

$$= -\frac{1}{3} - \underbrace{\sum_{k=3}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!}}_{>0} = -\frac{1}{3} - \left(\left(\frac{2^6}{6!} - \frac{2^8}{8!}\right) + \left(\frac{2^{10}}{10!} - \frac{2^{12}}{(12)!}\right) + \dots + \left(\frac{2^{2k}}{(2k)!} - \frac{2^{2k+2}}{(2k+2)!}\right) + \dots \right)$$

$$\frac{2^{2k}}{(2k)!} \left(1 - \frac{4}{(2k+2)(2k+1)}\right)$$

$k \ge 3$

$$> \frac{2^{2k}}{(2k)!} \left(1 - \frac{4}{56}\right)$$

$< -\frac{1}{3}$

$$\geq \frac{13}{14} \frac{2^{2k}}{(2k)!} > 0. \quad (k \geq 3)$$

Valori speciali $\cos\frac{\pi}{2} = 0$ (per definizione di π)

$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1.$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\sin x > 0$ per $x > 0$ vicino a zero; $\sin x < 0$ per $x < 0$ " "

F.S.A $\Rightarrow$ $\boxed{\cos\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2} \sin x}$ ($\Rightarrow \sin \in C(\mathbb{R})$) $\boxed{\cos\left(\frac{\pi}{2}-x\right) = \sin x}$

$\cos\left(x-\frac{\pi}{2}\right) = \sin x$

0 per $0 < x \leq \frac{\pi}{2}$

per $0 < x < \varepsilon$ $\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin x} = \sin\frac{\pi}{2} > 0 \Rightarrow \boxed{\sin\frac{\pi}{2} = 1}$

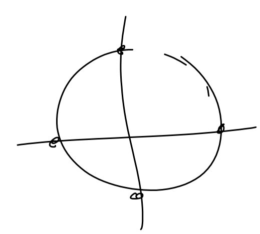
$\boxed{\sin x > 0 \text{ per } 0 < x \leq \frac{\pi}{2}}$

$\cos\pi = \cos\left(2\,\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2} = -1 \Rightarrow \sin\pi = 0$

$\cos 2\pi = \cos^2\pi - \sin^2\pi = 1$, $\sin 2\pi = 0$.

$\cos\frac{3}{2}\pi = \cos\left(2\pi - \frac{\pi}{2}\right) = \cos 2\pi \cancel{\cos\frac{\pi}{2}} + \cancel{\sin 2\pi} \sin\frac{\pi}{2} = 0$

$-1 = \cos\pi = \cos\left(\frac{3}{2}\pi - \frac{\pi}{2}\right) = \cancel{\cos\frac{3}{2}\pi \cos\frac{\pi}{2}} + \sin\frac{3}{2}\pi \sin\frac{\pi}{2} = \sin\frac{3}{2}\pi$



$\cos(2\pi + x) = \cos x$, $\cos x$ è una funzione periodica di periodo 2π.

$\sin(x+y) = \cos\left(x+y-\frac{\pi}{2}\right) = \cos\left(x-\frac{\pi}{2}+y\right) =$

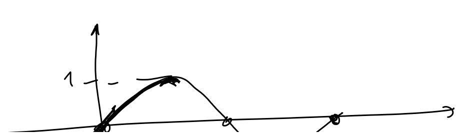
$= \cos\left(x-\frac{\pi}{2}\right)\cos y - \sin\left(x-\frac{\pi}{2}\right)\sin y$

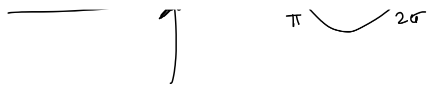
$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y}$

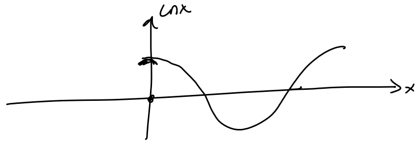
perché:

$\boxed{\sin\left(x-\frac{\pi}{2}\right) = -\cos x}$

$\cos x = \cos\left(\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}\right) = -\sin\left(x-\frac{\pi}{2}\right)\sin\frac{\pi}{2} = -\sin\left(x-\frac{\pi}{2}\right)$




π
2π


cos x
x