venerdì 16 ottobre 2020 08:07

$$(1)\quad \binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}, \qquad \binom{n}{k}=\frac{n!}{k!(n-k)!},\quad 0!=1 \qquad \forall\, 0\le k\le n,\quad k,n\in\mathbb{N}_0$$

$$\frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(k+1)\cancel{k!}}{(n-k)!\,\cancel{k(k-1)\cdots}} = \frac{n(n-1)\cdots(k+1)}{(n-k)(n-k-1)\cdots 1}$$

Nota. Dalla (1) segue che $\binom{n}{k}\in\mathbb{N}$ $\forall\ 0\le k\le n$

Per induzione su $n\in\mathbb{N}_0$

Base induttiva: $n=0$. $\binom{0}{0}=1\in\mathbb{N}$ ✓

Supponiamo $\binom{n}{k}\in\mathbb{N}$, $\forall\ 0\le k\le n$

Se $k=0$, $\binom{n+1}{0}=1$ ✓ $\qquad k\ge 1$. Per (1) possiamo scrivere $\binom{n+1}{k}=\underbrace{\binom{n}{k}}_{\in\mathbb{N}?}+\underbrace{\binom{n}{k-1}}_{\in\mathbb{N}?}\in\mathbb{N}$. ✓

Proposizione 144 (Formula del BINOMIO DI NEWTON).

$a,b\in\mathbb{R}$, $n\in\mathbb{N}_0$ Allora

$$(2)\qquad (a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}.$$

$n=0$ $\qquad (a+b)^0=1 \qquad \sum_{k=0}^{0}\underbrace{\binom{0}{0}}_{=1}\underbrace{a^0}_{=1}\underbrace{b^{0-k}}_{=0}=1$ ✓

$n=1$ $\qquad a+b=\sum_{k=0}^{1}\binom{1}{k}a^k b^{1-k}=\underbrace{\binom{1}{0}}_{=1}\underbrace{a^0}_{=1}\underbrace{b^{1-0}}_{=b}+\underbrace{\binom{1}{1}}_{=1}\underbrace{a^1}_{=a}b^{1-1}=b+a=a+b$ ✓

$n=2$ $\quad (a+b)^2 = \sum_{k=0}^{2} \binom{2}{k} a^k b^{2-k} = \underbrace{1 \cdot 1\, b^2}_{k=0} + 2ab + 1\, a^2 = a^2+b^2+2ab$ ✓

$$\binom{n}{1} = n \quad \forall n, \quad \frac{n!}{\cancel{(n-1)!}\,1!} = n$$

$$(a+b)^n = \overbrace{(a+b)(a+b)\cdots(a+b)}^{n \text{ volte}}$$

$ab \quad ab$

$a^k b^{n-k}$ ← monomio omogeneo di grado n

Dim per induzione su n.

I casi $n=0,1$ li abbiamo verificati. Assumiamo $n \geq 2$ e dimostriamo la (2) per $n+1$ (con n sostituito da $n+1$)

$$(a+b)^{n+1} = (a+b)(a+b)^n \underset{(2)_n}{=} (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\underset{(\text{sp})}{=} \sum_{k=0}^{n} \binom{n}{k} (a+b)\, a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left(a^{k+1} b^{n-k} + a^k b^{n-k+1}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \underset{k=n}{a^{n+1}} + \sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k} + \underset{k=0}{b^{n+1}} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

$$\sum_{k=0}^{n} (a_k+b_k) = \left(\sum_{k=0}^{n} a_k\right) + \left(\sum_{k=0}^{n} b_k\right)$$

si usa proprietà associativa e commutativa

$(a_0+b_0)+(a_1+b_1)$

$(a_0+a_1)+(b_0+b_1)$

$$= a^{n+1} + b^{n+1} + \sum_{h=1}^{n} \binom{n}{h-1} a^h b^{n-h+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

Cambio indice

$h = k+1$, $k = h-1$

$0 \le k \le n-1$, $1 \le h \le n$

Ho usato due cose

① $\sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k = \sum_{h=1}^{n} a_h$

② $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$

$\sum_{h=1}^{n} a_h + \sum_{j=1}^{n} b_j = \sum_{p=1}^{n} (a_p + b_p)$

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n-k+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) a^k b^{n+1-k}$$

$$\overset{(1)}{=} \underbrace{a^{n+1}}_{k=n+1} + \underbrace{b^{n+1}}_{k=0} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} . \quad \blacksquare$$

Somme geometriche (fatte mercoledì 14/10/20)

$$x \in \mathbb{R},\ n \in \mathbb{N}_0,\quad \sum_{k=0}^{n} x^k = 1^0 + x^1 + x^2 + \cdots + x^n = \begin{cases} \dfrac{1-x^{n+1}}{1-x} = \dfrac{x^{n+1}-1}{x-1}, & x \neq 1 \\ n+1, & x = 1 \end{cases}$$

$$\boxed{\sum_{k=1}^{n} x^k = \frac{x-x^{n+1}}{1-x}}\,, \quad x\neq 1$$

(x = "ragione della somma geometrica")

Dimostrazione alternativa

$$S = \sum_{k=1}^{n} x^k = x + x^2 + x^3 + \dots + x^n \qquad (1-x)S = S - xS = x - x^{n+1}$$

$$xS = x^2 + x^3 + \dots + x^n + x^{n+1}$$

$$\Downarrow x\neq 1$$

$$\boxed{S = \frac{x-x^{n+1}}{1-x}}$$

Proposizione 1.40 $x, y\in\mathbb{R}$, $n\in\mathbb{N}$. Allora

(4) $$x^n - y^n = (x-y)\sum_{k=0}^{n-1} \boxed{x^{n-1-k}\,y^k}$$

monomio generico (al variare di k) di grado $n-1$ in x e y

$$\sum_{k=0}^{n-1} x^{\overset{j}{n-1-k}}\,y^k \overset{?}{=} \sum_{k=0}^{n-1} y^{\overset{j}{n-1-k}}\,x^k$$

$x^0 = 1$, $\forall x\in\mathbb{R}$

$y=0$

$$x^n = x\sum_{k=0}^{n-1} x^{n-1-k}\,0^k = x\cdot(\underset{k=0}{x^{n-1}} + 0) = x^n \quad ✓$$

stesso caso per $x=0$

Da adesso studiamo $x\neq 0$ e $y\neq 0$

Divido per y^n

$$\left(\frac{x}{y}\right)^n - 1 = \left(\frac{x}{y}-1\right)\sum_{k=0}^{n-1} x^{n-1-k}\,y^{k-n+1}$$

$$k-n+1 = -(n-1-k)$$

$$\frac{1}{y} \qquad \frac{1}{y^{n-1}}$$

$$= \left(\frac{x}{y}-1\right)\left(\frac{x}{y}\right)^{n-1-k}$$

$$\sum_{k=0}^{n-1} x^{n-1-k}\,y^k = \sum_{j=0}^{n-1} x^j\,y^{n-1-j} = \sum_{k=0}^{n-1} y^{n-1-k}\,x^k$$

$j = n-1-k$

$k=0 \Longleftrightarrow j=n-1$

$k=n-1 \Longleftrightarrow j=0$

$t = \frac{x}{y} \in \mathbb{R}\setminus\{0\}$ la (4) è equivalente a

$$(5) \quad t^n - 1 = (t-1) \sum_{k=0}^{n-1} t^{n-1-k} \quad , \quad \left(t = \frac{x}{y}\right)$$

Se $t=1$ la (5) è vera

Se $t \neq 1$ $\quad (5) \Leftrightarrow \sum_{k=0}^{n-1} t^{n-1-k} = \frac{t^n - 1}{t-1} = \frac{1-t^n}{1-t}$

$$\sum_{h=1}^{n} t^{n-h} \neq \sum_{j=1}^{n} t^j \qquad \sum_{k=1}^{n} t^k \neq \frac{1-t^n}{1-t}$$

$h = k+1 \qquad j = n-h$

$$\underset{j=n-h}{\hookrightarrow} \quad \sum_{j=0}^{n-1} t^j \overset{(4)}{=} \frac{1-t^n}{1-t} \quad \checkmark$$

$h=1 \Leftrightarrow j=n-1$

$h=n \Leftrightarrow j=0$

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N}) \qquad \mathbb{Q} := \left\{x = \frac{p}{q} \;\middle|\; p \in \mathbb{Z},\ q \in \mathbb{N}\right\}$$

$$\frac{p}{q} = \frac{pn}{qn} \quad , \quad \frac{pn}{qn} := (pn)(qn)^{-1} = (pn)\cdot(q^{-1}n^{-1}) =$$

$$= (p\cdot q^{-1})(n\cdot n^{-1}) =: (p\, q^{-1})\, 1 = p\cdot q^{-1} = \frac{p}{q}$$

$$\frac{p}{q} + \frac{n}{m} = \frac{pm}{q\cdot m} + \frac{n\cdot q}{m\, q} = \frac{\overset{\mathbb{Z}}{pm} + \overset{\mathbb{Z}}{nq}}{\underset{\mathbb{N}}{m\cdot q}} \in \mathbb{Q}, \qquad q, m \in \mathbb{N}$$

$$-n = (-1)\cdot n$$

$$\frac{p}{q}\cdot\frac{n}{m} = p\, n\, q^{-1} m^{-1} = p\cdot n\,(qm)^{-1} = \frac{pn}{qm}$$

Morale: $\mathbb{Q}$ è un campo ordinato: verifica i primi 15 assiomi dei numeri reali. Ma NON il 16° assioma.

Esempio

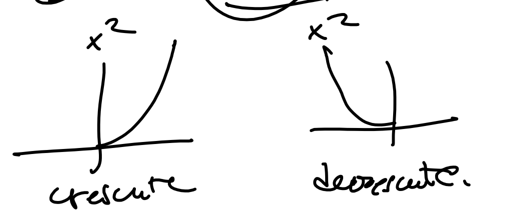
$$A = \{ r \in \mathbb{Q} \mid r < 0 \text{ oppure } r \geq 0 \text{ e } r^2 < 2 \}$$

$$B = \{ r \in \mathbb{Q}_+ \mid r^2 > 2 \}$$

$\mathbb{Q}_+$: $r \in \mathbb{Q}$ e $r > 0$.

$r \in A$ e $s \in B \Rightarrow r < s \Leftarrow r, s \geq 0$, $r \in A$: $r^2 < 2 < s^2 \Rightarrow s^2 - r^2 > 0$

$$s^2 - r^2 = (s-r)\underset{>0}{(s+r)} \Rightarrow s - r > 0$$



Oss: $a, b \geq 0$ allora $a^n \geq b^n \Longleftrightarrow a \geq b$. (Es.!)

$x^n : \{t \geq 0\} \to \{t \geq 0\}$ è una funzione strett. crescente

$\underset{\text{dom.}}{\uparrow}$ codominio $\quad (x < y \Longrightarrow f(x) < f(y))$

Ma $\exists ! \ \bar{x} \in \mathbb{R} \mid \ x < \bar{x} < y, \ \forall \ x \in A \ \forall \ y \in B \qquad (\bar{x} = \sqrt{2} \notin \mathbb{Q}.)$