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LIMITI "NOTEVOLI" DI SUCCESSIONI

$a: n \in \mathbb{N} \mapsto a_n \in \mathbb{R}$, $\mathcal{D}^* \mathbb{N} = \{+\infty\}$

$\lim_{n \to +\infty} a_n = L \in \mathbb{R}^* \iff \forall$ intorno U di L $\exists N > 0 \mid a_n \in U, \forall n \geq N$

Un esercizio dello esonero era trovare $N \mid 3^n > n^3 \quad \forall n \geq N$

esponenziale, potenza

Def P_n è DEFINITIVAMENTE VERA $\iff \exists N \mid P_n$ è vera $\forall n \geq N$.

(i) $a > 1$, $p \in \mathbb{N}_0$, $\lim \frac{a^n}{n^p} = +\infty$

$\boxed{\text{Es trovare } N \mid 2^n > n^{10}, \forall n \geq N}$ $\left(n = 10 \quad 2^{10} = 1024, \quad 10^{10} \right)$ $\frac{1024}{10^{10}} \sim \frac{1}{10^7}$

Vi ricordo il lemma di Bernoulli

$(1+x)^n \geq 1 + nx \quad \forall x \geq -1, \forall n \in \mathbb{N}$; $x \geq -1, x \neq 0$ $n \geq 2$ vale la dis stretta

$x > 0$ $n \geq 2$

esponenziale, retta (potenza)

Dim (i) $p = 0$ deriva da Bernoulli, $a = 1 + x$, $x > 0$

$p = \frac{1}{2}$ $\quad \frac{a^n}{\sqrt{n}} = \frac{(1+x)^n}{\sqrt{n}} \overset{Ber}{\geq} \frac{1+nx}{\sqrt{n}} > \frac{nx}{\sqrt{n}} = \sqrt{n}\, x \longrightarrow +\infty$, per $n \to \infty$

$x = a - 1 > 0$

$\left(\sqrt{n}\, x > M \iff n > \left(\frac{M}{x}\right)^2 = N. \right)$

$p \in \mathbb{N}$, $\quad \frac{a^n}{n^p} = \frac{a^n}{(\sqrt{n})^{2p}} = \left(\frac{\left(\sqrt[2p]{a}\right)^n}{\sqrt{n}} \right)^{2p} = \left(\frac{b^n}{\sqrt{n}} \right)^{2p} \xrightarrow{n \to +\infty} +\infty$ $\quad b = a^{\frac{1}{2p}} > 1$

usa l'algebra dei limiti

$a_n \to +\infty \implies a_n^m = \underbrace{a_n \cdots a_n}_{m \text{ volte}} \to +\infty$

Es $\frac{2^n}{n^{10}} = \frac{2^n}{\sqrt{n}^{20}} = \left(\frac{\left(2^{\frac{1}{20}}\right)^n}{\sqrt{n}} \right)^{20} \overset{B}{\geq} \left(\frac{1+nx}{\sqrt{n}} \right)^{20}$ $\quad 2^{\frac{1}{20}} = 1 + x$, $x = 2^{\frac{1}{20}} - 1$

$> (\sqrt{n}\, x)^{20} = n^{10} x^{20} > 1 \quad n > \frac{1}{x^2} = N$

$2^{\frac{1}{20}} - 1 = 0{,}0352649\ldots$

$\frac{1}{20} \quad 1{,}0352649\ldots$

$2^{..} = 1. ...$ $N = 817$

$\frac{1}{x^2} < \frac{1}{(0{,}035)^2} = \frac{1}{0.001225} \approx 816{,}32\ldots$

$\forall n \geq 817 \quad \frac{2^n}{n^{10}}$

(potreste verificare con conto e penna che $2^{\frac{1}{20}} - 1 > 0{,}035$

$2 > (1 + 0{,}035)^{20}$

(ii) $0 < a < 1$ $\quad a^n n^p \longrightarrow 0 \quad \forall p \in \mathbb{N}_0$ $\quad$ ($a_n \to L$ "tende")

$\left(\lim_{n\to+\infty} a^n n^p = 0\right)$

$a^n n^p > 0 \qquad a^n n^p \to 0 \iff \frac{1}{a^n n^p} \to +\infty$

ATTENZIONE $\quad a_n > 0 \qquad a_n \to 0 \iff \frac{1}{a_n} \to +\infty$

(In generale non è vero che $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$
$a_n = -\frac{1}{n} \to 0$ ma $\frac{1}{a_n} = -n$ non ha limite).

$\frac{1}{a^n} \frac{1}{n^p} = \left(\frac{1}{a}\right)^n \frac{1}{n^p} \to +\infty$, $\frac{1}{a} > 1$, per (i).

(iii) $\sqrt[n]{n} \longrightarrow 1$

$1 < \sqrt[n]{n}$. Sia $\varepsilon > 0$ vogliamo dimostrare che definitivamente

$1 - \varepsilon < \sqrt[n]{n} < 1 + \varepsilon \qquad n < (1+\varepsilon)^n \qquad 1 < \frac{(1+\varepsilon)^n}{n} \to +\infty$ (i)

(iv) $a^{\frac{1}{n}} \to 1$, $\forall a > 0$

$1 < a \qquad 1 < a^{\frac{1}{n}} < n^{\frac{1}{n}} \to 1 \qquad (\forall n > a)$

1

$a = 1$ banale

$0 < a < 1$ per punto precedente + A.d.L

$a^{\frac{1}{n}} = \frac{1}{..} \to \frac{1}{.}$

$(a^{-1})^n$

NUMERO DI NEPERO o DI EULERO

Lemma* $e_n := \left(1+\frac{1}{n}\right)^n$, $E_n := \left(1+\frac{1}{n}\right)^{n+1} = e_n\left(1+\frac{1}{n}\right)$ Allora

(a) $1 < e_n < E_n$ ✓

(ii) $e_n \nearrow$ (e_n è strett. crescente $\Leftrightarrow$ $e_n < e_{n+1}$ $\forall n$ $\Leftrightarrow$ $e_n < e_m$ $\forall$ $n<m$)

(iii) $E_n \searrow$ (E_n è strett decrescente).

$?\ \left(1+\frac{1}{n}\right)^n \to e \in [1,+\infty]$

DEF $e := \lim e_n = \lim E_n$ NUMERO DI NEPERO

$e^{\pi i} = -1$ Eulero

$e_n \nearrow$ $\forall m$ $e_n < E_m$ $\sup e_n \le E_m$ $\forall m$.

$\sup e_n = \lim e_n$.

$\Rightarrow \exists \lim e_n \le E_m \ \forall m$ $E_1 = 4$

$\exists \lim E_m \ge e_m \ \forall m$

$e_1 = 2 < e < 4 = E_1$

$e_2 = \left(1+\frac{1}{2}\right)^2 = \frac{9}{4} = 1.25$ $E_2 = \left(1+\frac{1}{2}\right)^3 = \frac{27}{8} = 3+\frac{3}{8}$

$2.25 < e < 3+\frac{3}{8}$ $E_5 = 2,985984$

$e = 2.71$

Dim. lemma ($n \ge 2$) $\frac{e_n}{e_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \left(1+\frac{1}{n}\right)^n\left(\frac{n-1}{n}\right)^{n-1} = \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^n \cdot \left(\frac{n}{n-1}\right)$

$1+\frac{1}{n-1} = \frac{n-1+1}{n-1} = \frac{n}{n-1}$

$= \left(\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\right)^n = \left(1-\frac{1}{n^2}\right)^n \frac{n}{n-1}$

$> \left(1-\frac{n}{n^2}\right)\frac{n}{n-1} = \left(1-\frac{1}{n}\right)\frac{1}{1-\frac{1}{n}} = 1$

Bernoulli $n \ge 1$ $x \ge -1, x \ne 0$

$n \ge 2$, $\frac{E_{n-1}}{E_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} \frac{\left(\frac{n-1+1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^n}\frac{1}{\left(1+\frac{1}{n}\right)} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^n}\frac{1}{1+\frac{1}{n}} = \frac{1}{\left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^n}\frac{1}{1+\frac{1}{n}}$

è giusto ma non serve

$$\frac{E_{n-1}}{E_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^n}{\underline{\left(1+\frac{1}{n}\right)}} = \frac{\left(\frac{n^2}{n^2-1}\right)^n}{1+\frac{1}{n}} = \frac{\left(\frac{n^2-1+1}{n^2-1}\right)^n}{1+\frac{1}{n}}$$

$$= \frac{\left(1+\frac{1}{n^2-1}\right)^n}{1+\frac{1}{n}} > \frac{1+\frac{n}{n^2-1}}{1+\frac{1}{n}} > \frac{1+\frac{1}{n}}{1+\frac{1}{n}} = 1 \quad \checkmark$$

ES 30 p 66 [GE]

$$\lim_{n\to+\infty} n\left(\sqrt{1+\frac{2}{n^2}} - \sqrt{1-\frac{4}{n}}\right)$$

$n \to +\infty$, $\sqrt{1+\frac{2}{n^2}} \to 1$, $\sqrt{1-\frac{4}{n}} \to 1$, $\underbrace{1 \quad 1}_{\to 0}$ alg. di limiti

$$n\left(\sqrt{1+\frac{2}{n^2}} - \sqrt{1-\frac{4}{n}}\right) = n\,\frac{\cancel{1}+\frac{2}{n^2}-\cancel{1}+\frac{4}{n}}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{4}{n}}} = \frac{\overset{0}{\frac{2}{n}}+4}{\underbrace{\sqrt{1+\frac{2}{n^2}}}_{\to 1}+\underbrace{\sqrt{1-\frac{4}{n}}}_{\to 1}} \to 2$$

24. $\lim_{n\to+\infty} \sqrt[n]{2n^5+1} = 1$

$$(2n^5+1)^{\frac{1}{n}} = (2n^5)^{\frac{1}{n}} \left(\left(1+\frac{1}{2n^5}\right)\right)^{\frac{1}{n}} \to 1$$

$$(2n^5)^{\frac{1}{n}} = \underbrace{2^{\frac{1}{n}}}_{\to 1}\underbrace{\left(n^{\frac{1}{n}}\right)^5}_{\to 1}$$

$$1 < \underbrace{\left(1+\frac{1}{2n^5}\right)^{\frac{1}{n}}}_{\to 1} < \underbrace{2^{\frac{1}{n}}}_{\to 1}$$

VENERDÌ

$$\boxed{\left(1+\frac{1}{a_n}\right)^{a_n} \to e \quad \text{se } a_n \to +\infty}$$