lunedì 2 novembre 2020 08:17

DEFINIZIONE DI LIMITE

Data una funzione $f : A \subseteq \mathbb{R} \to \mathbb{R}$,

$x_0 \in \mathcal{D}^* A$, $L \in \mathbb{R}^*$

{punti di accumulazione in $\mathbb{R}^*$ di A}.

$\lim_{x \to x_0} f(x) = L \iff \forall$ intorno V di L $\exists$ U intorno di x_0 :

$x \in \underbrace{U \cap A \setminus \{x_0\}}_{\neq \emptyset} \Rightarrow f(x) \in V$

$\neq$ ← PERCHÉ $x_0 \in \mathcal{D}^* A$

Caso finito $x_0, L \in \mathbb{R}$. $V = (L-\varepsilon, L+\varepsilon) = I_\varepsilon(L) = \{y \mid |y-L| < \varepsilon\}$

fissiamo scegliendo $\varepsilon > 0$

$\exists \quad U = (x_0-\delta, x_0+\delta) = I_\delta(x_0) = \{x \in \mathbb{R} \mid |x-x_0| < \delta\}$

troviamo δ DIPENDE da ε

$0 < |x - x_0| < \delta$, $x \in A \Rightarrow |f(x) - L| < \varepsilon$.

$x \neq x_0$ $x \in I_\delta(x_0)$

Casi in cui x_0 o L sono $\pm\infty$.

$x_0 = +\infty$, $\underline{L \in \mathbb{R}}$

$\lim_{x \to +\infty} f(x) = L \iff \forall \varepsilon > 0, \left(I_\varepsilon(L) \right) \; \exists \; \underline{U \text{ intorno di } +\infty}$

$\Updownarrow$

$(\forall \text{ intorno di } L)$

ossia $\exists \; \underline{M > 0}$ $\left(U = \{y > M\} \right)$ |

$\boxed{x > M}$ $\left(\underline{x \in U} \right)$ e $\underline{x \in A} \Rightarrow |f(x) - L| < \varepsilon$

$\Downarrow$ $\Updownarrow$

$x \in \mathbb{R}$ $f(x) \in V$

$(x \neq +\infty)$

$x_0 = -\infty$, $\underline{U = \{y < M\}}$, $\underline{M < 0}$

intorno di $-\infty$.

$\boxed{x < M}$

$x_0 \in \mathbb{R}$, $L = +\infty$

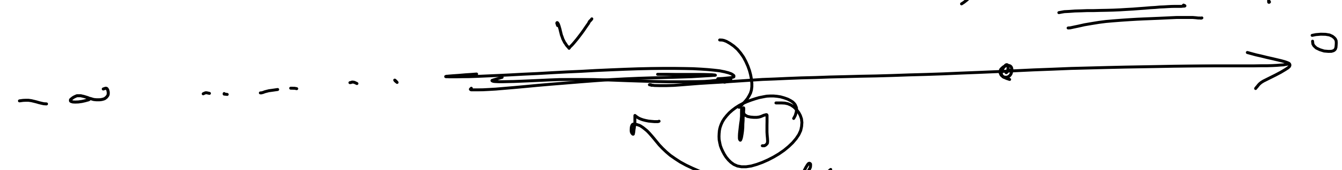
$\forall V \text{ intorno di } L \iff \forall M > 0$ $\left(V = (M, +\infty) = \{y > M\} \right)$

$\exists \; \delta > 0$ | $0 < |x - x_0| < \delta$, $x \in A \Rightarrow (f(x) > M)$

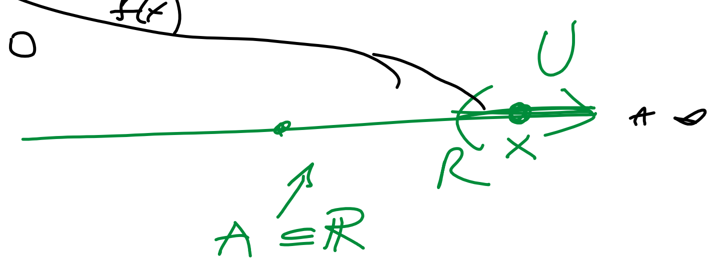
$L = -\infty \quad \forall M < 0 \qquad f(x) < M.$

$x_0 = +\infty, \quad L = -\infty$

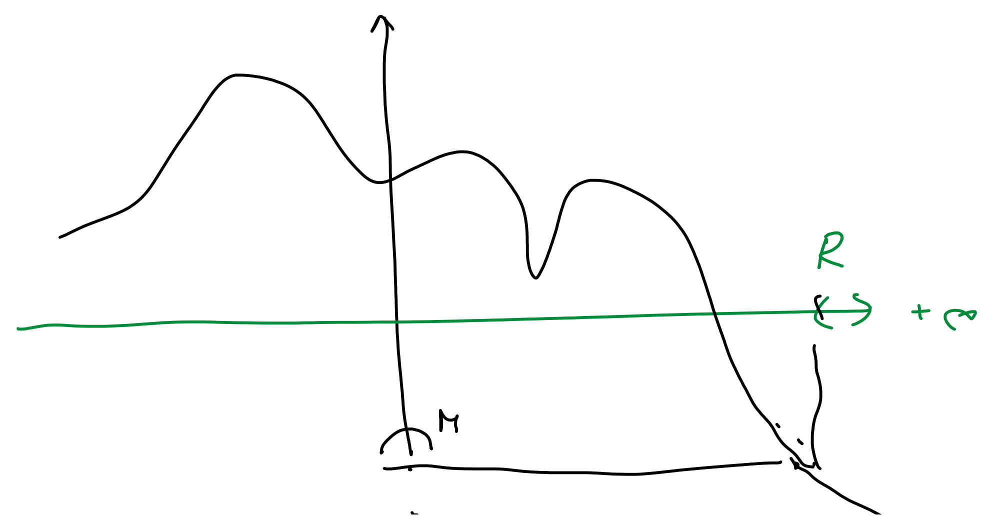
$\forall V$ intorno di $L \iff \forall M < 0 \quad ((-\infty, M))$ contenuto in $\mathbb{R}$



$\exists$ un intorno U di x_0 cioè $\exists R > 0$



$x > R, \; x \in A \implies f(x) < M.$



ESEMPI ED ESERCIZI

1. $f_n = x_n$ $\quad A = \mathbb{N} \Rightarrow \mathcal{D}^*\mathbb{N} = \{+\infty\} \Leftrightarrow +\infty$ è l'unico p.to di acc. di $\mathbb{N}$
$\Leftrightarrow \mathbb{N}$ non è limitato (proprietà archimedea)

$x_n = \frac{1}{n}$, $\quad \lim_{n\to+\infty} \frac{1}{n} = 0.$

$L = 0 \qquad x_0 = +\infty$

$\forall\, \varepsilon > 0$ dobbiamo trovare $N > 0 \;|\; -\varepsilon < \frac{1}{n} < \varepsilon$, $\quad \forall n \in \mathbb{N}, n > N$

$0 <$

$V = \{y > N\}$

$A = \mathbb{N}$

$V \cap A = \{n \in \mathbb{N} \;|\; n > N\}$

$n > \frac{1}{\varepsilon} = N$

se $n > N := \frac{1}{\varepsilon} > 0 \Longrightarrow 0 < \frac{1}{n} < \varepsilon$ ossia $x_n = \frac{1}{n} \in V$.

2. $x_n = \frac{n}{n^3 + n + 1}$, $\quad n \in \mathbb{Z}$ (n.b. $n^3 + n + 1 \neq 0$, $\forall n \in \mathbb{Z}$)

$\mathcal{D}^*\mathbb{Z} = \{-\infty, +\infty\}$

$\lim_{n\to+\infty} x_n \overset{?}{=} 0$, $\quad$ Fissiamo $\varepsilon > 0$ (ossia fissiamo un intorno arbitrario di 0)

$n \to +\infty$

Consideriamo solo $n > 0$

$$I_\varepsilon(0) = (-\varepsilon, \varepsilon)$$

$$x_n > 0$$

$$-\varepsilon \lneq x_n < 0$$

$$\frac{n}{n^3+n+1} < \varepsilon \quad ?$$

$$\frac{n}{n^3+n+1} < \frac{n}{n^3} = \frac{1}{n^2} < \varepsilon$$

$$\Updownarrow$$

$$n^2 > \frac{1}{\varepsilon} \iff n > \frac{1}{\sqrt{\varepsilon}} = N$$

Se $n > \frac{1}{\sqrt{\varepsilon}} \implies \frac{n}{n^3+n+1} < \varepsilon$

$$\lim_{n \to -\infty} \frac{n}{n^3+n+1} \overset{?}{=} 0$$

$$0 < \frac{n}{n^3+n+1} < \frac{n}{m}$$

$$n^3+n+1 < 0$$

$$n \leq -1$$

$$\boxed{n+1 \leq 0}$$

"Sottoesercizio" !!

$$\boxed{\frac{a}{b} < \frac{a}{c}}, \quad a, b, c < 0$$

moltiplico per $b \cdot c > 0$

$$ac < ab$$

$$\boxed{c > b}$$

$$\frac{n}{n^3+n+1} \leq \frac{n}{n^3} = \frac{1}{n^2} < \varepsilon \qquad n^3 \geq n^3+n+1$$

$$\boxed{\text{se } n \leq -1}$$

$$\cancel{n^3} \geq \cancel{n^3} + n \quad \forall\, n \leq 0 \Rightarrow \frac{n}{n^3+n+1} \leq \frac{n}{n^3+1} < \varepsilon$$

$$\frac{1}{n^2} < \varepsilon, \quad n^2 > \frac{1}{\varepsilon} \quad |n| > \frac{1}{\sqrt{\varepsilon}} \iff n < -\frac{1}{\sqrt{\varepsilon}}$$

$$|n| = -n$$

$$n \leq -1 \qquad n < -\frac{1}{\sqrt{\varepsilon}} \Longrightarrow 0 < \frac{n}{n^3+n+1} < \varepsilon$$

$$N = -\frac{1}{\sqrt{\varepsilon}} \left(\Longrightarrow n \leq -1 \right)$$

Soluzione alternativa

$$\frac{n}{n^3+n+1}, \quad n < 0 \qquad n = -m \qquad m := -n \qquad n < 0 \qquad -n = m \in \mathbb{N}$$

"cambio di variabile"

$x_n = \frac{n}{n^3+n+2} = \frac{-m}{-m^3-m+1} = \frac{m}{m^3+m-1} = y_m$ $\quad (n \to -\infty \iff m \to +\infty)$

$\frac{m}{m^3+m-1} \leq \frac{1}{m^2} < \varepsilon \qquad m > \frac{1}{\sqrt{\varepsilon}} \iff \boxed{-n > \frac{1}{\sqrt{\varepsilon}}} \qquad n < -\frac{1}{\sqrt{\varepsilon}} = N.$

a $m \geq 1$

ES 9 (I ES. AA 19-20) Sia $f(x) = \frac{x^2-1}{x^2+x}$. Dimostrare che $\forall \varepsilon > 0$

$\exists M \mid |f(x)-1| < \varepsilon, \ \forall x \geq M$

(OSS $L=1$, $x_0 = +\infty$, $\lim_{x\to+\infty} f(x) = 1$)

Ragionamento intuitivo $\quad x^2-1 \sim x^2$ per $x \to +\infty \qquad x^2+x \sim x^2$ per $x \to +\infty$

$\frac{x^2-1}{x^2+x} = \frac{x^2-1}{x(x+1)} = \frac{x-1}{x} = 1 - \frac{1}{x}$

$x > 0$, $|f(x)-1| = \frac{1}{x} < \varepsilon \qquad x > \frac{1}{\varepsilon} = M$

$$\lim_{x\to-\infty} 1-\frac{1}{x} = 1 \qquad |f(x)-1| = \left|\frac{1}{x}\right| = \frac{1}{-x} < \varepsilon \iff \frac{1}{\varepsilon} < -x$$

$$\left|-\frac{1}{x}\right| \qquad \underline{\underline{x<0.}} \qquad \iff x < -\frac{1}{\varepsilon} = M < 0$$

Es 10* dis $f(x) = \underbrace{(1-\sqrt{2}x)^{-2/3}}$. Trovare $\delta \mid |f(x)-1| < 10^{-10}$, $\forall\ |x|<\delta$

(I ES AA 18-19)

$$\left|\frac{1}{(1-\sqrt{2}x)^{2/3}} - 1\right| < 10^{-10}$$

$$\left|\frac{1-(1-\sqrt{2}x)^{2/3}}{(1-\sqrt{2}x)^{2/3}}\right| < 2\left|1-(1-\sqrt{2}x)^{2/3}\right|$$

se $\underline{\underline{x < \frac{1}{4}}}$

$(1-\sqrt{2}x)^{2/3} > \frac{1}{2} \iff \underbrace{1-\sqrt{2}x}_{>0} > \frac{1}{2^{3/2}}$

$(1-\sqrt{2}x)>0$

$1-\frac{1}{2^{3/2}} > \sqrt{2}x \qquad \underline{\frac{1}{\sqrt{2}}-\frac{1}{4} > x}$

$\frac{1}{\sqrt{2}} - \frac{1}{4} > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

si $x < \frac{1}{4} \Rightarrow x < \frac{1}{\sqrt{2}} - \frac{1}{4} \Rightarrow (1-\sqrt{2}x)^{2/3} > \frac{1}{2}$

$$\left| 1 - \underbrace{(1-\sqrt{2}x)^{2/3}}_{y} \right| < 2\,10^{-10}$$

$$|1-y| < 2\,10^{-10}$$

$$-2\,10^{-10} < y-1 < 2\,10^{-10} \qquad 1-2\cdot10^{-10} < y < 1+2\cdot 10^{-10}$$

$$1-2\cdot10^{-10} < (1-\sqrt{2}x)^{2/3} < 1+2\cdot10^{-10}$$

$$(1-2\cdot10^{-10})^{3/2} < 1-\sqrt{2}x < (1+2\cdot10^{-10})^{3/2}$$

$$\frac{-1+(1-2\cdot10^{-10})^{3/2}}{\sqrt{2}} < -x < \frac{-1+(1+2\cdot10^{-10})^{3/2}}{\sqrt{2}}$$

$$-\left(\frac{-1+(1+2\,10^{-10})^{3/2}}{\sqrt{2}}\right) < x < \frac{-(1-2\cdot10^{-10})^{3/2}+1}{\sqrt{2}} = \frac{1-(1-2\cdot10^{-10})^{3/2}}{\sqrt{2}}$$

$$\Big\{ 1-(1-2\cdot10^{-10})^{3/2} \quad \frac{(1+2\cdot10^{-10})^{3/2}-1}{} \Big\}$$

$$\delta = \min\left\{ \sqrt{2} \,,\quad \sqrt{2} \right\}$$

<u>Es.</u> Chi è δ?