mercoledì 2 dicembre 2020 08:15

FUNZIONI IPERBOLICHE

DEF seno iperbolico $\sinh x = \mathrm{sh}\, x := \frac{e^x - e^{-x}}{2}$

coseno " $\cosh x := \mathrm{ch}\, x := \frac{e^x + e^{-x}}{2}$

tangente " $\tanh x := \mathrm{th}\, x = \frac{\mathrm{sh}\, x}{\mathrm{ch}\, x}$

cotangente " $\mathrm{ctanh}\, x := \mathrm{cth}\, x = \frac{\mathrm{ch}\, x}{\mathrm{sh}\, x}$

Relazione con le funzioni trigonometriche è evidente in $\underline{\underline{\mathbb{C}}}$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$e^{i\pi} = -1$!!!

$$\boxed{\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

Proprietà:

(i) $\mathrm{ch}\, x = \mathrm{ch}(-x) \quad \forall x$ (funzione PARI)

$\mathrm{sh}(-x) = -\mathrm{sh}\, x$ (" DISPARI)

$\mathrm{th}(-x) = -\mathrm{th}\, x$ "

$\mathrm{cth}(-x) = -\mathrm{cth}(x)$

(ii) formule di addizione $\mathrm{ch}(x+y) = \mathrm{ch}\, x\, \mathrm{ch}\, y \underline{\underline{+}}\, \mathrm{sh}\, x\, \mathrm{sh}\, y \longleftarrow$

(iii) " " $\mathrm{sh}(x+y) = \mathrm{sh}\, x\, \mathrm{ch}\, y + \mathrm{sh}\, y\, \mathrm{ch}\, x$

(iv) $\mathrm{ch}^2 x \underline{\underline{-}} \mathrm{sh}^2 x = 1$

Verifiche elementari $\left(\overset{\text{regola da}}{e^a e^b = e^{a+b}} \right)$

Esempio $\mathrm{ch}^2 x - \mathrm{sh}^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - (\cancel{e^{2x}} + \cancel{e^{-2x}} - 2)}{4} = 1$$

(v) $x \in \mathbb{R} \mapsto \mathrm{sh}\, x$ è strett. crescente

$x \in [0, +\infty) \mapsto \mathrm{ch}\, x$ è " "

Dim $\frac{1}{2}(e^x - e^{-x})$

strett. crescente

strett. decresc.

strett. crescente

strett. crescente.

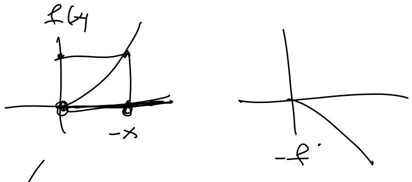
N.B. se $f(x)$ è crescente $\underline{f(-x) \text{ è decrescente}}$

$x < y$

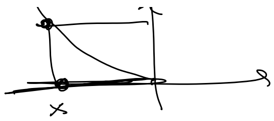
$-x > -y$

$- f(-x)$ crescente.

$- f(x)$ è $\underline{\text{decrescente.}}$



✓ $f(-x)$



$\underline{\underline{x \geq 0}}, \ y>0$ $\qquad$ $\text{th}x = \dfrac{\text{sh}x}{\text{ch}x}, \ \forall x \in \mathbb{R}$, $\quad \underbrace{\dfrac{e^x+e^{-x}}{2}}_{} \geq 1$ $\ (>0)$

$$\text{ch}(x+y) > \text{ch}\, x$$

$\text{sh}0 = \text{th}0 = 0$

Infatti,

$$\text{ch}x\,\text{ch}y + \text{sh}x\,\text{sh}y = \text{ch}x\left(\text{ch}y + \underbrace{\text{th}x\ \text{sh}y}_{\geq 0}\right) \geq \underset{\geq 0}{\text{ch}x}\ \underset{\geq 1}{\text{ch}y} > \text{ch}x$$

N.B

$$\frac{e^x + e^{-x}}{2} \overset{?}{>} 1 \quad \forall\ x>0 \quad (x \neq 0)$$

$$\Updownarrow$$

$$e^x + e^{-x} - 2 \overset{?}{>} 0$$

$$\|$$

$$\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2 > 0 \ \checkmark \qquad , \quad e^{\frac{x}{2}} > 1 > e^{-\frac{x}{2}}$$

Sono tutte funzioni continue sul dominio di definizione $\left(\text{cotanh}\,x = \dfrac{\text{ch}x}{\text{sh}x} \text{ definita in } \mathbb{R}\setminus\{0\}\right)$

Esempi di limiti a $\pm\infty$

$$\text{ch}x, \ \text{sh}x \sim \frac{e^x}{2} \quad \text{per } x \to +\infty$$

$$\lim_{x\to+\infty} \frac{\text{ch}x}{\frac{e^x}{2}} = 1$$

$f \sim g$ vicino a x_0

$$\Updownarrow$$

$$\lim_{x \to x_0} \frac{f}{g} = 1$$

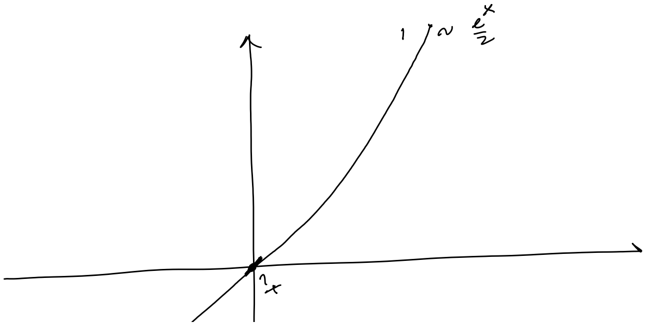
$$\frac{\frac{e^x + e^{-x}}{\cancel{2}}}{\frac{e^x}{\cancel{2}}} = (e^x + e^{-x})\,e^{-x} = 1 + e^{-2x} \xrightarrow[x\to+\infty]{} 1.$$

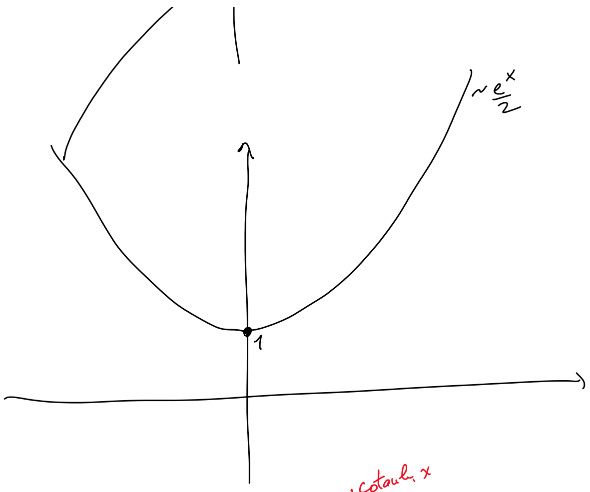
$$\lim_{x\to+\infty} \tanh x = 1$$

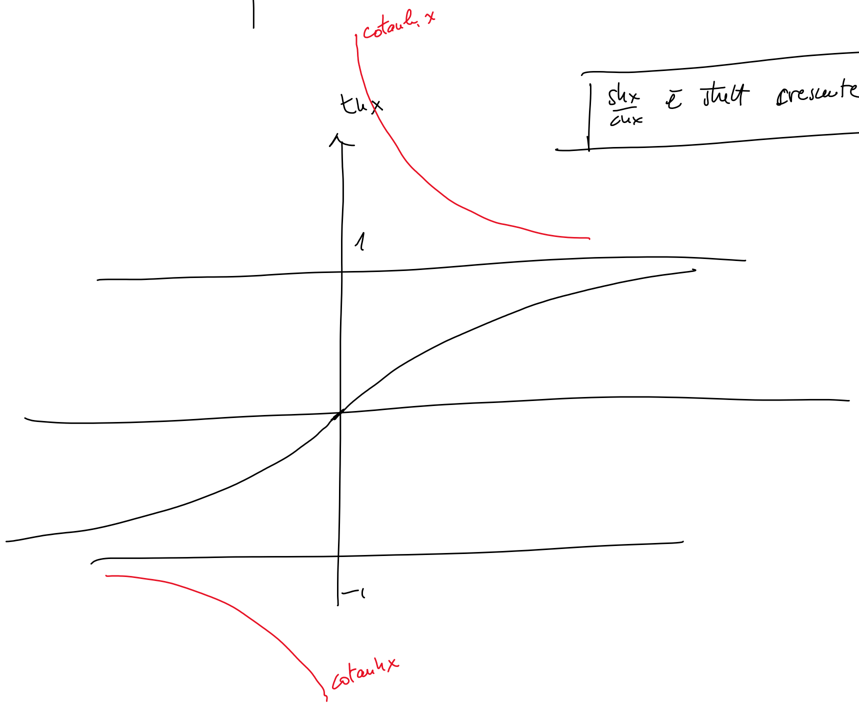
$$= \frac{\text{sh}x}{\text{ch}x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \xrightarrow[x\to+\infty]{} 1$$

GRAFICI

$\boxed{\text{sh}x \sim x \quad \text{per } x \to 0}$







$\boxed{\frac{\text{sh}x}{\text{ch}x} \text{ è strett crescente}}$

FUNZIONI IPERBOLICHE INVERSE

Es funzione inversa del sh x che è strett. cresc. su tutto $\mathbb{R}$.

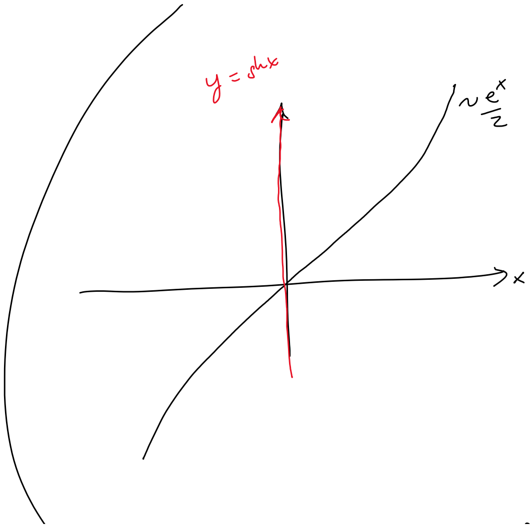
$x \in \mathbb{R} \mapsto \text{sh}^{-1} x = \text{arcsenh } x$

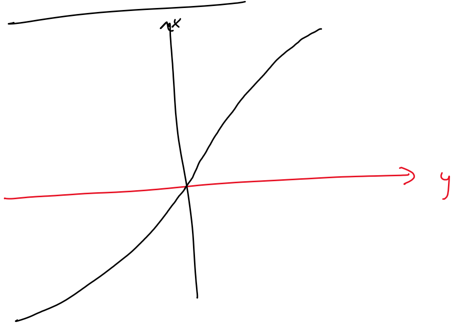
N.B. $\text{im}(\text{sh}) = \mathbb{R}$ — T.d.V.I.

$\sup_{x\in\mathbb{R}} \text{sh}\,x = +\infty$

$\inf_{x\in\mathbb{R}} \text{sh}\,x = -\infty$

ed è continua





$= x \overset{\text{def}}{\iff} e^y - e^{-y} = x \iff e^y - e^{-y} - 2x = 0$

$y = sh^{-1} x \underset{\text{biunivoca}}{\overset{\uparrow}{\Longleftrightarrow}} sh\, y$ − 2

$\Updownarrow$

$\underbrace{e^{2y} - 2x e^{y} - 1 = 0}$

se $z = e^{y} > 0$

$z^2 - 2xz - 1 = 0$

$z_{\pm} = x \pm \sqrt{x^2+1}$

$\iff z = x + \sqrt{x^2+1} > 0$

$\quad \overset{\shortparallel}{e^{y}}$

$\iff y = \log\left(x + \sqrt{x^2+1}\right)$

$$\boxed{sh^{-1} x = \log\left(\cancel{x} + \sqrt{x^2+1}\right)} \quad \underline{\underline{\forall\, x \in \mathbb{R}}}$$

LIMITI NOTEVOLI (§35).

(a) $\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^{x} = e$

Def. $e = \lim_{n \to +\infty} \underbrace{\boxed{\left(1 + \frac{1}{n}\right)^{n}}}_{e_n}$

Teorema ponte: sia (n_k) una qualunque succ. che tende a $+\infty$

allora $\lim_{k \to +\infty} \left(1 + \frac{1}{n_k}\right)^{n_k} = \lim_{k \to +\infty} e_{n_k} = e$

$\overset{T.P}{\Longrightarrow} \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = e$

Oss. $a_{\underline{\underline{n}}} \to L$

n_k è una succ a valori in $\mathbb{R}$ con $\boxed{\lim_{k \to +\infty} n_k = \infty}$

(oppure n_k è una sottosucc. di n)

$a_{\boxed{n_k}} \to L$

$\uparrow$

funzione composta

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} \underset{\substack{\uparrow \\ y = -x}}{=} \lim_{y \to +\infty} \boxed{\left(1 - \frac{1}{y}\right)^{-y}} = \lim_{y \to +\infty} \frac{1}{\left(1 - \frac{1}{y}\right)^{y}} = \lim_{y \to +\infty} \left(\frac{y}{y-1}\right)^{y}$$

T.d.C.V

$$= \lim_{y \to +\infty} \left(\frac{(y-1)+1}{y-1}\right)^{y} = \lim_{y \to +\infty} \left(1 + \frac{1}{\underline{y-1}}\right)^{y-1} \left(1 + \frac{1}{y-1}\right)$$

$$= \lim_{y \to +\infty} \left(1 + \frac{1}{y-1}\right)^{y-1} \lim_{y \to +\infty} \left(1 + \frac{1}{y-1}\right)$$

$$\underset{\substack{x = y-1 \\ \uparrow}}{=} \underbrace{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}}_{\downarrow\, e} \underbrace{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)}_{\downarrow\, 1} = e$$

(b) $\lim_{x \to 0} (1 + x)^{\boxed{\frac{1}{x}}} = e$ ✓

$$\lim_{\underline{\underline{x \to 0+}}} (1 + x)^{\frac{1}{x}} \underset{\substack{\uparrow \\ x = \frac{1}{y}}}{=} \lim_{\underline{\underline{y \to +\infty}}} \left(1 + \frac{1}{y}\right)^{y} \overset{(a)}{=} e$$

$$\lim (1 + x)^{\frac{1}{x}} = \lim \left(1 - \frac{1}{y}\right)^{-y} = e$$

$\lim_{x \to 0-}$ $(1+x)$ ... $\underset{\substack{\uparrow \\ y=-\frac{1}{x}}}{}$ $y \to +\infty$ ... $\uparrow$ come prima

(c) $$\boxed{\lim_{x\to 0} \frac{\log(1+x)}{x} = 1}$$ $\left(\log(1+x) \sim x \quad \text{per } x \to 0 \right)$

$$\Vert$$

$$\lim_{x\to 0} \log \underbrace{(1+x)^{\frac{1}{x}}}_{e} \underset{\substack{\uparrow \\ \text{T C.L} + (b)}}{=} \log e = 1$$

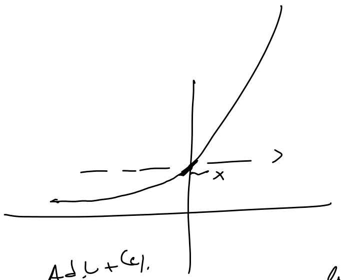
(d) $$\lim_{t\to+\infty} \left(1+\frac{x}{t}\right)^t = e^x \,, \quad \forall\, x \in \mathbb{R}$$

$x=0$ ovvio, sia $x \neq 0$

$$\lim_{t\to+\infty} \left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^x = \lim_{s\to\pm\infty} \left(\left(1+\frac{1}{s}\right)^s\right)^x \to e^x \quad \text{C.d.L.}$$

Oss. $$\lim_{n\to+\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

(e) $$\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = 1}$$



Ad L + (c).

$$\lim_{x\to 0} \frac{e^x-1}{x} \underset{\substack{\uparrow \\ y=e^x-1}}{=} \lim_{y\to 0} \frac{y}{\log(y+1)} = \lim_{y\to 0} \frac{1}{\frac{\log(1+y)}{y}} \overset{\downarrow}{=} 1$$

$y+1 = e^x$

$\lim_{x\to x_0} \frac{\sin x}{x}$!

(f) $a>0$ $$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{x\to 0} \frac{e^{(x \log a)}-1}{(x \log a)} \log a = \lim_{t\to 0} \left(\frac{e^t-1}{t}\right) \log a = \log a$$

$a=1$ Rer 0 . $a \neq 1$

(g) $$\lim_{x\to 0} \frac{(1+x)^\alpha-1}{x} \,, \quad \alpha \in \mathbb{R}$$

se $\alpha=0$ R. 0

$\alpha=1$ R. 1

$\alpha \neq 0,1$ $$\lim_{x\to 0} \frac{e^{\alpha\log(1+x)}-1}{x} \underset{\substack{\uparrow \\ x \sim \log 1+x \\ \text{per } x\to 0}}{=} \alpha \left(\lim_{x\to 0} \frac{e^{\overset{t}{\boxed{\alpha \log(1+x)}}}-1}{\alpha\log(1+x)} \right) = \alpha \lim_{t\to 0} \left(\frac{e^t-1}{t}\right) = \alpha$$

ATTENZIONE il fatto che nei limiti posso

ATTENZIONE

sostituire a f una funzione g quando $f \sim g$ lo posso fare **SOLO SE** f appare a fattore non in addendo.

Osservo è vero che:

$$\lim_{x \to x_0} f \cdot g \underset{\uparrow}{=} \boxed{\lim_{x \to x_0} h \cdot g}$$

$f \sim h$ per $x \to x_0$

$$\lim_{x \to x_0} \boxed{f \cdot g} = \lim_{x \to x_0} \frac{f}{h} \cdot gh = \lim_{x \to x_0} \boxed{gh}$$

ma, in generale, **NON è vero** che

$$\lim_{x \to x_0} f + g \underset{\uparrow}{=} \lim_{x \to x_0} (h+g)$$

$f \sim h$ per $x \to x_0$

$$\lim_{x \to x_0} (f+g) = \lim_{x \to x_0} \Big(h \underset{\substack{\downarrow \\ 1}}{\left(\frac{f}{h}\right)} + g \Big)$$

(vedi p 83 del libro)

ESEMPIO

$x_0 = +\infty$ $f = x + \sqrt{x}$, $g = x$, $h = -x$

$f \sim g$ ✓ $\lim_{x \to +\infty} \frac{x + \sqrt{x}}{x} = \lim \left(1 + \frac{1}{\sqrt{x}}\right) = 1.$

$\lim f + h = \lim (x + \sqrt{x} - x) = \lim \sqrt{x} = +\infty$, $\lim g + h = 0$

(h) $\frac{\operatorname{sh} x}{x} \sim 1$ per $x \to 0$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x \to 0} \frac{e^x - 1 + 1 - e^{-x}}{2x} = \frac{1}{2} \lim_{x \to 0} \Big(\underset{\substack{\downarrow \\ 1}}{\frac{e^x - 1}{x}} + \underset{\substack{\downarrow \\ 1}}{\frac{e^{\boxed{-x}} - 1}{\boxed{-x}}} \Big) = 1$$

(i) $\lim_{x \to 0} \frac{\operatorname{ch} x - 1}{x^2} = \frac{1}{2}$

$$\| $$

$$\lim_{x \to 0} \frac{(\operatorname{ch} x - 1)(\operatorname{ch} x + 1)}{x^2} \boxed{\frac{1}{\operatorname{ch} x + 1}} = \frac{1}{2} \lim_{x \to 0} \left(\frac{\operatorname{ch}^2 x - 1}{x^2} \right) = \frac{1}{2} \lim_{x \to 0} \frac{\operatorname{sh}^2 x}{x^2} = \frac{1}{2} \lim \left(\frac{\operatorname{sh} x}{x} \right)^2 = \frac{1}{2}$$ ✓

(j) $\lim_{x \to 0} \frac{\operatorname{arcsh} x}{x} = 1$ $y = \operatorname{arcsh} x$

$\lim e^x - 1 = 1$

Def f è DERIVABILE in x_0 se $\exists$ $\lim \frac{f(x_0+h) - f(x_0)}{} =: (Df)(x_0)$

$x \to 0 \quad \overline{x} \quad \parallel$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \underset{x=x_0+h}{=} \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

$$(D e^x)(0) = 1$$

$$\lim \frac{\lg(1+\overset{h}{x})}{x} = 1 = \underline{(D \lg x)(1)}$$

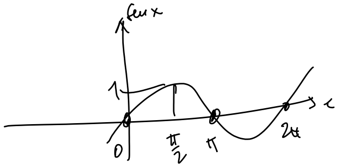
$$\underline{D e^x = e^x}$$

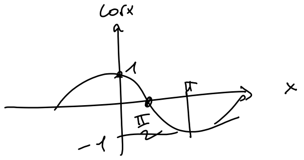
$$\frac{e^{x+h} - e^x}{h} = e^x \, \frac{e^h - 1}{h} \to e^x .$$

Limiti notevoli per sen x e cos x

sono funzioni periodiche di periodo $2\pi = 6,\ldots$

$\operatorname{sen}(x+2\pi) = \sin x \qquad \cos(x+2\pi) = \cos x \quad \forall x \in \mathbb{R}$





$\underline{x \in \mathbb{R}} \qquad x \ \underline{\text{RADIANTI}}$

$\operatorname{sen}(-x) = -\sin x$

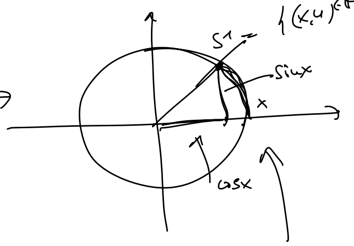
$\cos(-x) = \cos x$

$\operatorname{sen}(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\underline{\sin^2 x + \cos^2 x = 1}$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$



(A) $\dfrac{\sin x}{x} \sim 1$ per $x \sim 0$

(B) $\boxed{\dfrac{1-\cos x}{x^2} \sim \dfrac{1}{2}}$ per $x \sim 0$

(A) ⟹ (B)

$$\frac{1-\cos x}{x^2} = (1+\cos x)\,\frac{(1-\cos x)}{x^2}\,\frac{1}{(1+\cos x)} = \frac{(1-\cos^2 x)}{x^2}\,\frac{1}{1+\cos x}$$

$=$