SERIE NUMERICHE

Def. Data una successione di numeri reali $\{a_k\}$ si dice SERIE con termini a_k la successione $S_n := \sum_{k=1}^{n} a_k$.

1) La serie geometrica di ragione x, $\sum_{k=1}^{n} x^k$

$$x S_n = \sum_{k=1}^{n} x^{k+1} = \sum_{k=2}^{n+1} x^k \qquad \underbrace{x S_n - S_n}_{S_n(x-1)} = \sum_{2}^{n+1} x^k - \sum_{1}^{n} x^k = x^{n+1} - x$$

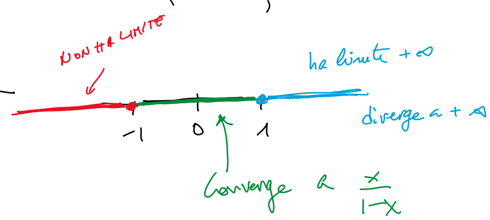
se $x \neq 1$ $\quad S_n = \frac{x^{n+1} - x}{x-1} = \frac{x - x^{n+1}}{1-x} = \frac{x}{1-x}(1-x^n)$

se $x=1$ $\quad \sum_{k=1}^{n} 1^k = n$ $\qquad S_n(x) = \begin{cases} \frac{x}{1-x}(1-x^n), & x \neq 1 \\ n, & \text{se } x=1 \end{cases}$

Oss. $x^n \to 0$, se $|x|<1$

$x^n \to +\infty$, se $x>1$

x^n NON CONVERGE (NON HA LIMITE) se $x \leq -1$

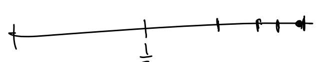


Quindi

$$\lim_{n\to+\infty} \sum_{k=1}^{n} x^k =: \sum_{k=1}^{\infty} x^k = \frac{x}{1-x} \quad \text{se } |x|<1$$

non converge altrimenti $\Leftrightarrow$ non ha limite finito. Per $x \geq 1$, $\sum_{k=1}^{\infty} x^k = +\infty$ — la serie diverge a $+\infty$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 = \frac{\frac{1}{2}}{1-\frac{1}{2}}$$



$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} = \frac{-\frac{1}{3}}{1+\frac{1}{3}} = -\frac{1}{4}$$

Serie di MENGOLI

$$a_n = \frac{1}{n(n+1)} \sim \frac{1}{n^2} \qquad \left(\lim n^2 a_n = 1\right)$$

$$\boxed{S_n = \sum_{k=1}^{n} a_k} = a_1 + a_2 + \dots + a_n$$

In generale

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\left(\sum_{k=1}^{n} a_n = a_n \cdot n\right)$$

SOMMA PARZIALE
o SOMMA RIDOTTA

$$= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{2}^{n+1} \frac{1}{k} = 1 + \cancel{\sum_{k=2}^{n} \frac{1}{k}} - \cancel{\sum_{2}^{n} \frac{1}{k}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$j = k+1$

$$\sum_{j=2}^{n+1} \frac{1}{j} = \sum_{k=2}^{n+1} \frac{1}{k}$$

$$S_n = 1 - \frac{1}{n+1} \nearrow 1.$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2)}$$

$$\boxed{\zeta(x) := \text{SERIE DI RIEMANN} := \sum_{n=1}^{+\infty} \frac{1}{n^x}}$$

è una serie a termini positivi $\left(\sum_{k=1}^{\infty} a_k \text{ con } a_k > 0\right)$.

Se $\sum_{k=1}^{\infty} a_n$ è una serie a termini positivi $\Rightarrow$ S_n è strettamente crescente.

$S_n = a_1 + \dots + a_n$

$S_{n+1} = S_n + a_{n+1} > S_n$.

$\Rightarrow$ S_n ha limite in $\mathbb{R}^*$

$S_n \to s \in \mathbb{R}_+$ se S_n è limitata.

$S_n \to +\infty$ altrimenti.

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots \quad \sum_{k=1}^{n} a_k < \sum_{k=1}^{\infty} a_k$$

$$\sum_{k=1}^{\infty} a_k = \lim S_n$$

$$a_k = \frac{1}{k^2} \to 0.$$

$$\sum_{k=1}^{n} \left(\frac{1}{k^2}\right) < 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} < 2, \quad \left(\frac{1}{k^2} < \frac{1}{k(k-1)}\right)$$

$j = k-1$

$$n \geq 2, \quad 1 + \sum_{k=2}^{n} \frac{1}{k^2}$$

Quindi $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = 1.6\dots$$

$$1 - \frac{1}{n} < 1$$

$\alpha \leftarrow a_n < b_n \to \beta$

$a_n \to \alpha \leq \beta$

$\zeta(1) = \sum_{k=1}^{\infty} \frac{1}{k} = \begin{cases} +\infty \\ ? \\ \alpha > 0 \end{cases}$ $\Big|\ a_k = \frac{1}{k} \to 0$ $\quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ SERIE ARMONICA

ESERCIZI Abbiamo visto che $\boxed{\frac{1}{n+1} < \log\left(1+\frac{1}{n}\right) < \frac{1}{n}}$

$\left(\log\left(1+\frac{1}{n}\right) \sim \frac{1}{n} \Leftarrow \log(1+x) \sim x \text{ per } x \to 0. \quad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1. \right)$

$\Leftrightarrow \quad e^{\frac{1}{n+1}} < 1 + \frac{1}{n} < e^{\frac{1}{n}}$

$\Updownarrow$

$e < \left(1+\frac{1}{n}\right)^{n+1} \qquad \left(1+\frac{1}{n}\right)^n < e$

$\qquad E_n \qquad\qquad\qquad e_n$

$e_n \nearrow e \swarrow E_n$

$$\sum_{k=1}^{n} \log\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \log\frac{k+1}{k} = \sum_{k=1}^{n} \log(k+1) - \log k$$

$$= \sum_{k=1}^{n} \log(k+1) - \sum_{k=1}^{n} \log k = \sum_{k=2}^{n+1} \log k - \sum_{k=1}^{n} \log k = \log(n+1)$$

$$\boxed{\sum_{k=1}^{n} \log\left(1+\frac{1}{k}\right) = \log(n+1)}$$

$$\sum_{k=1}^{n} \log\left(1+\frac{1}{k}\right) = \log(n+1) \to +\infty$$

$\left(\frac{1}{n+1} < \log\left(1+\frac{1}{n}\right) \qquad \boxed{\frac{1}{n} < \log\left(1+\frac{1}{n-1}\right)}, \ n \geq 2 \right)$

$\boxed{\log\left(1+\frac{1}{n}\right) < \frac{1}{n}.}$

$\Downarrow$

$$\sum_{k=1}^{n} \log\left(1+\frac{1}{k}\right) < \sum_{k=1}^{n} \frac{1}{k} \to +\infty.$$

$+\infty$

Quindi la serie armonica diverge (a $+\infty$).

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty \qquad \sum_{k=1}^{n} \frac{1}{k} > \log(n+1)$$

Qual è l'andamento di $\sum_{k=1}^{n} \frac{1}{k}$

$n \geq 2$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} < 1 + \sum_{k=2}^{n} \log\left(1+\frac{1}{k-1}\right) = 1 + \sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right)$

$= 1 + \log n$

$$\log n - \log\left(1+\frac{1}{n}\right) = \log(n+1) < \sum_{k=1}^{n} \frac{1}{k} = S_n < 1 + \log n$$

$\log\left(1+\frac{1}{n}\right) \approx \frac{1}{n}$

$\log n - \frac{1}{n}$.

NOTA BENE: $\frac{1}{k} \to 0$ ma $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.

Proposizione Se $\displaystyle\sum_{1}^{\infty} a_k$ converge (ha limite finito) $\Longrightarrow$ $a_k \to 0$

(ma il viceversa NON È VERO!!!)

Dim. $S_n := \displaystyle\sum_{k=1}^{n} a_k$ e so che $S_n \to L \in \mathbb{R}$

Quindi anche $S_{n-1} \to L$ $\quad a_n = S_n - S_{n-1} \longrightarrow L - L = 0$. ∎

VEDI ERRATA CORRIGE

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

a_k

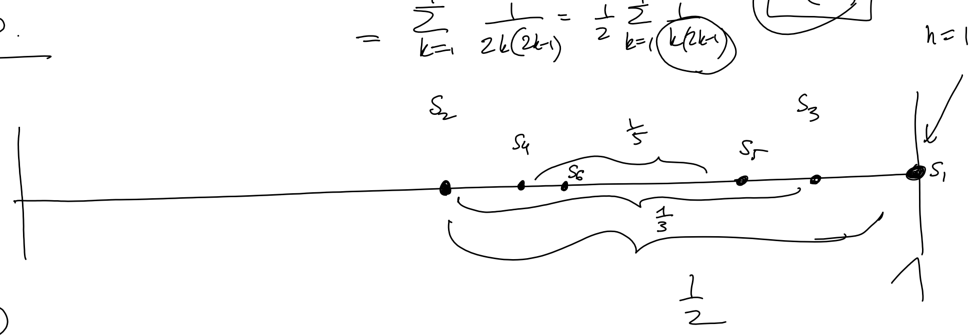
$$S_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = \left(1-\frac{1}{2}\right) + \dots + \boxed{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)}$$

$\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$

$$= \frac{1}{2} + \dots + \boxed{\frac{1}{2n(2n-1)}}$$

$a_k \to 0$.

$$= \sum_{k=1}^{n} \frac{1}{2k(2k-1)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k(2k-1)}$$



Sembrerebbe che:

$\frac{1}{2}+\frac{1}{3}$

1) $S_{2k} \nearrow$ strett. crescente. $\quad S_{2k} \nearrow \alpha$ ✓

2) $S_{2k+1} \searrow$ strett. decr. $\quad S_{2k+1} \searrow \beta$

3) $S_{2k} < S_{2k+1}$ $\quad \alpha \leq \beta$

$a_k = \frac{(-1)^{k-1}}{k}$

$S_{2k+1} = S_{2k} + \frac{1}{2k+1}$

$= S_{2k+2} - \frac{1}{2k+2}$

4) $\underline{s_{2k+1} - s_{2k} \longrightarrow 0}$

$\alpha = \beta$

DIMOSTREREMO IN SEGUITO

5) $s_k \longrightarrow \underline{\alpha = \log 2}$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2}$$

✓

$s_{2(k+1)+1} < s_{2k+1}$

$\shortparallel$

$$s_{2k+1} + a_{2k+2} + a_{2k+3} = s_{2k+1} - \frac{1}{2k+2} + \frac{1}{2k+3}$$

$$= s_{2k+1} - \Big(\underbrace{\frac{1}{2k+2} - \frac{1}{2k+3}}_{>0}\Big)$$

$$s_{2k+1} = s_{2k} + a_{2k+1} = s_{2k} + \frac{(-1)^{2k}}{2k+1} = s_{2k} + \frac{1}{2k+1}$$

$\Rightarrow \quad s_{2k+1} > s_{2k} \quad$ e che $\quad s_{2k+1} - s_{2k} = \frac{1}{2k+1} \longrightarrow 0$.

$\Longmapsto \quad s_n \longrightarrow \lim s_{2k+1} = \lim s_{2k} \quad \left(\overset{?}{=} \log 2 \right)$

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty \quad \left(\sum_{k=1}^{n} \frac{1}{k} \sim \log n. \right) \quad , \quad \underline{\frac{1}{k} \to 0.}$$

$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converge.

$a_k \longrightarrow 0$ COND. NECESSARIA PER CONVERGENZA

Vediamo che $\sum_{k=3}^{\infty} \frac{(-1)^k}{\log(\log(\log k))}$ CONVERGE

$\log(\log 5) > 0 \iff \log 5 > 1$

$\underline{5 > e}$. ✓