venerdì 8 gennaio 2021 08:19

1. $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} = \lim \frac{(\cancel{1}+x+x^2)-\cancel{1}}{x\cdot(\sqrt{1+x+x^2}+1)} = \lim \frac{1+x}{\sqrt{1+x+x^2}+1} = \frac{1}{2}$

2. $\lim_{x\to 0} \frac{e^{x+x^2}-1}{\mathrm{th}\, x}$

$\frac{e^{x+x^2}-1}{\mathrm{th}\,x} \sim \frac{x+x^2}{x} = 1+x \to 1$

$e^y - 1 \sim y, \quad y \to 0$

3 $\lim_{n\to+\infty} \left(\frac{n+3}{n+4}\right)^n = \frac{1}{e}$ $\qquad e^{-1}$

$\left(\frac{n+3}{n+4}\right)^n = \left(1-\frac{1}{n+4}\right)^{n+4} \left(1-\frac{1}{n+4}\right)^{-4}$

scrivere come $(1+a_n)^n$ $\qquad \downarrow 1$

$\left(\frac{n+3}{n+4}\right)^n = e^{n\log\frac{n+3}{n+4}} = e^{n\log\left(1+\left(\frac{n+3}{n+4}-1\right)\right)} = e^{n\log\left(1-\frac{1}{n+4}\right)}$

$\frac{\cancel{n}+3-\cancel{n}-4}{n+4} = -\frac{1}{n+4}$

$\sim e^{n\left(-\frac{1}{n+4}\right)} \to e^{-1}$

A.d.L.

$\tanh x \sim x, \quad x\to 0$

$\left(f \sim g, \; x\to x_0 \overset{\text{def.}}{\iff} \lim_{x\to x_0}\frac{f(x)}{g(x)} = 1\right)$

$\boxed{e^x - 1 \sim x}, \quad x\to 0$

$\left(1+\frac{1}{n}\right)^n \to e$

$\left(1+\frac{\alpha}{a_n}\right)^{a_n} \to e$, se $a_n \to +\infty$

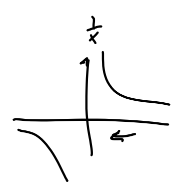
$(1+x)^{\frac{1}{x}} \to e, \quad x\to 0$

$a_n^{b_n} = e^{\boxed{b_n \log a_n}}$, $a_n > 0$

$\log(1+x) \sim x, \quad x\to 0$

4. $\lim_{x\to 0\pm} \frac{1+2^{\frac{1}{x}}}{1+x^2} \quad f(x)$

$\frac{1+2^{\frac{1}{x}}}{1+x^2} \sim 1+2^{\frac{1}{x}} \qquad x\to 0+$



$\lim_{x\to 0+} 1+2^{\frac{1}{x}} = +\infty$

$\lim_{x\to 0-} 1+2^{\frac{1}{x}} = 1$ comp. di limiti $\frac{1}{x} \to -\infty$

0 è una sing. essenziale per $f(x)$

5. $\lim_{x\to\frac{\pi}{2}} (\mathrm{tg}\, x)\left(x-\frac{\pi}{2}\right)$

$$\left(x-\frac{\pi}{2}\right) tg\, x \sim \frac{x-\pi/2}{\cos x} \quad \text{per } x \to \frac{\pi}{2}$$

Cambio variabile $y = x - \frac{\pi}{2}$

$$\underset{x\to\frac{\pi}{2}}{\frac{x-\frac{\pi}{2}}{\cos x}} = \underset{y\to 0}{\frac{y}{\cos\left(y+\frac{\pi}{2}\right)}} = \frac{y}{\cos y \underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\sin\frac{\pi}{2}}_{1} \sin y} = -\frac{y}{\sin y} \to -1$$

6. $$\lim_{x\to 1} x^{\frac{1+x}{x-1}} = \lim_{y\to 0} (1+y)^{\frac{2+y}{y}} = \lim_{y\to 0} (1+y)^{\frac{2}{y}} \cdot (1+y) = e^2$$

C.d.V $x = 1+y$ $\quad y \to 0$

$x \to 1.$

$$(1+y)^{\frac{2}{y}} = \left((1+y)^{\frac{1}{y}}\right)^2 \to e^2.$$

7. $$\lim_{n\to\infty} \frac{(\log n)^2 + n}{\sqrt{n}} = \lim_{n\to\infty} \underbrace{\frac{(\log n)^2}{\sqrt{n}}}_{0} + \underbrace{\sqrt{n}}_{+\infty}$$

A.d.L. estesa

$a_n + b_n \to +\infty$, $\quad a_n \to +\infty$, $b_n \geq M$

$M \leq b_n$.

$$\frac{(\log n)^\alpha}{n^\beta} \to 0, \quad \beta > 0$$

$$\frac{a^n}{n^\alpha} \to +\infty \quad a > 1$$

8. $$\sum_{n=1}^{\infty} \frac{n!}{(\sqrt{2}\, n)^n}$$

cr. rapp. (per la presenza di fattoriali e potenze).

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(\sqrt{2})^{n+1} (n+1)^{n+1}} \cdot \frac{(\sqrt{2})^n n^n}{n!} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{\sqrt{2} \cdot e} < 1$$

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1. $$\lim_{x\to 0} \frac{e^{tg^3 x} - 1}{x(\cos x - e^{x^2})}$$

$e^{tg^3 x} - 1 \sim tg^3 x \sim x^3$ per $x \to 0$.

$$\frac{e^{tg^3 x} - 1}{x\left((\cos x - 1) + (1 - e^{x^2})\right)} \sim \frac{x^3}{x\left((\cos x - 1) + (1 - e^{x^2})\right)}$$

$$= \frac{1}{\frac{\cos x - 1}{x^2} + \frac{1 - e^{x^2}}{x^2}} \longrightarrow \frac{1}{-\frac{1}{2} - 1} = -\frac{2}{3}$$

A.d.L

Attenzione in generale

$f \sim F$ e $g \sim G$

non è vero

$f + g \sim F + G$ NO

$\frac{\cos x - 1}{x^2} \to -\frac{1}{2}$

$\frac{1 - \cos x}{x^2} \sim \frac{1}{2}$

x^2 $\qquad e^x - 1 \sim x$

2. $\displaystyle \lim_{x\to 0} \frac{\log(\cos x)}{x^2}$

$$\frac{\log(\cos x)}{x^2} = \frac{\log(1+\cos x - 1)}{x^2} \sim \frac{\cos x - 1}{x^2} \to -\frac{1}{2}$$

3. $\displaystyle \lim_{x\to 0} \frac{\sin\left(\sqrt{1+x^2}-1\right)}{x}$

$$\frac{\sin\left(\sqrt{1+x^2}-1\right)}{x} \sim \frac{\sqrt{1+x^2}-1}{x} = \frac{x^{\cancel{2}}}{\cancel{x}\left(\sqrt{1+x^2}+1\right)} \to 0.$$

4. $\displaystyle \lim_{x\to 0} \frac{\sin(\pi\cos x)}{x\sin x}$

$$\frac{\sin(\pi\cos x)}{x\sin x} \sim \frac{\sin(\pi\cos x)}{x^2} = \frac{\sin\left(\pi(\cos x - 1) + \pi\right)}{x^2}$$

$$= -\frac{\sin\pi(\cos x - 1)}{x^2} \sim -\frac{\pi(\cos x - 1)}{x^2} \to \frac{\pi}{2}$$

$\sin y \sim y$ per $y \to 0$

$y = \pi(\cos x - 1)$

5. $\displaystyle \lim_{n\to +\infty} \boxed{\frac{n^{\log n}}{(\log n)^n}}$

PORTARE A ESPONENTE

$$= \frac{e^{(\log n)^2}}{e^{n\log\log n}} \longrightarrow 0$$

$\log n = e^{\log\log n}$

$\boxed{a = e^{\log a}}$

$$= \frac{1}{e^{n\log\log n - (\log n)^2}} \longrightarrow 0.$$

$e^{n(\log\log n)}$ $\downarrow$ $+\infty$

$e^{1 - \frac{(\log n)^2}{n(\log\log n)}}$ $\downarrow$ e

$\frac{(\log n)^2}{n} \to 0$

$\Rightarrow \frac{(\log n)^2}{n(\log\log n)} \to 0$

6. $\lim \boxed{(\log n)^{\sqrt{n}}}$

2^n

$$\frac{e^{\sqrt{n}(\log \lg n)}}{e^{n \lg 2}} = \frac{1}{e^{n \lg 2 - \sqrt{n} \lg \lg n}} = \frac{1}{e^{n \lg 2}\left(e^{1 - \frac{\lg \lg n}{2\sqrt{n}}}\right)} \to 0$$

$\underbrace{e^{1 - \frac{\lg \lg n}{2\sqrt{n}}}}_{\to e}$

7. $\lim_{x \to 0} x \, |\log x| = \lim_{y \to +\infty} e^{-y} y = 0$

$y = -\lg x$, $y \to +\infty$, $x \to 0+$.

$x = e^{-y}$

$\frac{y^\alpha}{a^y} \xrightarrow[y \to \infty]{} 0$ $a > 1$.

8 Calcolare lim sup/inf di a_n

$a_n = \sqrt[n]{(-1)^n n} = (-1)^n \sqrt[n]{n}$

$n^{\frac{1}{n}} \to 1$

$a_{2k} = \sqrt[2k]{2k} \to 1.$

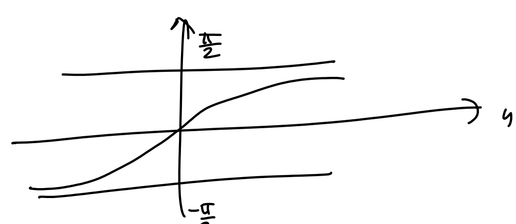
$a_{2k+1} = -\sqrt[2k+1]{2k+1} \to -1$

$-\sqrt[n]{n} \leq a_n \leq \sqrt[n]{n}$

se $a_{n_k} \to \alpha$

$-1 \leq \alpha \leq 1$

9. Come 8 con $a_n = \frac{(-1)^{n^2} + 1}{\operatorname{arctg} n}$



$(-1)^{n^2} = (-1)^n$

$n^2 = \begin{cases} \text{pari} & \text{se } n \text{ è pari} \\ \text{dispari} & \text{se } n \text{ è dispari} \end{cases}$

$\overline{\lim}\, a_n = \frac{4}{\pi}$ $\underline{\lim}\, a_n = 0$

$a_{2k} = \frac{2}{\operatorname{arctg} 2k} \to \frac{4}{\pi}$, $a_{2k+1} = 0$

$0 \leq a_n \leq \frac{2}{\pi}$ & $a_{n_k} \to L$

arcsin

$$0 \leq L \leq \frac{4}{\pi}$$

Serie con parametro

1 $\sum (nx^2)^{n^2 x}$, $x \neq 0$ (per $x=0$ $(nx^2)^{n^2 x} = 1$, serie diverge)

$x \neq 0$, serie a termini positivi

Cr. radice $(nx^2)^{nx} \longrightarrow \begin{cases} +\infty, & x>0 \\ 0, & x<0 \end{cases}$

La serie diverge per ($x \geq 0$) $x>0$

converge per $x<0$

2. $\sum \frac{n \sin x^n}{n + x^{2n}}$

Studiamo la serie in valore assoluto

$a_n = \frac{n |\sin x^n|}{n + x^{2n}}$ $x = \pm 1$, $a_n \not\to 0 \Rightarrow \sum a_n = +\infty$.

$a_n \sim \frac{n |x|^n}{n + x^{2n}} \sim \frac{n |x|^n}{n} = |x|^n$ $|x|<1 \Rightarrow \sum a_n < +\infty$ per il CRITERIO CONFRONTO ASINT

per confronto asintotico $\sum a_n < +\infty$ la serie data conv ass per $|x|<1$

$|x|>1$, $|a_n| \leq \frac{n}{x^{2n}}$ conv. per Cr. radice $\sqrt[n]{\frac{n}{x^{2n}}} \to \frac{1}{x^2} < 1$

3. $\sum \underbrace{\frac{1 - x^n}{1 + x^{2n}}}_{a_n}$

$x = \pm 1$ $x=1$ converge, non è infinitesimo $\Rightarrow$ non converge

$|x|<1$ $|x|<1$ $a_n \not\to 0$ $\Rightarrow$ diverge

$|x|>1 \qquad a_n \sim -\frac{x^n}{x^{2n}} = -\frac{1}{x^n}$

$\sum a_n$ converge assolutamente per $|x|>1$

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4 $\displaystyle\sum_{n=1}^{\infty} n^\alpha x^{\sqrt{n}}$, $x\geq 0$

[GE, Es 55 p 91]

non funziona Cr. RAO nè Cauchy

Per $x=1$ conv. $\iff \alpha<-1$ (serie di Riemann).

Se $x>1 \qquad n^\alpha x^{\sqrt{n}} \to +\infty \implies$ serie diverge

$$e^{\alpha \log n + \sqrt{n}\log x} = e^{\sqrt{n}\log x\left(1+\frac{\alpha \log n}{\sqrt{n}\log x}\right)} \to +\infty.$$

$\frac{\alpha \log n}{\sqrt{n}\log x} \to 0$

$x<1 \qquad n^\alpha x^{\sqrt{n}} = e^{-\sqrt{n}|\log x|\left(1-\frac{\alpha\log n}{\sqrt{n}|\log x|}\right)}$

$\log x = -\log\frac{1}{x} \qquad \left(1-\frac{\alpha\log n}{\sqrt{n}|\log x|}\right) \to 1$

$$\underbrace{\sum n^\alpha x^{\sqrt{n}} \approx \sum \frac{1}{a^{\sqrt{n}}}}_{\text{C. confr. as.}} , \quad a>1. \quad a=e^{|\log x|}>1.$$

$$\boxed{\sum \frac{1}{a^{\sqrt{n}}} < +\infty}$$

$\frac{a^{\sqrt{n}}}{n^2} \to +\infty \implies a^{\sqrt{n}} > n^2$ definitivamente.

$e^{\sqrt{n}\log a - 2\log n} \to +\infty \qquad n^2 = e^{\log n^2} \qquad \implies \frac{1}{a^{\sqrt{n}}} < \frac{1}{n^2}$

certamente con $\sum \frac{1}{n^2} < \infty$.