1-Funzioni

martedì 22 settembre 2020 15:55

FUNZIONI REALI DI VARIABILE REALE

Def. $f : x \in D \subseteq \mathbb{R} \longmapsto f(x) \in \mathbb{R}$

dominio — codominio

più precisamente una funzione è un GRAFICO

$G_f \subseteq D \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}$ t.c. (tale che)

(prodotto cartesiano $:= \{(x,y) \mid x \in D,\ y \in \mathbb{R}\}$)

$\forall\ x \in D\ \exists!\ y \in \mathbb{R} \mid (x,y) \in G_f$

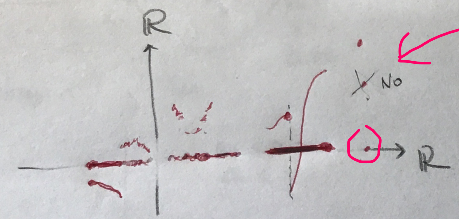
per ogni — esiste unico — tale che

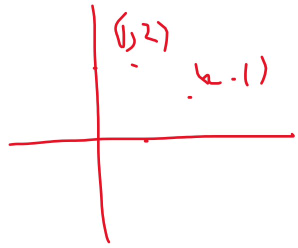
o ancora:

se $(x,y), (x,z) \in G_f \Longrightarrow y = z$

allora

NOTAZIONE: $G_f := \{(x, f(x)) \mid x \in D\}$





2-esempi

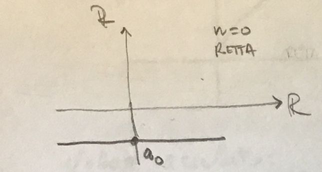
martedì 22 settembre 2020 15:56

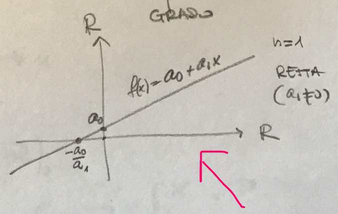
ESEMPI

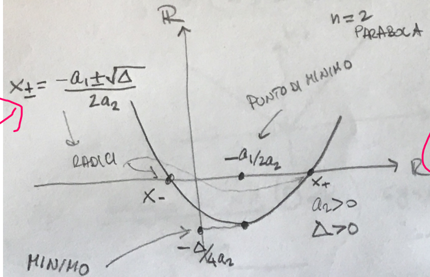
1. Polinomi di grado $n \in \mathbb{N}_0 := \{0,1,2,\dots\} = \{0\} \cup \mathbb{N}$.

$$f : x \in \underset{D}{\mathbb{R}} \longmapsto f(x) = a_0 + a_1 x + \dots + a_n x^n, \quad a_n \neq 0, \quad (a_i \in \mathbb{R})$$

GRADO







$$f(x) = a_0 + a_1 x + a_2 x^2 \quad (a_2 \neq 0)$$

$$= a_2\left(x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2}\right)$$

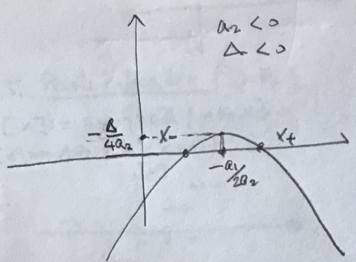
completamento del quadrato

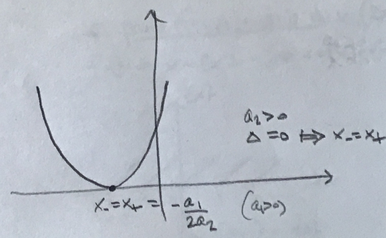
$$= a_2\left(\left(x + \frac{a_1}{2a_2}\right)^2 - \frac{a_1^2}{4a_2^2} + \frac{a_0}{a_2}\right)$$

$$= a_2\left(\left(x + \frac{a_1}{2a_2}\right)^2 - \frac{\Delta}{4a_2^2}\right)$$

con $\Delta := a_1^2 - 4a_0 a_2$

DISCRIMINANTE





$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$a > 0, \quad \sqrt{a} > 0$$

3-esempi

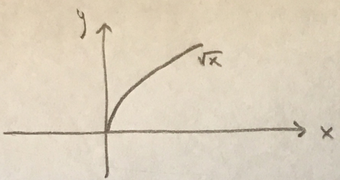
martedì 22 settembre 2020 15:57

2. Radici $n \geq 2$, $f: x \in D := [0,+\infty) := \{x \mid x \geq 0\} \mapsto \sqrt[n]{x} \in \mathbb{R}$

0 è incluso

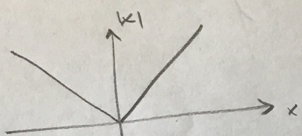
(N.B. l'immagine di $f := f(D) = \{y = f(x) \mid x \in D\}$
$= [0,+\infty)$)

t.c.



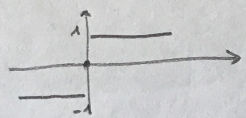
3. Valore assoluto: $(D=\mathbb{R})$,

$$f(x) = |x| = \max\{x,-x\} = \begin{cases} x, & \text{se } x \geq 0 \\ -x, & \text{se } x < 0 \end{cases}$$



4. Segno di x: $(D=\mathbb{R})$, $\operatorname{sgn}(x) = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \\ 0 & \text{se } x = 0 \end{cases}$, per $x \neq 0$ $\operatorname{sgn}(x) = \frac{x}{|x|}$

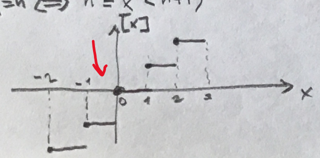
"convenzione"



5. Parte intera di x $(D=\mathbb{R})$

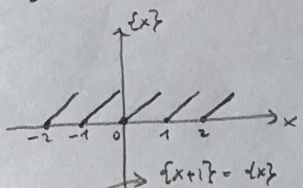
$[x] := \sup\{n \in \mathbb{Z} \mid n \leq x\}$

$([x] = n \iff n \leq x < n+1)$



6. parte frazionaria di x. $(D=\mathbb{R})$

$\{x\} := x - [x]$ $(\Rightarrow x = [x] + \{x\})$



$\{x+1\} = \{x\}$

FUNZIONE PERIODICA DI PERIODO 1

4-esempi

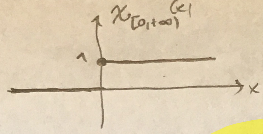
martedì 22 settembre 2020 15:59

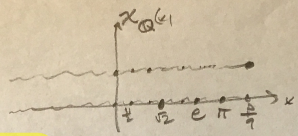
6. FUNZIONI CARATTERISTICHE $(D=\mathbb{R})$

$\emptyset \neq A \in \mathbb{R}$, $\chi_A(x) := \begin{cases} 1 & \text{se } x \in A \\ 0 & \text{se } x \notin A \end{cases}$

insieme vuoto

$im(\chi_A) = \{0,1\}$



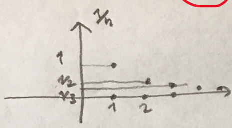


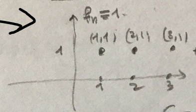
7. SUCCESSIONI $D=\mathbb{N}$

$f : n \in \mathbb{N} \longmapsto f_n \in \mathbb{R}$. notazioni comuni: $\{f_n\}, \{x_n\}, \{a_n\}$ $\neq$

(ma non confondere con $\{f_n \mid n \in \mathbb{N}\}$)

immagine





$\{1\} = \{f_n \mid n \in \mathbb{N}\}$

$G_f = \{(1,1), (2,1), (3,1), (4,1), \dots\}$

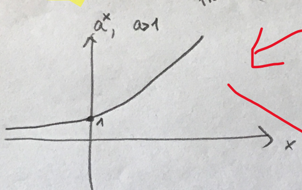
8. FUNZIONI ESPONENZIALI $D=\mathbb{R}$

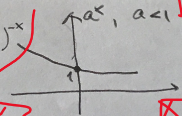
$a > 1$, $x \in \mathbb{R} \longmapsto a^x \in (0,+\infty)$, $a^x := \sup\{a^r \mid r \in \mathbb{Q},\ r \leq x\}$

BASE fisso varia ("0 escluso")

[N.B. se $a > 0$, $a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = (a^{\frac{1}{q}})^p$
e $a^{\frac{1}{q}} := \sqrt[q]{a}$]

(se $a < 1$ $a^x := (a^{-1})^{-x}$





$p < 0$, $p \in \mathbb{Z}$, $a \neq 0$

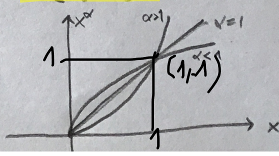
$a^p := \frac{1}{a^{-p}} = \left(\frac{1}{a}\right)^{-p}$

$2^{-3} = \frac{1}{8}$

9. POTENZE REALI $D = (0,+\infty) = \{x > 0\}$, $\alpha \in \mathbb{R}$

$x \in D \longmapsto x^\alpha$ ← fisso

varia



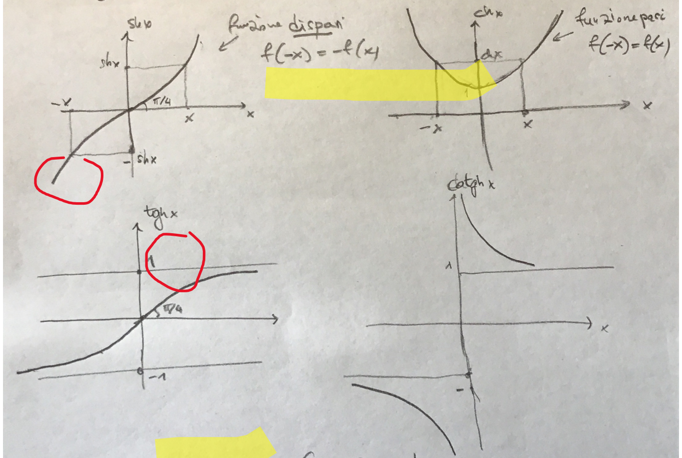
5-esempi

martedì 22 settembre 2020 15:59

10. FUNZIONI IPERBOLICHE (seno iperbolico, coseno iperb., etc)

$\mathrm{sh}\,x = \mathrm{senh}\,x := \frac{e^x - e^{-x}}{2}$, $\mathrm{ch}\,x := \cosh x = \frac{e^x + e^{-x}}{2}$

$\mathrm{tgh}\,x = \frac{\mathrm{sh}\,x}{\mathrm{ch}\,x}$ $\quad$ $\mathrm{cotgh}\,x = \frac{\mathrm{ch}\,x}{\mathrm{sh}\,x} \leftarrow D = \mathbb{R} \setminus \{0\}$



Proprietà algebriche (verificare!)

$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$

$\mathrm{ch}\,(x+y) = \mathrm{ch}\,x \cdot \mathrm{ch}\,y + \mathrm{sh}\,x\,\mathrm{sh}\,y$

$\mathrm{sh}\,(x+y) = \mathrm{ch}\,x\,\mathrm{sh}\,y + \mathrm{ch}\,y\,\mathrm{sh}\,x$

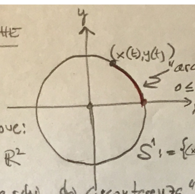
6-esempi

martedì 22 settembre 2020 16:01

10*. FUNZIONI TRIGONOMETRICHE

Definizione geometrica:

$\cos t := x(t)$, $\sin t := y(t)$ dove:

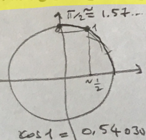


$\mathbb{R}^2$ $\quad S^1 := \{(x,y) \mid x^2+y^2=1\}$

! → Va definita la "lunghezza di archi di circonferenza"

→ Al contrario di tutte le altre funzioni qui non stiamo dando una definizione "funzionale" o "algoritmica" diretta:

ad esempio: come calcolare $\cos 1$?



La definizione che daremo nel corso è analitica:

$$\cos x := 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin x := x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Proprietà:

$\cos^2 x + \sin^2 x = 1$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

Valori speciali

$\cos 0 = 1$, $\cos \frac{\pi}{2} = 0$, $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, etc

$\sin 0 = 0$, $\sin \frac{\pi}{2} = 1$, $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

7-funzioni composte e inverse

martedì 22 settembre 2020 16:02

FUNZIONI COMPOSTE E FUNZIONI INVERSE

FUNZIONI COMPOSTE:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ x & & y=f(x) & & z=g(y) \end{array}$$

$$g \circ f(x) := g(f(x))$$

$\Rightarrow$ im f = immagine di f $\subseteq$ dom(g) ← dominio di g

Esempio: $F(x) = \sqrt{\frac{1}{1-x}}$

$F = g \circ f$ con $g(y) = \sqrt{y}$ e $f(x) = \frac{1}{1-x}$

im $f \subseteq$ dom $g = [0, +\infty) \iff \frac{1}{1-x} \geq 0 \iff x \neq 1$ e $1-x \geq 0$
$\iff x < 1$

dom$(f) = \{x \in \mathbb{R} \mid x < 1\}$

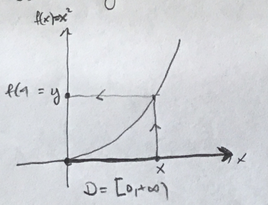
FUNZIONI INVERSE

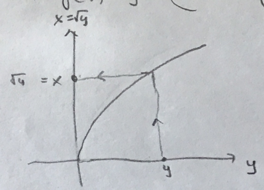
$f: D \to \mathbb{R}$ iniettiva (cioè, $x \neq x' \Rightarrow f(x) \neq f(x')$)

f iniettiva è invertibile: $\exists$ funzione inversa g

$g: \text{im}(f) = \{f(x) \mid x \in D\} \to D: \quad \forall y \in \text{im}(f),\ g(y) = x \iff f(x) = y$

ossia $g \circ f(x) = x$ e $f \circ g(y) = y$ (notazione $g = f^{-1}$)





(n.b.: x^2 non è iniettiva su $\mathbb{R}$)

8-esempi

martedì 22 settembre 2020 16:03

Esempi:

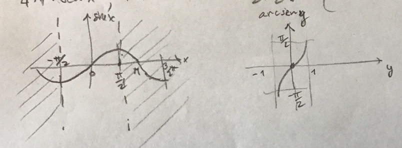
1. retta $y = ax + b$ con $a \neq 0$ $D = \mathbb{R} = im(f)$, $f^{-1}(y) = \frac{y-b}{a}$

2. $f = e^x$: $D = \mathbb{R}$, $im(f) = (0, +\infty)$, $f^{-1}(y) =: \log(y)$

(va dimostrato!)

3. $f = x^\alpha$, $D = (0, +\infty)$, $im(f) = (0, +\infty)$ $f^{-1}(y) = y^{\frac{1}{\alpha}}$

4. $f = \operatorname{sen} x$, iniettiva su $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (o su $[\frac{\pi}{2}, \frac{3}{2}\pi]$)



etc.