Lezione 10 13/10/21

mercoledì 13 ottobre 2021 08:17

Def $f: A \subseteq \mathbb{R} \to \mathbb{R}$

Sia A non limitato superiormente $(\forall M \ \exists x \in A \mid x > M)$

Sia $L \in \mathbb{R}$, $\lim_{x \to +\infty} f(x) = L \overset{def}{\iff}$

$$\forall \varepsilon > 0 \ \exists M \mid |f(x) - L| < \varepsilon, \ \forall \underline{x \in A \text{ e } x > M}$$

(cfr. la def. di $\lim_{n \to +\infty} x_n$, dove $A := \mathbb{N}$, $x_n \leftrightarrow f(x)$, $x(n)$)

Es $\lim_{x \to +\infty} \frac{e^{-x}}{2 + \sin x} = 0$ infatti se $\varepsilon > 0$

$$0 \underset{=}{<} \frac{e^{-x}}{2 + \sin x} \leq e^{-x} < \varepsilon \iff e^x > \frac{1}{\varepsilon} \iff \log x > \underbrace{\log \frac{1}{\varepsilon}}_{M}$$

$2 + \sin x \geq 1 \ (-1 \leq \sin x \leq 1)$

$M = \max\{1, \log \frac{1}{\varepsilon}\}$

Come per le succ vale el A.d.L (Algebra dei limiti)

Teor • A non limitato sup, $f, g: A \to \mathbb{R}$, $\lim_{x \to +\infty} f(x) = \alpha$, $\lim_{x \to +\infty} g(x) = \beta$

Allora, $\lim_{x \to +\infty} f(x) + g(x) = \alpha + \beta$, $\lim_{x \to +\infty} f(x) g(x) = \alpha \cdot \beta$; $\alpha \neq 0$, $\lim_{x \to +\infty} \frac{1}{f} = \frac{1}{\alpha}$

• (teor. confronto) $f \leq h \leq g$ $\alpha = \beta \Rightarrow \lim_{x \to +\infty} h(x) = \alpha$

Analog. si definisce il limite per $x \to -\infty$ per funzioni con dominio A non limitato inf. $(\iff \forall M < 0 \ \exists x \in A \mid x < M)$

Eser. Scrivere la def. di $\lim_{x \to -\infty} f(x) = L$, con $L \in \mathbb{R}$

Es $\lim_{x \to -\infty} \frac{\sin x}{x} = 0$ $(x \neq 0)$

Dato $\varepsilon > 0$ $\left| \frac{\sin x}{x} \right| < \varepsilon$

$\frac{\sin x}{x}$ è una funzione pari

$$\frac{\sin x}{x} = \frac{\sin(-x)}{(-x)}$$

$$\lim_{\underline{\underline{x \to -\infty}}} \frac{\sin x}{x} \underset{\underline{\underline{y = -x}}}{=} \lim_{\underline{\underline{y \to +\infty}}} \frac{\sin y}{y}$$

Dato $\varepsilon > 0$, $\left|\frac{\sin x}{x}\right| \overset{?}{<} \varepsilon$, $\left|\frac{\sin x}{x}\right| \leq \frac{1}{|x|} < \varepsilon$

$\iff |x| > \frac{1}{\varepsilon}$ $\quad (x \to -\infty)$ $\quad \iff -x > \frac{1}{\varepsilon} = M$

$\iff -x > M$, $\underline{\underline{x < -M}}$

$\forall \; x < -M \quad \left|\frac{\sin x}{x}\right| < \varepsilon$

Limiti notevoli

$$\lim_{x \to +\infty} \frac{a^x}{x^b} = +\infty , \quad \underline{\underline{a > 1}}$$

$$\lim_{x \to +\infty} \frac{\log_a x}{x^b} = 0 , \quad b > 0$$

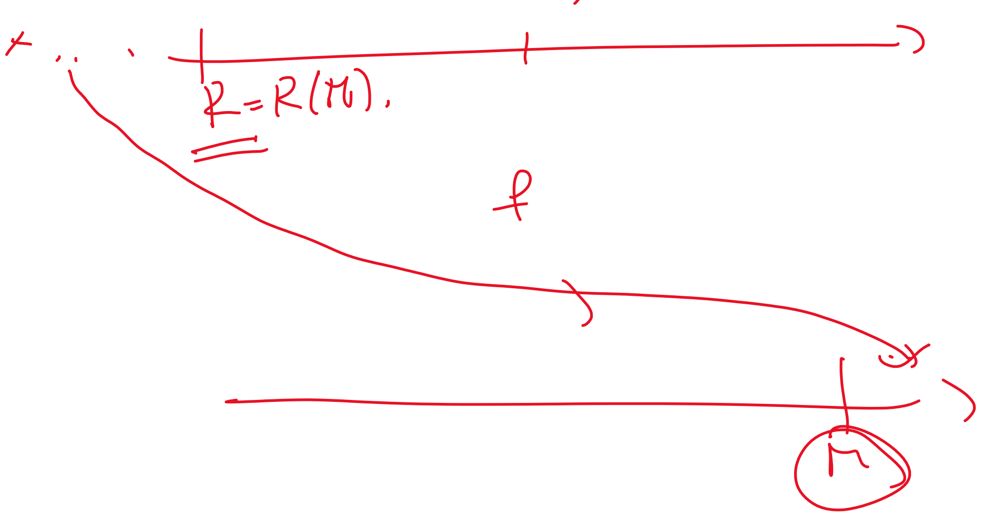
$$\lim_{x \to \pm\infty} \left(1 + \frac{\alpha}{x}\right)^x = e^\alpha , \quad \forall \; \alpha \in \mathbb{R}$$

(nuovo)

Abbiamo usato anche la def. di $\lim_{x \to \pm\infty} f(x) = \pm\infty$ (4 definizioni scrivetele per esercizio tutte e 4 !!)

P.e. $\lim_{x \to -\infty} f(x) = +\infty \iff f: A \to \mathbb{R}$, con A un illimitato inf.

$\underline{\underline{\forall \; M > 0}} \; \exists \; R < 0 \; \big| \; f(x) > M , \; \underline{\underline{\forall \; x}} \in A$ con $x < \underline{\underline{R}}$



ESERCIZI (DEMIDOVIC 215)

$$\lim_{x\to+\infty}\left(x+\sqrt[3]{1-x^3}\right)=\lim_{x\to+\infty}\left(x-\sqrt[3]{x^3-1}\right)$$

$x \to +\infty$, $\sqrt[3]{1-x^3} \to -\infty$

$x^3-1 = x^3\left(1-\frac{1}{x^3}\right)$

$$=\lim_{x\to+\infty} x\left(1-\sqrt[3]{1-\frac{1}{x^3}}\right)=x(a-b) \quad a=1,\ b=+\sqrt[3]{1-\frac{1}{x^3}}$$

$$a^3-b^3=(a-b)(a^2+ab+b^2) \qquad \boxed{a-b}=\frac{a^3-b^3}{a^2+ab+b^2}$$

DIFFERENZA DI CUBI

$$=\lim_{x\to+\infty}\cancel{x}\,\frac{\frac{1}{x^{\cancel{3}2}}}{1+\sqrt[3]{1-\frac{1}{x^3}}+\left(1-\frac{1}{x^3}\right)^{\frac{2}{3}}}$$

$\frac{1}{x^2} \to 0$

$$a+b=\frac{a^3+b^3}{a^2-ab+b^2}$$

$= 0$

$b=\sqrt[3]{\ldots}$

$\to 3$ PERCHÉ $\frac{1}{x^3}\to 0$

$$a^3-b^3=1^3-\left(1-\frac{1}{x^3}\right)=\frac{1}{x^3}$$

DEMIDOVIC 248

$$\lim_{x\to+\infty}\left(\frac{x}{x+1}\right)^x \quad \text{“}\left(\frac{\infty}{\infty}\right)^{+\infty}\text{”}$$

$$=\lim_{x\to+\infty}\left(\frac{1}{1+\frac{1}{x}}\right)^x=\lim_{x\to+\infty}\frac{1}{\left(1+\frac{1}{x}\right)^x}\overset{\text{ALGEBRA D. LIMITI}}{=}\frac{1}{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x}\longrightarrow\frac{1}{e}$$

$x = x\left(1+\frac{1}{x}\right)$

$1^x=1 \quad \forall x$

DEMIDOVIC 245

$$\lim_{x\to+\infty}\left(\frac{x^2+2}{2x^2+1}\right)^{x^2} \qquad x^2\to+\infty$$

$$\left(\lim_{x\to+\infty}\frac{x^2+2}{2x^2+1}=\frac{\cancel{x^2}\left(1+\frac{2}{x^2}\right)}{\cancel{x^2}\left(2+\frac{1}{x^2}\right)}\longrightarrow\frac{1}{2}\right) \quad \text{“}\left(\frac{1}{2}\right)^{\infty}=0\text{”}$$

$$\varepsilon=\frac{1}{4} \quad \exists M:\ x>M \implies \frac{1}{4}<\frac{x^2+2}{2x^2+1}<\frac{3}{4}=\frac{1}{2}+\varepsilon<1$$

$\frac{1}{2}$, $\frac{3}{4}$, 1

$$\left(\frac{1}{4}\right)^{x^2} \quad \left(\frac{x^2+2}{\ldots}\right)^{x^2} \quad \left(\ldots\right)^{x^2} \quad \ldots^{x^2}$$

$$\underset{\downarrow\\0}{\left(\frac{\cdot}{4}\right)} < \left(\frac{x^2+2}{2x^2+1}\right) < \underset{\downarrow\\0}{\left(\frac{3}{4}\right)} \Longrightarrow \lim_{x\to+\infty}\left(\frac{x^2+2}{2x^2+1}\right) = 0$$

METODO PER $\lim_{x\to x_0} f(x)^{g(x)}$ $\quad x_0\in\mathbb{R}$ OPPURE $+\infty$ OPPURE $-\infty$

SE $f(x)\xrightarrow[x\to x_0]{} a\in(0,1)$ $\;(0<a<1)$ $\;g(x)\xrightarrow[x\to x_0]{}+\infty$

ALLORA $f(x)^{g(x)}\xrightarrow[x\to x_0]{} 0$

SE $f(x)\xrightarrow[x\to x_0]{} a>1$, $g(x)\xrightarrow[x\to x_0]{}+\infty$ ALLORA $f(x)^{g(x)}\xrightarrow[x\to x_0]{}+\infty$

GIUSTI ESERCIZI CAP. 5 N. 101 $\quad \lim_{x\to+\infty}\frac{\log(3+\sin x)}{x^3}$

$$-1\le\sin x\le 1 \Rightarrow \frac{\log 2}{x^3} \le \frac{\log(3+\sin x)}{x^3} \le \frac{\log 4}{x^3}\xrightarrow[x\to+\infty]{}0$$

$$\frac{\log 2}{x^3}\xrightarrow[x\to+\infty]{}0$$

PERCHÉ $\frac{1}{x^3}\xrightarrow[x\to+\infty]{}0$

TEO. DEL CONFRONTO $\Rightarrow \lim_{x\to+\infty}\frac{\log(3+\sin x)}{x^3}=0$

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90) $$\lim_{x\to+\infty}\frac{6x^4-x^2}{x-x^3} = \lim_{x\to+\infty}\frac{6x^4\left(1-\frac{1}{6x^2}\right)}{-x^3\left(-\frac{1}{x^2}+1\right)}$$

$$=\lim_{x\to+\infty}\frac{6x^4}{-x^3}\cdot\lim_{x\to+\infty}\frac{1-\frac{1}{6x^2}}{1-\frac{1}{x^2}} \longrightarrow 1$$

$\|$

$-6x \to -\infty$

$$\Rightarrow \lim_{x\to+\infty} \frac{6x^4-x^2}{x-x^3} = -\infty$$

$\boxed{95}$ $$\lim_{x\to+\infty}\left(x-(\sin x)^2 \lg x\right) = \lim_{x\to+\infty} \underbrace{x}_{\downarrow\, +\infty}\Big(\underbrace{1-\frac{\sin^2 x \lg x}{x}}_{\downarrow\, 1}\Big)$$

$$\left|\frac{\sin^2 x \lg x}{x}\right| \le \frac{|\lg x|}{x} \xrightarrow[x\to+\infty]{} 0$$

$$|f(x)| \xrightarrow[x\to+\infty]{} 0 \quad \Downarrow$$

$$\Rightarrow \lim_{x\to+\infty}\left(x-\sin^2 x \lg x\right) = +\infty$$

$$\underset{\downarrow\, 0}{-|f(x)|} \le \underset{\downarrow\, 0}{f(x)} \le \underset{\downarrow\, 0}{|f(x)|}$$

(CONFRONTO)