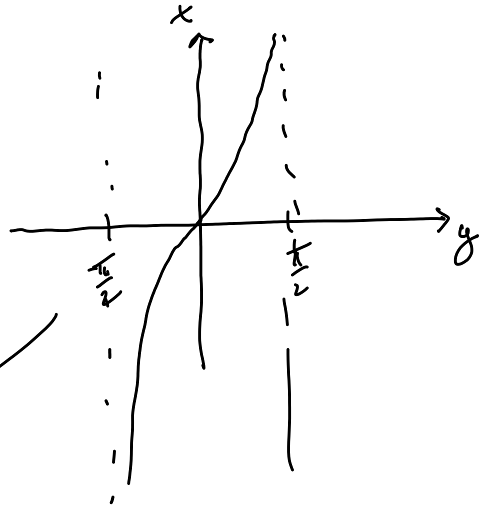
Lezione 14 22-10-21

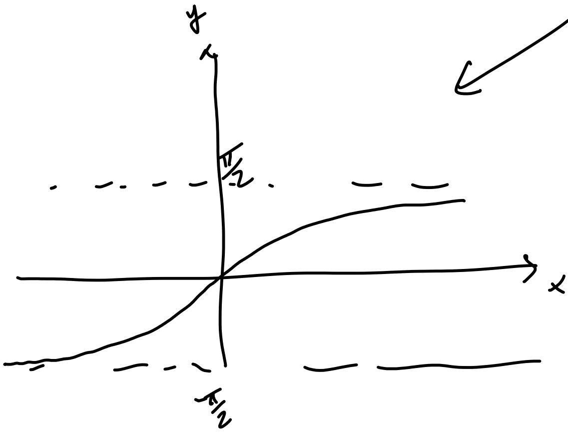
venerdì 22 ottobre 2021 08:12

Derivata di $\operatorname{arctg} x = y$

$\Updownarrow$

$\operatorname{tg} y = x$




Oss. Se $f$ è dispari $(f(-x) = -f(x))$ e invertibile $\Rightarrow$ $f^{-1}$ è dispari

Regola di derivazione funzione inversa

$y = y(x)$ con inversa $x = x(y)$

$$y'(x) = \frac{1}{x'(y)} = \frac{1}{x'(y(x))} \leftarrow x' \text{ calcolata in } y(x)$$

$(x' \circ y)(x)$

( $\boxed{x(y(x)) = x}$ $\leftarrow$ $x \circ y = id$ derivo rispetto ad $x$

funzione, variabile indipendente

$x' \circ y \cdot y' = 1$

$y' = \frac{1}{x' \circ y(x)}$ )

$$D \operatorname{arctg}(x) = \frac{1}{(D_y \operatorname{tg})(y(x))} = \cos^2 y = \frac{1}{1+x^2}$$

$D_x$

$$\left(\frac{\sin y}{\cos y}\right)' = \frac{1}{\cos^2 y}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

$y = \operatorname{arctg} x \Leftrightarrow \boxed{x = \frac{\sin y}{\cos y}} \Rightarrow x^2 = \frac{\sin^2 y}{\cos^2 y} = \frac{1-\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} - 1$

$1 + x^2 = \frac{1}{\cos^2 y}$

$$\boxed{D \operatorname{arctg} x = \frac{1}{1+x^2}}$$

---

[D] 396 $\quad y = e^x \arcsin x$ $\qquad \boxed{D \arcsin x = \frac{1}{\sqrt{1-x^2}}}$

Regola di Leibniz $\quad y' = e^x \arcsin x + e^x \frac{1}{\sqrt{1-x^2}} =$

$$= e^x \left( \arcsin x + \frac{1}{\sqrt{1-x^2}} \right)$$

Es 379 $\quad y(x) = \frac{1}{x} + 2\log x - \frac{\lg x}{x}$

$$y' = -\frac{1}{x^2} + 2\,\frac{1}{x} - \frac{1-\lg x}{x^2}$$

$$= \frac{2x + \lg x - 2}{x^2}$$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$(x^{-1})' = \frac{-1}{x^2}$.

$D x^\alpha = \alpha x^{\alpha-1}$

$D \lg x = \frac{1}{x}$

$\left(\lg x \, \frac{1}{x}\right)'$

$= \frac{1}{x^2} + \lg x \left(-\frac{1}{x^2}\right)$

$= \frac{1-\lg x}{x^2}$

## Es 435

$$y = \frac{1+\cos 2x}{1-\cos 2x}$$

. Si possono prendere tante strade

$z(t) = \frac{1+t}{1-t}$, $y = z(\cos 2x)$ $\qquad D y(x) = (z' \circ \cos 2x) \cdot (-2\sin 2x)$

$z'(t) = \frac{(1-t)+(1+t)}{(1-t)^2} = \frac{2}{(1-t)^2}$ $\qquad (1+t)'(1-t) - (1+t)(-1)$

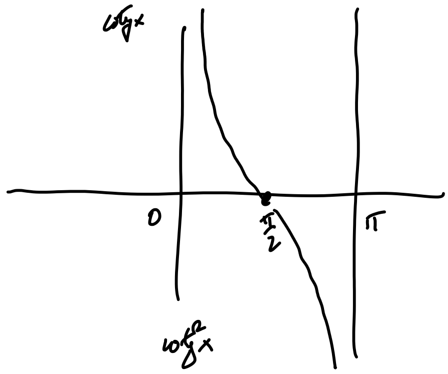
Altra strategia $\quad \boxed{\frac{1+\cos 2x}{1-\cos 2x}}$ .

$\cos 2x = \cos^2 x - \sin^2 x$

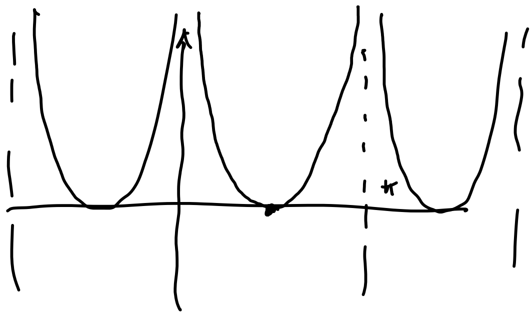
$$= \frac{1+\cos^2 x - \sin^2 x}{\phantom{x}} = 2\cos^2 x$$

$$1 - \cos^2 x + \sin^2 x \qquad \cancel{2\sin^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x} = (\cot g\, x)^2$$




$\frac{\cos x}{\sin x}$




$$\left(\frac{\cos x}{\sin x}\right)' = -\frac{1}{\sin^2 x}$$

$$D(\cot g\, x)^2 = \underbrace{2\cot g\, x}\cdot\left(-\frac{1}{\sin^2 x}\right)$$

$$z(t) = t^2 \qquad (z \circ \cot g\, x)' = \underbrace{z' \circ \cot g\, x} \cdot D(\cot g\, x)$$

$$= -2\,\frac{\cos x}{\sin^3 x}$$

---

$$\boxed{\sqrt{1+e^{\sin x^2}}}$$

$$x \xrightarrow[f_1]{(\ )^2} x^2 \xrightarrow[f_2]{\sin} \sin x^2$$

$$\xrightarrow[f_3]{e} e^{\sin x^2} \xrightarrow[f_4]{1+(\ )} 1+e^{\sin x^2} \xrightarrow[f_5]{\sqrt{\ }} \sqrt{1+e^{\sin x^2}}$$

$$\sqrt{1+e^{\sin x^2}} = f_5 \circ \underbrace{f_4 \circ f_3 \circ f_2}_{g} \circ f_1\,(x)$$

$$D\sqrt{1+e^{\sin x^2}} = f_5' \circ \underbrace{f_4 \circ \dots f_1} \cdot f_4' \circ f_3 \dots \cdot f_3' \circ f_2 \circ f_1 \cdot f_2' \circ f_1 \cdot f_1'$$

$$f_5'(y) = \frac{1}{2\sqrt{y}} \qquad f_4'(y) = 1 \qquad f_3(y) = e^y$$

$$f_2'(y) = \cos y \qquad f_1'(y) = 2y$$

$$\frac{1}{\cancel{2}\sqrt{1+e^{\sin x^2}}} \cdot e^{\sin x^2} \quad \cos x^2 \cdot \cancel{2}x = \frac{e^{\sin x^2} \cos x^2}{\sqrt{1+e^{\sin x^2}}}$$

---

La derivata è legata intrinsecamente alla crescita/decrescita delle funzioni

$$R_f(y,x) \qquad y \neq x$$

$$f'(x) = \lim_{y \to x} \overbrace{\frac{f(y)-f(x)}{y-x}}^{R_f(y,x)}$$

$$y > x \quad \begin{cases} R_f \geq 0 \iff f(y) \geq f(x) \\ R_f > 0 \iff f(y) > f(x) \\ R_f \leq 0 \ (<) \iff f(y) \leq f(x) \end{cases}$$

$y < x$

$$R_f(y,x) = \frac{f(x)-f(y)}{x-y} = \frac{f(y)-f(x)}{y-x}$$

Quindi se $f : [a,b]$ ed è crescente ($f(y) \leq f(x)$ se $y < x$)

strettamente cresc. $f(y) < f(x)$ se $y < x$

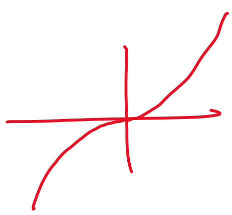
$$\iff \boxed{R_f(x,y) \geq 0, \ \forall\, x \neq y}$$

Teorema (i) Se $f$ è crescente e derivabile su $[a,b]$ $\Rightarrow f'(x) \geq 0$

e "analog" le altre affermazioni (ii) $f$ decresc $\Rightarrow f'(x) \leq 0$

(iii)* $f$ strett. cresc. $\Rightarrow f' \geq 0$

Es $x^3$ è strett. cresc. su $\mathbb{R}$



$$D x^3 = 3x^2 \Big|_{x=0} = 0$$

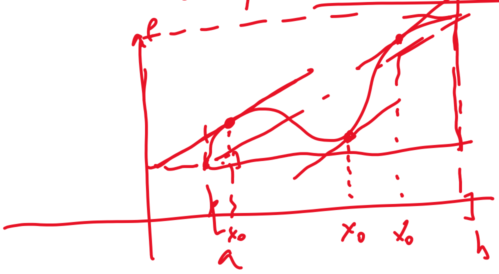
**NB** $\boxed{f(x) > \alpha \quad \forall x \implies \lim_{x\to x_0} f(x) \geq \alpha \ \text{(se esiste)}}$

$f(x) = 1-x^2$ , $1-x^2 > 1$, ma $\lim_{x\to 0} 1-x^2 = 1$.

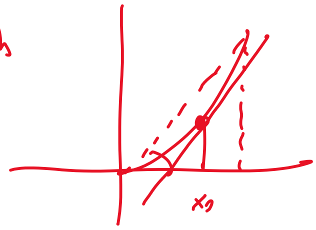
## Teorema del valor medio (Lagrange)

$f$ cont su $[a,b]$ , derivabile in $(a,b)$. Allora

$\exists \ x_0 \in (a,b) \ \Big| \ \boxed{f(b) - f(a) = f'(x_0)(b-a)}$




$$\frac{f(b)-f(a)}{b-a} = f'(x_0)$$




Se $f' \underset{=}{>} 0$ su $(a,b)$ $\implies$ $f$ è $\underset{=}{\text{strett}}$ crescente.

$x<y$ in $(a,b)$ applico Lagrange a $[x,y]$

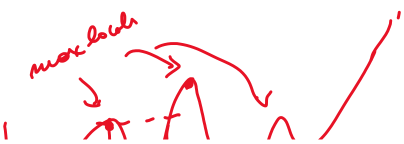
$$f(y) - f(x) = (y-x)\, f'(x_0) > 0 \iff f(y) > f(x).$$

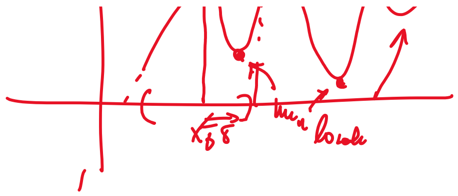
## Studio dei massimi/minimi di una funzione

**Def** $f: A \subseteq \mathbb{R} \to \mathbb{R}$

$x_0$ è un MASSIMO GLOBALE (MIN) di $f$ se $f(x_0) \underset{\leq}{\geq} f(x)$, $\forall x \in A$

$x_0$ è un MASSIMO LOCALE se $\exists \ \delta > 0 \ \Big| \ f(x_0) \geq f(x)$
$\forall x \in A, \ |x-x_0| < \delta$






Se $x_0$ è un max (min) locale interno ossia

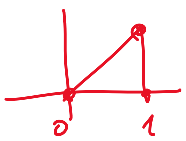
$(x_0-\delta, x_0+\delta) \subseteq A$

$$\frac{R_f(x,x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} \begin{cases} \le 0 & x>x_0 \\ \ge 0 & x<x_0 \end{cases}$$

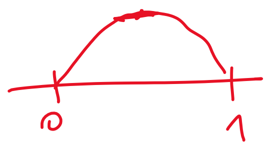
Teo (Fermat): Se $f$ è der. in $x_0$ $\Rightarrow$ $f'(x_0)=0$ $\boxed{\text{CONDIZIONE NECESSARIA}}$

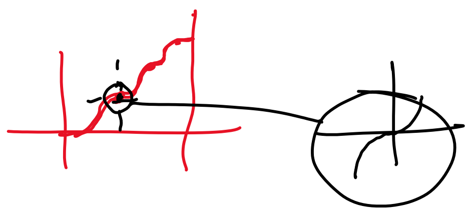
$x_0$ pto int. di max (min) locale.

Es. $f(x)=x$ su $[0,1]$



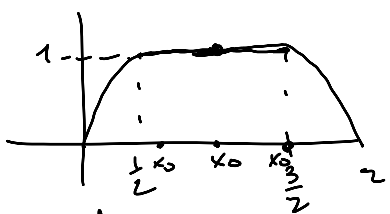
$x_0=1$



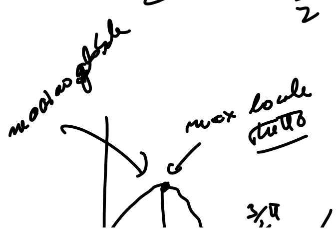


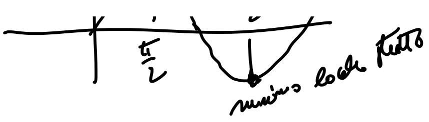
Def. $x_0$ è un max locale stretto

$\exists\, \delta>0 \mid f(x_0)>f(x) \quad \forall\, x\in A,\ 0<|x-x_0|<\delta$



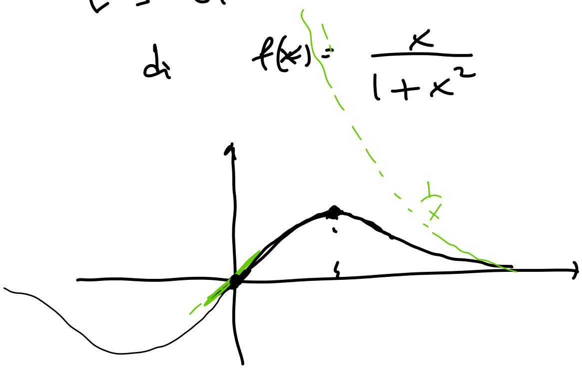
$\forall\ \frac{1}{2}\le x_0\le\frac{3}{2}$ $x_0$ è pto di max globale (non stretto)






[D] 849 Determinare massimo/minimo assoluti (o globale)

di $f(x) = \dfrac{x}{1+x^2}$  $f$ è dispari



$f > 0$ per $x > 0$

$$\lim_{x \to +\infty} \frac{x}{1+x^2} = 0$$

$f(x) \sim x$ vicino a $0$

$f(x) \sim \frac{1}{x}$ vicino a $+\infty$

$$f'(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \quad \frac{x}{1+x^2}$$

$$f'(x) = \frac{(1-x)(1+x)}{(1+x^2)^2} = \begin{cases} 0 & \iff x = \pm 1 \\ > 0 & -1 < x < 1 \\ < 0 & |x| > 1 \end{cases}$$

$1$ è un pto di max globale stretto ( $f(1) = \frac{1}{2}$
$-1$ " min " " $f(-1) = -\frac{1}{2}$

per $x \in (0,1)$ $f$ è strett. cres partendo da $0 = f(0)$

e arriva in $f(1) = \frac{1}{2}$  $\frac{1}{2} > f(x)$ $\forall x \in (0,1)$

$x > 1$ $f$ è strett. decre. $f(x) < \frac{1}{2}$ $\forall x > 1$

$\Rightarrow$ $\frac{1}{2}$ è max globale.

OSS. $[a,b]$  $f$ cont. in $[a,b]$ e deriv.

il max può essere o in $a$ o in $b$ o

nei punti interni dove si annulla la derivata