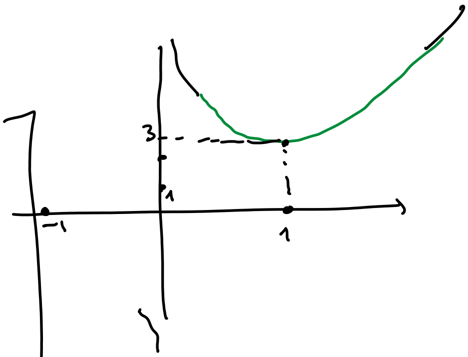
Lezione 16 27-10-21

mercoledì 27 ottobre 2021 08:27

[T] 924 $f = x^2 + \frac{2}{x}$ $A = \{x \in \mathbb{R} \mid x \neq 0\}$

Vicino a 0. $\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to 0^-} f(x) = -\infty$



$\lim_{x \to \pm\infty} f(x) = +\infty$.

Se $x > 0$, $f > 0$.

$f = 0 \iff x^2 + \frac{2}{x} = 0 \iff x^3 + 2 = 0 \iff x^3 = -2 \iff x = -\sqrt[3]{2}$

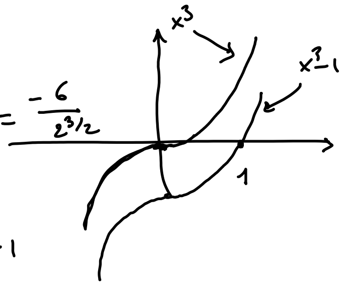
Studio $f'(x) = 2x - \frac{2}{x^2}$

$\left((-2x^{-2})' = (-2)(-2)x^{-3} = \frac{4}{x^3} \right)$

$f' = 0 \iff 2\underbrace{\frac{x^3 - 1}{x^2}}_{f'} = 0 \iff x^3 = 1 \iff x = 1$

$\boxed{f'' = 2 + \frac{4}{x^3}}$

$f'(-\sqrt[3]{2}) = \frac{-6}{2^{3/2}}$



$f' > 0 \iff x^3 - 1 > 0$

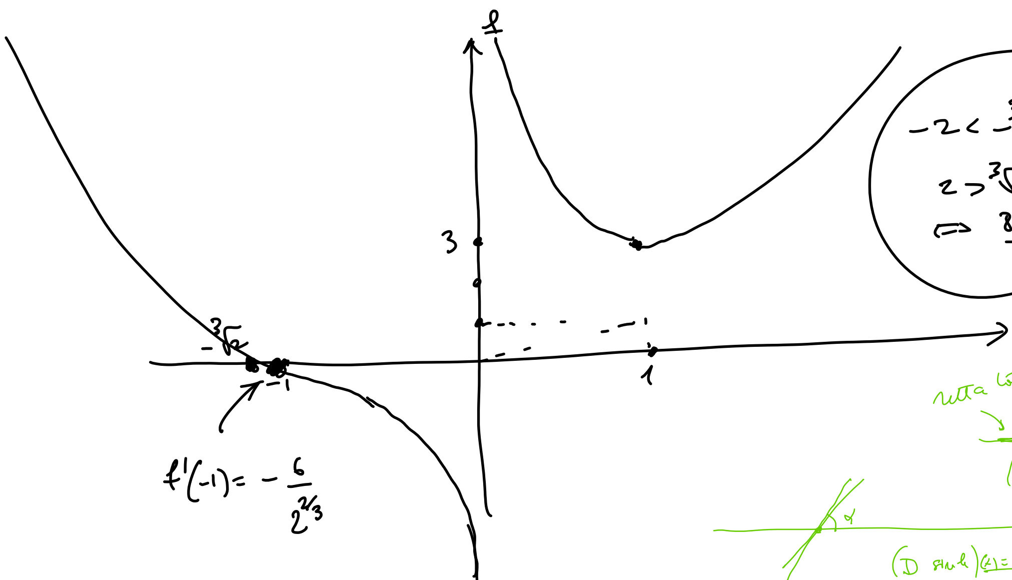
$\left(-2^{\frac{1}{3}}\right)^2 = 2^{\frac{2}{3}}$

$\boxed{\sqrt[3]{2} = 2^{\frac{1}{3}}}$

$f' > 0$ se $x > 1$, $f' < 0$ se $x < 1$

f strett cres per $x > 1$ strett. decr per $x < 1$ $\Rightarrow$ $x_0 = 1$ è un pto di min stretto su $(0, +\infty)$ con valore 3

Per $x < 0$, $f' < 0 \Rightarrow f$ strett decres (ci non ci sono locali)

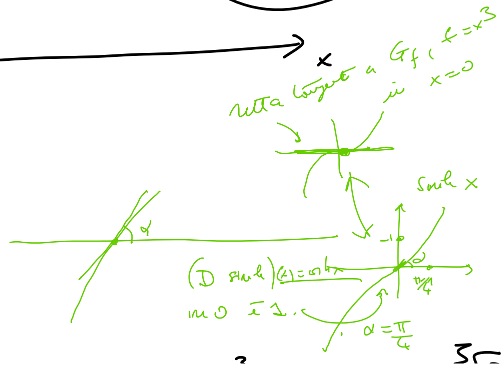


Perché

$-2 < -\sqrt[3]{2} < -1$

$2 > \sqrt[3]{2}$

$\iff 8 > 2$



$f'' = 2\left(1+\frac{2}{x^3}\right)$, $f''=0 \iff 1+\frac{2}{x^3}=0 \iff x^3=-2 \iff x=-\sqrt[3]{2}$

$f''=0 \iff x^3=-2 \iff x=-\sqrt[3]{2}$

$f''>0$ per $x>0$ strett. convessa in $(0,+\infty)$

$f''<0$ strett. concava per $-\sqrt[3]{2}<x<0$

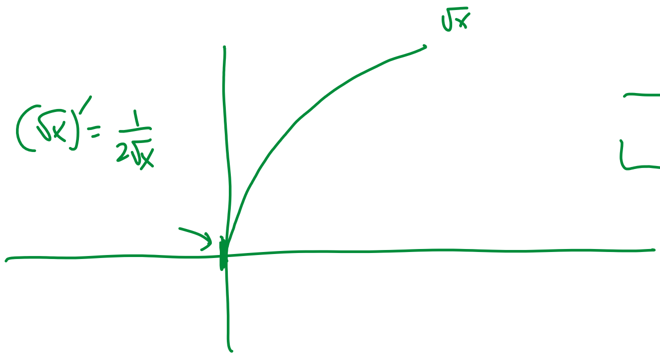
$f''>0$ " convessa per $x<-\sqrt[3]{2}$

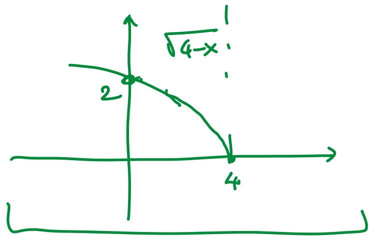
[D] 932 $f(x)=\sqrt{x}+\sqrt{4-x}$

Dominio di f: $x\geq 0$ & $4-x\geq 0$

$A=[0,4]=\{0\leq x\leq 4\}$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

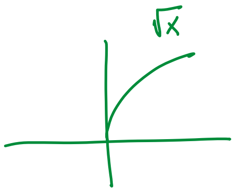




$f(x)=\sqrt{x-4}$

$g(x) \rightarrow g(x-x_0)$

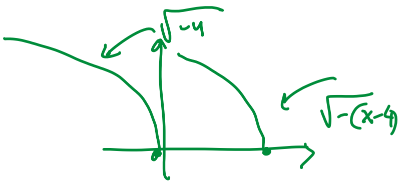
$g(x-1)$

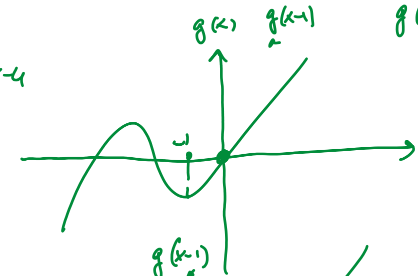


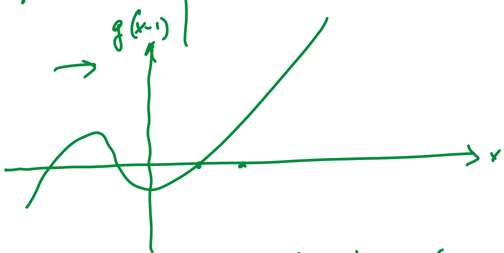
$\sqrt{4-x}$

$=\sqrt{-(x-4)}$

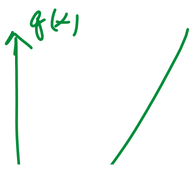
$=\sqrt{-y}$, $y=x-4$

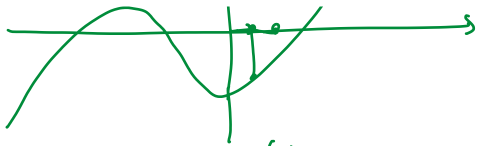


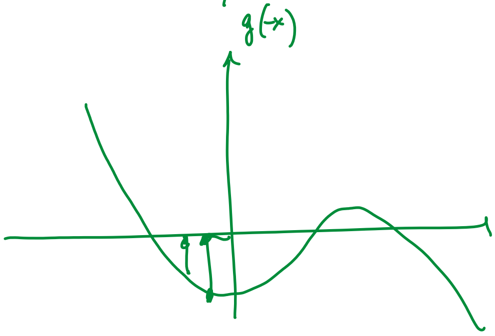




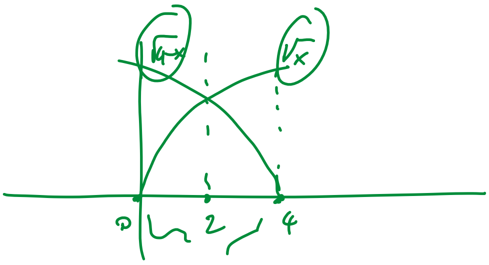
il grafico di $g(x-x_0)$ è il grafico di $g(x)$ traslato a destra di x_0





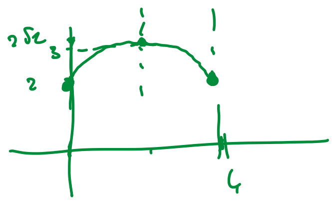


$f(x) = \sqrt{x} + \sqrt{4-x}$



$f(0) = 2 = f(4)$

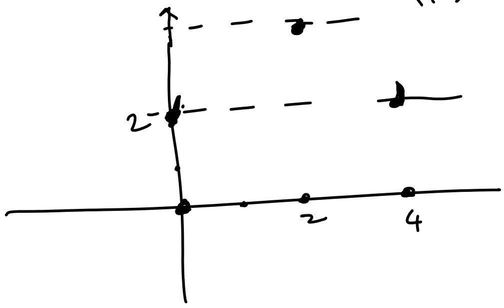
$f(2) = \underline{2\sqrt{2}}$



$f(x) = \sqrt{x} + \sqrt{4-x}$

$f(x) > 0$ in $(0,4)$

$f(0) = f(4) = 0$



$$f' = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{4-x}} = \frac{1}{2}\,\frac{\sqrt{4-x} - \sqrt{x}}{\sqrt{x(4-x)}}$$

$f' = 0 \iff \sqrt{4-x} = \sqrt{x} \iff 4 - x = x \Rightarrow x = 2$ ✓

$f(2) = 2\sqrt{2}$ $\Rightarrow$ 2 è un max globale (anche lo)

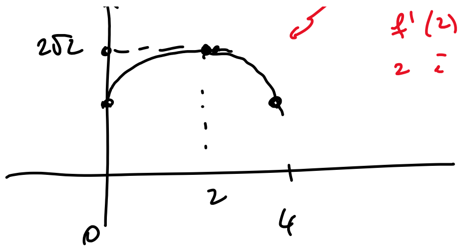
$$f'' = -\frac{1}{4x^{3/2}} + \frac{1}{4}\,\frac{1}{(4-x)^{3/2}} = \frac{1}{4}\left(\frac{1}{(4-x)^{3/2}} - \frac{1}{x^{3/2}}\right)$$

$$\left(\frac{1}{2}x^{-\frac{1}{2}}\right)' = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}\,\frac{1}{x^{3/2}} \quad ✓$$

$$\left(\frac{1}{2\sqrt{4-x}}\right)' = \frac{1}{2}\left((4-x)^{\frac{1}{2}}\right)' = -\frac{1}{2}\cdot\frac{1}{2}\,\frac{1}{(4-x)^{3/2}} = -\frac{1}{4}\,\frac{1}{(4-x)^{3/2}}$$

$$f'' = -\frac{1}{4}\,\frac{x^{3/2} + (4-x)^{3/2}}{(\quad)} \leq 0 \quad \text{per } 0 < x < 4$$

$f'' < 0$



$f'(2)=0$

2 è un max globale

Teorema 1 $\quad f \in C^2((a,b))$, stett. convessa, $f'(x_0)=0$, $a<x_0<b$

$\Rightarrow$ x_0 è minimo assoluto (globale) in (a,b) ed è l'unico PUNTO CRITICO o stazionario ($f'(y)=0$)

Teorema 1' $\quad f \in C^2((a,b))$ strett. concava, $f'(x_0)=0$

$\Rightarrow$ x_0 è massimo assoluto in (a,b) ed è l'unico ...

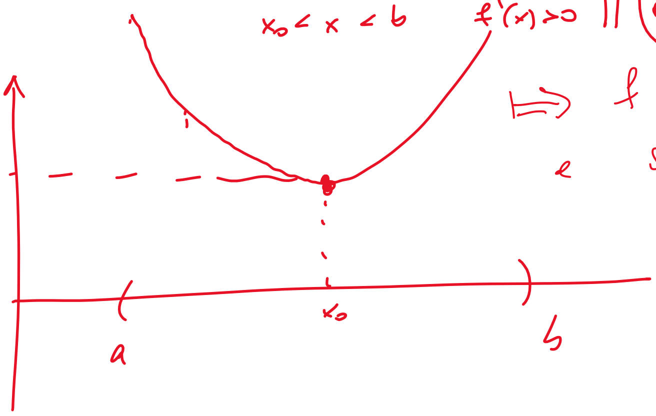
Dim $\quad f''>0 \iff f'$ è strett. cresc. in (a,b) e $f'(x_0)=0$

$(f')'$

$\Rightarrow$ $a<x<x_0$, $f'(x)<0$

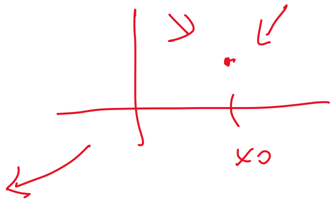
$x_0<x<b$ $\quad f'(x)>0$

$f'(x)\neq 0$ se $x\neq x_0$



$\Rightarrow$ f stett. decresc. per $x<x_0$

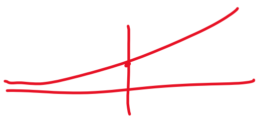
e stretti crescente per $x>x_0$



$\Rightarrow$ x_0 è un min assoluto stett

Naturalmente ci sono funzioni stett convesse senza punti critici

ad esempio $\quad e^x$



$(e^x)' = e^x > 0$

Teorema 2 (Teorema della permanenza del segno)

f è cont. in A e $f(x_0)\neq 0$ $\quad (x_0 \in A)$.

Allora, localmente f mantiene lo stesso segno di $f(x_0)$

cioè $\exists\, \delta>0 \mid f(x)\,f(x_0)>0, \ \forall\, x\in A$ e $|x-x_0|<\delta$,

$y, z \neq 0$

$y\cdot z>0$

$\Rightarrow$ y e z hanno lo stesso segno

Dim. Esercizio! [Sugg. usare la definizione di limite con $\varepsilon = \frac{|f(x_0)|}{2} > 0$]

Teorema 3 $f \in C^2((a,b))$, e $x_0 \in (a,b)$ | $f'(x_0) = 0$ e $f''(x_0) > 0$.

Allora, x_0 è un minimo locale stretto

Dim f'' è cont in x_0 per P.d.S. (Teorema 2) $f'' > 0$ in $(x_0-\delta, x_0+\delta)$ per un qualche $\delta > 0$ suff. piccolo

e ora applico il Teorema 1 a f su $(x_0-\delta, x_0+\delta)$. ■

Asintoti obliqui

DEF. f ha un asintoto obliquo in $+\infty$

$\Longleftrightarrow \exists\, a \neq 0$ e b | $f(x) - (ax+b) \xrightarrow[x\to+\infty]{} 0$.

[analog. asint. obliquo in $-\infty$]

Quindi $\frac{f(x) - (ax+b)}{ax} \to 0$

$$\Updownarrow$$

$$\frac{f(x)}{ax} - 1 + \underbrace{\frac{b}{ax}}_{\to 0} \Rightarrow \frac{f(x)}{ax} \xrightarrow[x\to+\infty]{} 1 \Longleftrightarrow \boxed{f(x) \sim ax} \text{ per } x \to +\infty$$

ed inoltre

$$b = \lim_{x\to+\infty} f(x) - ax$$

Es $f(x) = \frac{x^2+1}{2x-1}$ ha asint. obliquo in $\pm\infty$

$$= \frac{x^2\left(1+\frac{1}{x^2}\right)}{2x\left(1-\frac{1}{2x}\right)} \sim \frac{x}{2} \text{ per } x \to \pm\infty$$

$x \to$

$$f(x) - \frac{x}{2} = \frac{x^2+1}{2x-1} - \frac{x}{2} = \frac{2(\cancel{x^2}+1) - 2\cancel{x^2} + 2x}{(2x-1)\,2} =$$

$$= \frac{\cancel{2}(1+x)}{\cancel{2}(2x-1)} \xrightarrow[x \to \pm\infty]{} \frac{1}{2}$$

$\frac{x+1}{2}$ è un asint. obliquo a $\pm\infty$

finire lo studio di funzi.