Lezione 17 29-10-21

venerdì 29 ottobre 2021 08:12

Es 1 Dimostrare per induzione che

$$\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}, \quad \forall n \in \mathbb{N}.$$

$\{a_k\}$ $\quad \sum_{k=1}^{n} a_k \overset{DEF}{=} \begin{cases} a_1, & n=1 \\ \sum_{k=1}^{n-1} a_k + a_n, & n \geq 2 \end{cases}$ $\qquad \sigma_n = \sum_{k=1}^{n} a_k$

sommatoria

$$\boxed{\sigma_n = \begin{cases} a_1 & n=1 \\ \sigma_{n-1} + a_n, & n \geq 2 \end{cases}}$$

Esempio $a_k = \frac{1}{k}$

$\sigma_1 = 1 \qquad \sigma_2 = 1 + \frac{1}{2} = \frac{3}{2} \qquad \sigma_3 = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$

$\sigma_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \sigma_4 = \frac{11}{6} + \frac{1}{4} = \frac{50}{24} = \frac{25}{12}$

$$\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2} \quad (*)_n$$

Base induttiva $\quad n=1 \quad \sum_{k=1}^{2} \frac{1}{k} = 1 + \frac{1}{2} = \frac{3}{2} = 1 + \frac{1}{2}$

$(*)_1$ è vera.

Nel nostro esercizio $\sigma_n = \sum_{k=1}^{2^n} \frac{1}{k}$

Dimostriamo che $(*)_n \Rightarrow (*)_{n+1}$:

$$\sigma_{n+1} = \sum_{k=1}^{2^{n+1}} \frac{1}{k} = \underbrace{\sum_{k=1}^{2^n} \frac{1}{k}}_{\sigma_n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \overset{(*)_n}{\geq}$$

$\boxed{\sigma_n \geq 1 + \frac{n}{2}}$

$$1 + \frac{n}{2} + \underbrace{\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}}_{} \overset{?}{\underset{(*)_{n+1}}{\geq}} 1 + \frac{n+1}{2}$$

$$\iff \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq \frac{1}{2}$$

$$\underbrace{\underbrace{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n}}_{\sigma_n} + \underbrace{\frac{1}{2^n+1} + \dots + \frac{1}{2^{n+1}}}_{\text{quanti sono questi termini?}}}_{\sigma_{n+1}}$$

Numero di termini

$\# \; k \mid 1 \leq k \leq 2^{n+1}, \qquad \# \{k \mid 1 \leq k \leq N\}$

N

$\#\{k \mid 2^n+1 \le k \le 2^{n+1}\}$ numero di termini in questa

$= 2^{n+1} - 2^n = 2^n(2-1) = 2^n$

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}}$$

$$= \frac{1}{2^{n+1}}\Big(\underbrace{2^{n+1}}_{M} - \underbrace{(2^n+1)}_{N} + 1\Big)$$

$$= \frac{1}{2^{n+1}}\big(\underbrace{2^{n+1} - 2^n}_{2^n}\big)$$

$$= \frac{1}{2} \checkmark$$

$a_k \geq b_k$, $M \geq N$

$$\sum_{k=N}^{M} a_k \geq \sum_{k=N}^{M} b_k$$

$$\boxed{\sum_{k=N}^{M} a = (M-N+1)\,a}$$

$$\sum_{k=2}^{5} a = \overset{k=2}{a} + \overset{k=3}{a} + \overset{k=4}{a} + \overset{k=5}{a} = 4a$$

$$\sum_{k=2}^{5} a_k = a_2 + a_3 + a_4 + a_5$$

$a_k = a$ $= 4a$

$\#\{N \le k \le M\}_{k \in \mathbb{N}} = M - N + 1$

Es dimostrare per induzione su $M \geq N$

Base induttiva $M = N$

$\#\{M \le k \le M\} = 1$

. . . . .

Abbiamo dimostrato che

$$\boxed{\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}}$$

Facciamo il lim per $n \to +\infty$

$$\sum_{k=1}^{\infty} \frac{1}{k} := \lim_{N \to +\infty} \sum_{k=1}^{N} \frac{1}{k} = \infty$$

serie ARMONICA

$\frac{1}{k} \to 0$ per $k \to +\infty$

Per def. $\forall M > 0 \ \exists N \mid \sum_{k=1}^{N} \frac{1}{k} > M$

$$\sum_{k=1}^{N} \frac{1}{k} \geq \sum_{k=1}^{N} \frac{1}{k} > M$$

$$\sum_{k=1}^{N} \frac{1}{2k} = \sum_{k=1}^{N} \frac{1}{2} \cdot \frac{1}{k} = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{k}$$

$$\forall N' \geq N$$

$$\sum_{k=1}^{N} \frac{1}{k} > 100$$

$$2^{200} = 2^{20\cdot 10} = (2^{10})^{20} = (1024)^{20} \sim 10^{3\cdot 20} = 10^{60}$$

$$= 1\underbrace{0000\ \ldots\ 0}_{60 \text{ zeri}}$$

$$\sum_{k=1}^{2^n} \frac{1}{k} > 1 + \frac{n}{2} = 101 \quad \text{se } n = 200$$

$$\sum_{k=1}^{2^{200}} \frac{1}{k} > 101$$

2^{200} termini

$$\boxed{2^{200} > 10^{60}}$$

$$N = 2^n \qquad \boxed{n = \log_2 N.}$$

$$\sum_{k=1}^{2^n} \frac{1}{k} > 1 + \frac{\log_2 2^n}{2} \rightsquigarrow \boxed{\sum_{k=1}^{N} \frac{1}{k} > 1 + \frac{\log_2 N}{2}}$$

$$\zeta(1) = \sum_{k=1}^{\infty} \frac{1}{k}$$

DEF. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^s} =: \zeta(s)$

= funzione zeta di Riemann calcolata in s

$$\boxed{s \in \mathbb{C}}$$

Teorema I numeri primi sono infiniti.

def $p \in \mathbb{N}$, $p \geq 2$, che è "divisibile" solo per se stesso e 1.

$p = kn$ con $k, n \in \mathbb{N}$

1 non è numero primo.

2, 3, 5, 7, 11, 13, 17, 19, ...

Dimo di Euclide per assurdo supponiamo che i numeri siano finiti, $\{p_1, p_2, p_3, \ldots, p_n\} = \{\text{numeri primi}\}$

primi stanno tutti $\{\underbrace{11, 12, 13, \dots, N}\}$

$$2 = p_1 < \underset{3}{\underset{\shortparallel}{p_2}} < \dots < p_N$$

$$p = p_1 \cdot p_2 \cdots p_N + 1 > p_N$$

p dovrebbe essere un numero non primo

$$\frac{p}{p_i} = \underbrace{p_1 \cdot p_2 \cdots \cancel{p_i} \cdots p_N}_{\in \mathbb{N}} + \frac{1}{p_i}$$

che non è un numero naturale $\quad p_i \geq 2$

$$0 < \frac{1}{p_i} < 1$$

Contraddizione!

Un algoritmo

$k \to p_k$ non esiste

Se conosciamo

$p_1 - p_N$

ovvero i primi N numeri primi.

$$p_1 \cdots p_N + 1$$

$2 \cdot 3 \cdot 5 + 1 = 31$

$\underbrace{2 \cdot 3 \cdot 5}_{30} \cdot 7 + 1 = 211$

$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$

$\underbrace{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}_{} + 1 = 30031$

$= 59 \cdot 509$

tutti i numeri primi $\leq N$

$p_1, \text{——}, p_N$

p numero più grande conosciuto al momento.

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2 ES Dim per induzione

$$(*)_n \quad \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

$$\sum_{k=1}^{n} \frac{1}{k^2}$$

$(\sum^{\infty} \dots)$

$$\left(S(2) = \sum_{k=1} \frac{1}{k^2} \right) \quad \rightarrow \quad \underbrace{\underset{k=1}{1} + \underset{k=2}{\frac{1}{4}} + \underset{k=3}{\frac{1}{9}} + \underset{k=4}{\frac{1}{16}}}_{\sum_{k=1}^{4} \frac{1}{k^2}} + \frac{1}{25} + \dots \frac{1}{n^2}$$

$$(*)_1 \qquad \sum_{k=1}^{1} \frac{1}{k^2} = 1 = 2-1 = 2 - \frac{1}{n} \quad \text{per } n=1. \quad ✓$$

$$\underbrace{\sum_{k=1}^{n+1} \frac{1}{k^2}}_{\sigma_{n+1}} = \underbrace{\sum_{k=1}^{n} \frac{1}{k^2}}_{\sigma_n} + \underbrace{\frac{1}{(n+1)^2}}_{a_{n+1}} \overset{(*)_n}{\le} 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \overset{?}{\le} \underbrace{2 - \frac{1}{n+1}}_{(*)_{n+1}}$$

$$a_k = \frac{1}{k^2}.$$

$$\rightarrow \quad 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \overset{?\ \text{sì}}{\le} 2 - \frac{1}{n+1}$$

$$\Leftrightarrow \quad \frac{1}{(n+1)^2} \le \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\Leftrightarrow \quad n(n+1) \le (n+1)^2 \quad ✓$$

$$\Updownarrow$$

$$n \le n+1$$

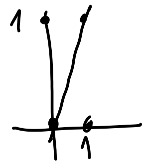
Abbiamo dimostrato che $\sum_{k=1}^{n} \frac{1}{k^2} \le 2 - \frac{1}{n}$

prendendo il limite per $n \to +\infty$.

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k^2} := \zeta(2) \le 2 < +\infty$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$$

$f(x) = x e^{-x}$ $\qquad A = \mathbb{R}$

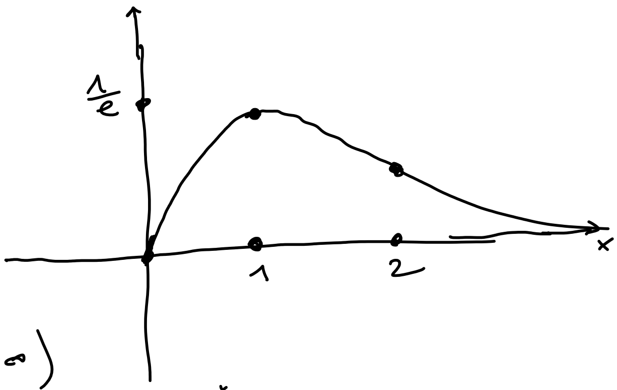


$\lim_{x \to +\infty} x e^{-x} = 0, \quad \lim_{x \to -\infty} x e^{-x} = -\infty$

$f(x) < 0$ per $x < 0$.

$f(x) > 0$ per $x > 0$,

$f(0)=0$, $f(x) \to 0$



$$f'(x) = e^{-x} - xe^{-x} = \boxed{e^{-x}(1-x)}$$

$f'=0 \iff x=1$ l'unico pto critico

$\Rightarrow$ $x=1$ è il massimo assoluto di f

(per $x<0$, $f<0$; $f(0)=0$, $f(x)\to 0$ per $x \to +\infty$)

$xe^{-x} \sim x$ vicino a 0

$$\left.\begin{array}{l} f'>0 \text{ per } x<1 \\ f'<0 \text{ per } x>1 \end{array}\right\} \Rightarrow x=1 \text{ è un max assoluto.}$$

$$\boxed{f'(0)=1}$$

$$f'' = e^{-x}(x-1-1)$$

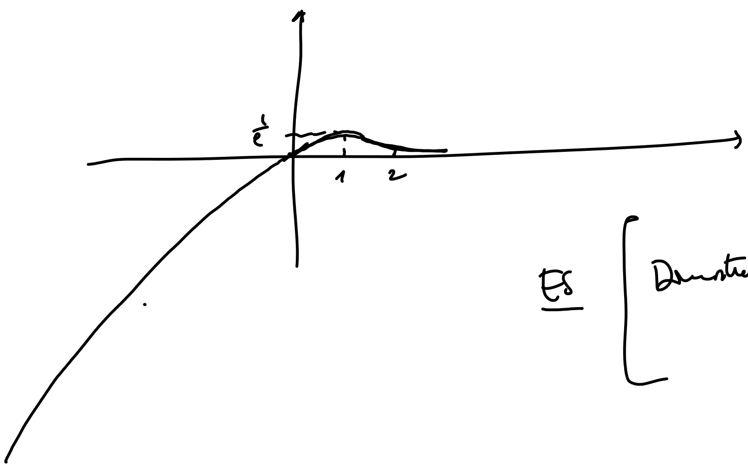
$$\left(e^{-x}(1-x)\right)' = (e^{-x})'(1-x) + e^{-x}(1-x)'$$

$(fg)' = f'g + fg'$

$$= (-e^{-x})(1-x) + e^{-x}(-1)$$

$$= e^{-x}(x-1-1) = e^{-x}(x-2)$$

$x>2$ f è convessa $\quad x<2$ concava



Es [Dimostra che $e^x \geq 1+x \quad \forall x \in \mathbb{R}$

$e^x > 1+x \quad \forall x \neq 0$]