Lezione 19 5-11-21

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Es (facile) Dimostrare che $\forall \; 1 \le k \le n \in \mathbb{N}$, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

$$\binom{n}{k-1} + \binom{n}{k} \overset{DEF}{=} \frac{k \; n!}{k\,(k-1)!\,(n-k+1)!} + \frac{(n-k+1)\; n!}{k!\,(n-k+1)(n-k)!} = \frac{k \cdot n! + (n-k+1)\, n!}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

Es* FORMULA DEL BINOMIO DI NEWTON

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad \forall \; a,b \in \mathbb{R},\; n \in \mathbb{N}} \quad (*)_n$$

Dimostrare per induzione n.

Pensiamo:

$$(a+b)^n = \overbrace{(a+b)(a+b)(a+b)\cdots(a+b)}^{n \text{ volte}}$$

$$a \cdot \underbrace{\quad}_{n-1} + b \underbrace{\quad}_{n-1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

non scelgo mai a $\to$ $k=0$ b^n

scelgo sempre a $\to$ $k=n$ a^n

$0! = 1$

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$\binom{n}{0} = 1$

$\binom{n}{n} = 1$

$\boxed{\binom{n}{k} = \binom{n}{n-k}}$

Facciamo: Base induttiva $n=1$

$$(a+b) = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0$$

$$= b + a$$

Assumiamo $(*)_n$.

$$(a+b)^{n+1} \overset{def}{=} (a+b)\,(a+b)^n \overset{(*)_n}{=} (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

prop. distrib. usata 2 volte

$j = k+1$

$$a \sum_{k=1}^{n} c_k = \sum_{k=1}^{n} a c_k$$

proprietà distrib.

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} a^j b^{n-j+1} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

nella prima somma pongo $j = k+1 \iff k = j-1$

$0 \le k \le n \quad , \quad 1 \le j \le n+1$

$$\sum_{k=1}^{N} a_k = \sum_{j=1}^{N} a_j = \sum_{\ell=1}^{N} a_\ell$$

$a_1, \dots, a_N$

$$\underset{k=j}{=} \sum_{k=1}^{n+1} \binom{n}{k-1} \underbrace{a^k b^{n-k+1}}_{= a^k b^{n+1-k}} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$\sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k = \sum_{k=1}^{N} (a_k + b_k)$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

$k = n+1$ 1ª somma

$k = 0$ nella 2ª somma

$$\sum_1^N a_k + \sum_1^N b_k$$

$$(a_1 + a_2 + \dots + a_N) + (b_1 + \dots + b_N)$$

ass. $= a_1 + \dots + a_N + b_1 + \dots + b_N$

comm. $= a_1 + b_1 + a_2 + b_2 + \dots + a_N + b_N$

$$= (a_1 + b_1) + (a_2 + b_2) + \dots + (a_N + b_N)$$

$$= \sum_{k=1}^{N} a_k + b_k$$

$$= a^{n+1} + \left(\sum_{k=1}^{n} \underbrace{\left(\binom{n}{k-1} + \binom{n}{k} \right)}_{} a^k b^{n+1-k} \right) + b^{n+1}$$

fs. Stifel

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \quad ✓$$

il termine $k=0$ è b^{n+1}

il termine $k=n+1$ è a^{n+1}

Es Dimostrare che $e^x > 1 + x \quad \forall \quad x \neq 0$.

$e^x \ge 1 + x, \ \forall x$

Ricordiamo la def. di funzione C^2 strettamente convessa $f(x)$ in (a,b)

$$f(x) > ax + b \quad \forall\, x \neq x_0$$

retta tg. al G_f nel punto $(x_0, f(x_0))$

$$:= f'(x_0)(x - x_0) + f(x_0)$$

$$\boxed{\text{Teorema} \quad f \in C^2 \text{ è strett. convessa} \iff f'' > 0}$$

retta tg a G_{e^x} in $(0,1) = (0, e^0)$ è

$(x_0 = 0,\ f(x) = e^x)$ $\quad x + 1$

$$(e^x)'' = e^x > 0. \Rightarrow e^x > x + 1 \quad \forall\, x \neq 0.$$

Altra soluzione $\quad e^x \geq (x+1),\ \forall x \quad f(x) = e^x - x - 1 \geq 0$

$$\begin{cases} f(0) = 0 \\ f(x) > 0,\ \forall\, x \neq 0. \end{cases}$$

cioè 0 è un pto di minimo globale.

Studiamo la funzione $e^x - x - 1$

$f' = e^x - 1 \qquad f'(x) = 0 \iff e^x = 1 \iff x = 0.$

$$\begin{cases} f'(x) > 0 \iff e^x > 1 \iff x > 0 \Rightarrow f \text{ è strett cresc in } (0, +\infty) \\ f'(x) < 0 \iff e^x < 1 \iff x < 0 \Rightarrow f \text{ è strett. decresc in } (-\infty, 0) \\ \Rightarrow 0 \text{ è un minimo globale} \end{cases}$$

O ancora $\quad \lim_{x \to +\infty} f(x) = +\infty, \quad \lim_{x \to -\infty} f(x) = +\infty$

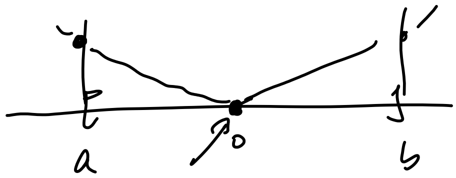
0 è l'unico punto critico.

$\Downarrow$

$\exists\ a<b$ t.c.

$f(x) \geq 1 \quad \forall x \leq a,$

$f(x) \geq 1 \quad \forall\ x \geq b > a$

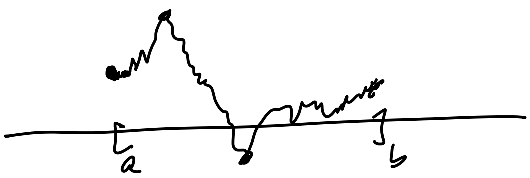


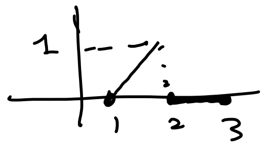
$a, b \in \mathbb{R}$
$a \neq b.$

$a = b$

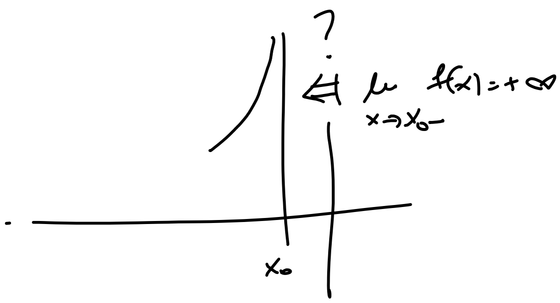
Teorema di Weierstrass f continua su $[a,b]$

ha un massimo e un minimo in $[a,b]$





$[1, 3]$



Risp NO

$$f(x) = \left(\frac{2}{1-x} + \sin \frac{1}{1-x} \right) \qquad 0 \leq x < 1.$$

Weg in $x=1$

$$\lim_{x \to 1-} f(x) = +\infty$$

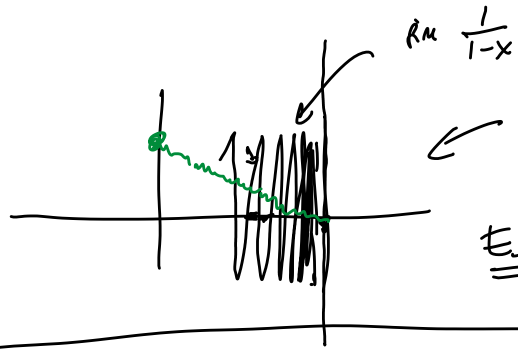
$$f'(x) = \frac{1}{(1-x)^2} + \left(\cos\frac{1}{1-x}\right)\frac{1}{(1-x)^2}$$

$(1-x)$ $(1-x)^2$

$$= \frac{2 + \cos\frac{1}{1-x}}{(1-x)^2} \geq \frac{1}{(1-x)^2} > 0.$$

$$f''(x) = \frac{-\sin\left(\frac{1}{1-x}\right)\cancel{\frac{1}{(1-x)^2}}\cancel{(1-x)^2} + \left(2+\cos\frac{1}{1-x}\right)2(1-x)}{(1-x)^4}$$

$$= \frac{2\left(2+\cos\frac{1}{1-x}\right)(1-x) - \sin\frac{1}{1-x}}{(1-x)^4}$$



f'' cambia segno ∞ volte

ES.** (AM120). derivate.