CRITERI DI CONVERGENZA PER SERIE A TERMINI POSITIVI

$$\sum_{n=1}^{\infty} a_n \,, \quad a_n > 0 \quad (o \ a_n \geq 0)$$

Il criterio fondamentale è il CRITERIO DEL CONFRONTO

$$0 \leq a_n \leq b_n$$

$$0 \leq s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k = t_n$$

Sappiamo che s_n e t_n sono strett. crescenti e che quindi $\exists$ $s = \lim s_n \overset{\text{Teo.}}{=} \sup\{s_n \mid n \in \mathbb{N}\} \leq t = \lim t_n = \sup\{t_n \mid n \in \mathbb{N}\}$

Quindi, (1) se $t < +\infty$ $\quad (t_n \in [0, \infty))$

$\Rightarrow$ $s < +\infty$ (criterio del confronto per successioni)

Ossia se converge $\sum b_k \Rightarrow$ converge $\sum a_k$.

(2) Se $s = +\infty \Rightarrow t = +\infty$ (criterio del confr. per succ.).

Ossia, se diverge $\sum a_k \Rightarrow$ diverge $\sum b_k$

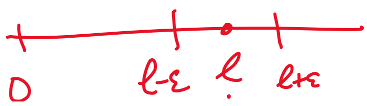
CRITERIO CONFRONTO ASINTOTICO

$a_k, b_k > 0$ e $\lim \frac{a_k}{b_k} = l \in [0, +\infty) \cup \{+\infty\}$

se $\underline{l \in (0, +\infty)} \Rightarrow \sum a_k \approx \sum b_k$

def. le due serie si comportano nello stesso modo, cioè,
- o convergono entrambe
- o divergono entrambe

$$\frac{a_k}{b_k} \to l \in (0, +\infty)$$



$\Rightarrow$

$\exists N \mid \frac{a_k}{b_k} \in (\ell-\varepsilon, \ell+\varepsilon) \quad \forall\ k \geq N$

$0 < \ell - \varepsilon \quad (\text{e.g. } \varepsilon = \frac{\ell}{2})$

$\frac{a_k}{b_k}$ appartiene definitivamente (per def da un N in poi) all'intervallo $(\ell-\varepsilon, \ell+\varepsilon) = (a, b)$, $0 < a < b$

$a < \frac{a_k}{b_k} < b, \ \forall k \geq N$

$a b_k < a_k < b\, b_k$

Quindi se $\sum a_k$ converge $\Rightarrow$ converge $\sum a b_k = a \sum b_k$ (C.C.)

$\Rightarrow$ converge $\sum b_k$

$\sum a_k$ diverge. $\Rightarrow$ $\sum a_k \leq b \sum b_k \Rightarrow \sum b_k = +\infty$

Se $\ell = 0$ $\quad \lim_{k\to\infty} \frac{a_k}{b_k} = 0$, $0 \leq \frac{a_k}{b_k} < 1$ per $k \geq N$

$a_k < b_k \Rightarrow \sum b_k$ conv $\Rightarrow \sum a_k$ conv

$\sum a_k$ div. $\Rightarrow \sum b_k$ diverge

Se $\ell = +\infty$ $\quad \exists N \mid \frac{a_k}{b_k} > 1 \quad \forall\ k \geq N$

$a_k > b_k \ldots$

CRITERIO DELLA RADICE

$a_k > 0$ supponiamo che $\exists \lim_{k\to+\infty} a_k^{\frac{1}{k}} = \lim_{k\to\infty} \sqrt[k]{a_k} = \theta$

$\theta \in [0, +\infty] := [0, +\infty) \cup \{+\infty\} = \{x > 0\} \cup \{+\infty\}$

se $\theta < 1 \Rightarrow$ la serie converge

se $\theta > 1 \Rightarrow$ la serie diverge

se $\theta = 1$ non si sa.

Esempi (i) $\sum x^k$ ← serie geometrica di ragione $x > 0$

$a_k = x^k$ $\quad a_k^{\frac{1}{k}} = x$ e sappiamo la serie conv. se $x < 1$

e diverge se $x \geq 1$

$\theta = \lim a_k^{\frac{1}{k}} = x.$

$\ldots \quad \frac{1}{k} \quad \sqrt[k]{2^k+1} = \sqrt[k]{\left(\frac{2}{3}\right)^k \left(1 + \frac{1}{2^k}\right)}$

(ii) $a_n = \frac{2^n+1}{n+3^n}$ $\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n+1}{3^n+n}} = \sqrt[n]{\frac{2^n\left(1+\frac{1}{2^n}\right)}{3^n\left(1+\frac{n}{3^n}\right)}}$

$$= \frac{2}{3}\sqrt[n]{\frac{1+\frac{1}{2^n}}{1+\frac{n}{3^n}}} \longrightarrow \frac{2}{3} < 1$$

la serie $\sum a_n$ converge per C.d.R.

N.B $\lim \frac{a_n}{\left(\frac{2}{3}\right)^n} = 1$ ora $a_n \sim \left(\frac{2}{3}\right)^n$

converge per C.C.A. con la serie $\sum \left(\frac{2}{3}\right)^n$

(iii) $\sum \frac{1}{k} = +\infty$, $\sum \frac{1}{k^2} < \infty$

ma $\lim_{k\to+\infty} \left(\frac{1}{k}\right)^{\frac{1}{k}} = 1 = \lim_{k\to+\infty} \left(\frac{1}{k^2}\right)^{\frac{1}{k}} = 1$

($\left(\frac{1}{k^p}\right)^{\frac{1}{k}} \to 1 \quad \forall\, p \in \mathbb{R}$

Idea dietro il C.R.

$(a_k)^{\frac{1}{k}} \to \underline{\ell \neq 1} \quad \Rightarrow \quad a_k \sim \ell^k$

CRITERIO RAPPORTO

Se $\exists \lim \frac{a_{k+1}}{a_k} = \ell \in [0, +\infty]$

allora se $\ell < 1$ la serie converge

$\ell > 1$ la serie diverge

$\ell = 1$ non si sa

[Segue da $\exists \lim \frac{a_{k+1}}{a_k} = \ell \Rightarrow \lim \sqrt[k]{a_k} = \ell$]

Idea $\ell < 1$

$$\frac{a_{k+1}}{a_k} \to \ell < 1$$

$\ell \quad \frac{\ell+1}{2} \quad 1$

$\frac{\ell+1}{2} = \theta < 1$

$$\exists N \mid \frac{a_{k+1}}{a_k} < \theta < 1 \quad \forall k \geq N.$$

$$\frac{a_{N+1}}{a_N} < \theta \qquad \frac{a_{N+2}}{a_{N+1}} < \theta \quad , \quad \frac{a_{N+3}}{a_{N+2}} < \theta , \dots$$

$$a_{N+3} < \theta \, a_{N+2} < \theta^2 \, a_{N+1} < \theta^3 a_N$$

$$a_{N+k} < \theta^k \, a_N$$

$$\sum_{k=N}^{\infty} a_k \leq \sum_{k=N}^{+\infty} \theta^k \, a_N < \infty$$

è una serie geometrica di ragione $\theta < 1$.

OSS.
$$\sum_{k=1}^{\infty} a_k = \sum_{k=N}^{\infty} a_k + \underbrace{\left(\sum_{k=1}^{N-1} a_k \right)}_{\in \mathbb{R}_+}$$

$$\Rightarrow \sum a_k \approx \sum b_k$$

ES pensare al caso $\ell > 1$

CRITERIO DI CONDENSAZIONE DI CAUCHY

$a_k > 0$, $a_k \to 0$, $a_{k+1} \leq a_k \ \forall k$ (ovvia a_k decrescente)

$$\sum_{k=1}^{\infty} a_k \approx \sum 2^k a_{2^k}$$

ES $$\sum_{2}^{\infty} \frac{1}{n \log n} \qquad \sum \frac{1}{n \log n} = +\infty.$$

$$\frac{1}{n \log n} \searrow 0$$

$$\sum \frac{1}{n} = +\infty \quad \text{lo sappiamo}$$

$$a_n < \frac{1}{n \lg n} < \frac{1}{n} \quad \text{ma non ci serve}$$

$n^{1+\varepsilon}$, $n \cdot n^{\varepsilon} > n \lg n > n$, (se $\varepsilon > 0$ $\frac{n^{\varepsilon}}{\lg n} \to +\infty$ strada sbagliata.

C.R. $\sqrt[n]{\frac{1}{n \lg n}} = \frac{1}{\sqrt[n]{n}\,\sqrt[n]{\lg n}} \to 1$

$\sqrt[n]{n} \to 1$, $\sqrt[n]{\lg n} \to 1$

Proviamo il criterio di condensazione.

ATTENZIONE!!

$\frac{1}{n \lg n} \searrow 0$ essendo $n \lg n \nearrow +\infty$

$$\sum_2 \frac{1}{n \lg n} \approx \sum \frac{\cancel{2^n}}{\cancel{2^n} \lg 2^n} = \sum \frac{1}{n \lg 2}$$

$$= \frac{1}{\lg 2} \sum \frac{1}{n} = +\infty$$

Quindi $\sum \frac{1}{n \lg n} = +\infty$

Es. $\sum_2^{\infty} \frac{1}{n \log^2 n}$

$\left(\frac{1}{n \lg^2 n}\right)^{\frac{1}{n}} \to 1$. no go

$\frac{1}{n \lg^2 n} \searrow 0$

CCC $\sum \frac{1}{n \lg^2 n} \approx \sum \frac{\cancel{2^n}}{\cancel{2^n} \log^2 2^n} =$

$$\sum \frac{1}{(\lg 2^n)^2} = \sum \frac{1}{n^2 (\lg 2)^2} = \frac{1}{(\lg 2)^2} \sum \frac{1}{n^2} < \infty$$

e quindi $\sum \frac{1}{n \lg^2 n} < +\infty$

Abbiamo dimostrato che $\sum \frac{1}{n} = +\infty$ $\sum \frac{1}{n^2} = +\infty$

$\sum \frac{1}{n^\alpha}$ converge se $\alpha \geq 2$ $\quad \frac{1}{n^\alpha} \leq \frac{1}{n^2}$.

$\sum \frac{1}{n^\alpha}$ diverge $\alpha \leq 1$ $\quad \frac{1}{n^\alpha} \geq \frac{1}{n}$

e per $\underline{1 < \alpha < 2}$?

Proviamo CCC

$\alpha \in (1,2)$
$\frac{1}{n^\alpha} \searrow 0$, $\sum \frac{1}{n^\alpha} \approx \sum \frac{2^n}{(2^n)^\alpha} = \sum \frac{1}{2^{n(\alpha-1)}}$

$= \sum \left(\frac{1}{2^{\alpha-1}}\right)^n$

è una serie geometrica che converge $\Leftrightarrow$ $\frac{1}{2^{\alpha-1}} < 1$

cioè $\underline{2^{\alpha-1}} > 1$ cioè $\underline{\alpha - 1 > 0}$, ovvero $\underline{\alpha > 1}$

Teorema (!) $\boxed{\sum \frac{1}{n^\alpha} = \zeta(\alpha) \text{ converge } \Leftrightarrow \alpha > 1.}$

[D 2467] $\sum_1^\infty \frac{3^n n!}{n^n}$

C.R. $\frac{a_{n+1}}{a_n} = \frac{3^{n+1} \cancel{(n+1)!}}{(n+1)^{n+1}} \frac{n^n}{\cancel{3^n} \cancel{n!}}$

$= \frac{3}{\left(\frac{n+1}{n}\right)^n} = \frac{3}{\left(1+\frac{1}{n}\right)^n} \to \frac{3}{e} > 1$ $\quad (n+1)! = (n+1)\, n!$

$\boxed{2 < e < 3}$ e quindi diverge

[D 2466] $\sum \frac{2^n n!}{n^n}$ $\quad \frac{a_{n+1}}{a_n} \to \frac{2}{e} < 1$

quindi converge

[D 2468] $\sum \frac{e^n n!}{n^n}$

Usiamo Stirling

Teorema** (Stirling) $\boxed{n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}$ difficile

e usiamo confronto asintotico $\Rightarrow$

$$\sum \frac{e^n n!}{n^n} \approx \sum \frac{\cancel{e^n}}{\cancel{n^n}} \cancel{\left(\frac{n}{e}\right)^n} \sqrt{2\pi n}$$

Quindi diverge.

Senza usare Stirling $\boxed{\left(\frac{n}{e}\right)^n \cdot e \le n! \le \left(\frac{n}{e}\right)^n \cdot n e}$ ES.

$n! \geq \left(\frac{n}{e}\right)^n \cdot e$ $\sum \frac{e^n n!}{n^n} \geq \sum e$ diverge.