Serie a termini in $\mathbb{R}$

① CRITERIO DI CONVERGENZA ASSOLUTA

$a_k \in \mathbb{R}$, $\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k|$ da questo segue che

$$\boxed{\text{Se } \sum |a_k| < +\infty \Rightarrow \sum a_k \text{ converge}}$$

e $\left| \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=1}^{\infty} |a_k|$

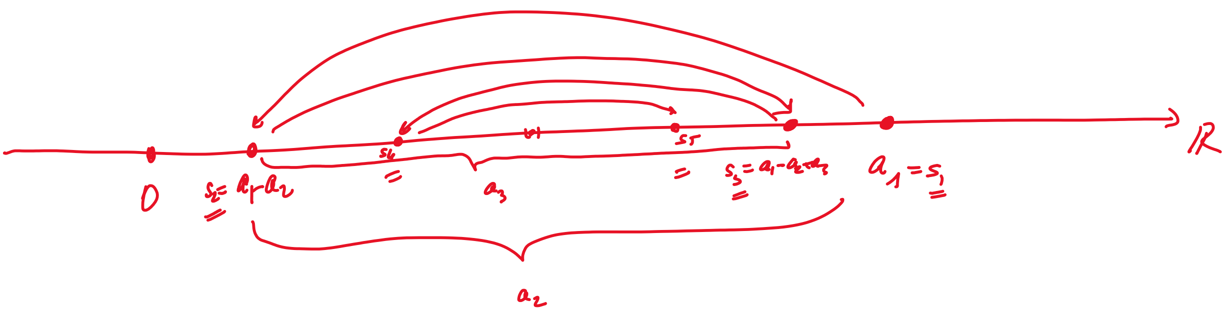
② CRITERIO DI LEIBNIZ

Se $a_n \searrow 0$ (monotona decrescente cioè $a_{n+1} \leq a_n$ e $a_n \to 0$)

allora la serie a segni alterni $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converge

$= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$

(oss $-\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} (-1)^n a_n$) $\qquad s_n = \sum_{k=1}^{n} (-1)^{k-1} a_k$



$s_{2n} \nearrow \qquad \swarrow s_{2n+1} \qquad s_{2n} < s_{2m+1} \quad \forall n, m$

$s_{2n} \nearrow \alpha \leq \beta \swarrow s_{2n+1}$

$s_{2n+1} - s_{2n} = a_{2n+1}$

per ipotesi $a_n \searrow 0$

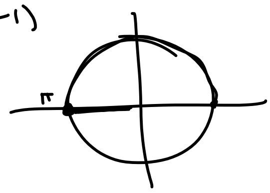
$\beta \quad \alpha$

$\Rightarrow \alpha = \beta$

[GE] Cap 4

Es 1 $\displaystyle\sum_{n=0}^{\infty} \frac{\cos \pi n}{n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$

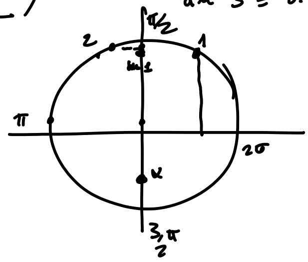
$\cos \pi n = (-1)^n$



$\frac{1}{n+2} \searrow 0 \overset{\text{Leib}}{\Longrightarrow}$ Converge.

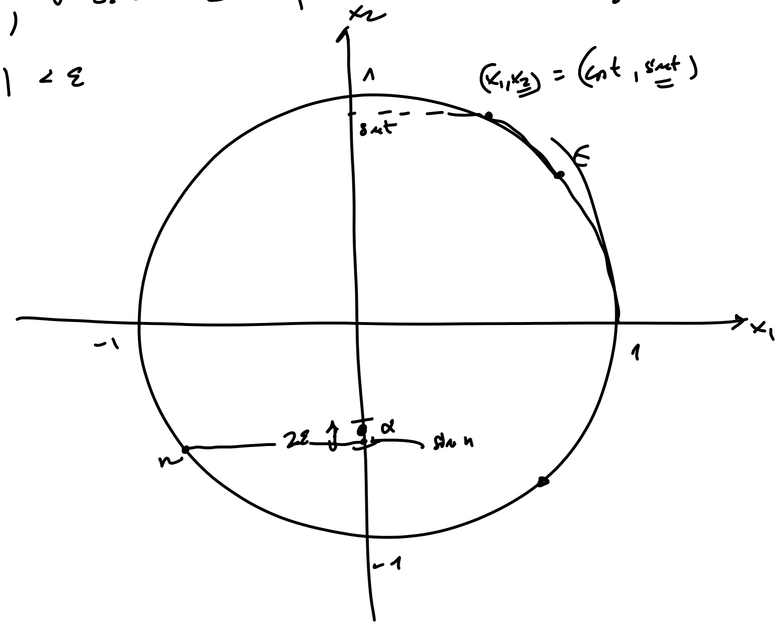
Es 10 $\displaystyle\sum_{n=1}^{\infty} \underline{[\sin(\sin n)]^n}$ (Es** $\displaystyle\sum_{n=1} (\sin n)^n$)

$\sin 1 = 0.84\ldots$
$\sin 2 = 0.9$
$\sin 3 = 0.1411$



$\{\sin n\}$ è una successione non banale

Vale** $\forall\ \alpha \in [-1,1],\ \forall\ \varepsilon > 0\ \exists\ n \mid$
$|\sin n - \alpha| < \varepsilon$

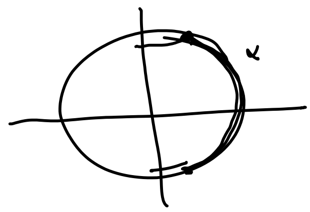


$\Sigma\ (\sin(\sin n))^n$ Converge an.

$|\sin n| < 1 \quad \forall n$

Studiamo la serie dei moduli $\Sigma\ |\underbrace{\sin(\sin n)}_{|\uparrow| < 1}|^n$, $\quad |\sin \alpha| \le \underline{\sin 1 < 1.}$
se $|\alpha| < 1$

$\underline{\Sigma\ |\sin(\sin n)|^n} < \Sigma\ (\sin 1)^n$
$\uparrow$
serie geom conv



per controllo $\sum |\text{Arc}(\sin n)|$ converge

$\overset{C.C.A.}{\Longrightarrow}$ $\sum \left(\text{Arc}(\sin n)\right)^n$ converge.

OS. Una serie può convergere ma non convergere assolutamente

$\sum_1^n a_k \to \alpha$ ma $\sum_1^n |a_n| \to +\infty$

Per esempio $\sum_1^n \frac{(-1)^{n-1}}{n}$ per Leibniz conv ma $\sum \frac{1}{n}$ diverge.

Dire per quali $x \in \mathbb{R}$ convergono le seguenti serie

Es 19. $\sum_{n=1}^{\infty} \frac{x^n}{1+x^{2n}}$

Studiamo la convergenza assoluta $\sum \frac{|x|^n}{1+|x|^{2n}}$

Proviamo C.R. $\left(\frac{|x|^n}{1+|x|^{2n}}\right)^{\frac{1}{n}} = \frac{|x|}{(1+|x|^{2n})^{\frac{1}{n}}} = \theta_n$

Se $|x|=1$, $\theta_n \to 1$

$|x|=2$ $\frac{2}{(1+2^{2n})^{\frac{1}{n}}} \to \frac{1}{2}$

$= \frac{1}{2} \frac{1}{\left(1+\frac{1}{2^{2n}}\right)^{\frac{1}{n}}}$

$1 = 1^{\frac{1}{n}} < \left(1+\frac{1}{2^{2n}}\right)^{\frac{1}{n}} < (2)^{\frac{1}{n}}$, con $\left(1+\frac{1}{2^{2n}}\right)^{\frac{1}{n}} \to 1$ e $(2)^{\frac{1}{n}} \to 1$.

Se $|x|=2$ la serie conv ass

Analog. Se $|x|>1$ la serie conv., $\theta_n \to \frac{1}{|x|} < 1$.

Se $|x|=1$ la serie div. assolutamente $\left(\frac{|x|^n}{1+|x|^{2n}} = \frac{1}{2}\right)$

Se $|x|=0$ $\theta_n \equiv 0$ la serie converge

$0 < |x| < 1$

$\theta_n = \left(\frac{|x|^n}{1+|x|^{2n}}\right)^{\frac{1}{n}} = \frac{|x|}{(1+|x|^{2n})^{\frac{1}{n}}} \to |x| = \theta$, converge

$(1+|x|^{2n})^{\frac{1}{n}} \to 1$

$\sum |x|^n$ conv $\forall x \mid |x| \neq 1$ $\overset{C.C.A.}{\Longrightarrow}$ $\sum \frac{x^n}{\dots^{2n}}$ converge $\forall x \neq \pm 1$

$1 + x^{2n}$ $\qquad\qquad\qquad$ $1+x$

se $x=1$ $\quad \sum \left(\frac{1}{2}\right)$ diverge a $+\infty$

$x=(-1)$, $\sum \frac{(-1)^n}{2} = \frac{1}{2}\sum_1^\infty (-1)^n$ è serie indeterminata

o, meglio,

non convergente.

Os. se $a_n \geq 0$ $\quad \sum a_n$, $\quad \underset{=}{\sqrt[n]{a_n}} \to \theta$, $\theta > 1$.

$\sqrt[n]{a_n}$ per $n \geq N$

$\Longrightarrow$ $a_n \to +\infty$

$1 < \bar{\theta} = \theta - \varepsilon$

Analog. se $\frac{a_{n+1}}{a_n} \to \theta > 1 \Longrightarrow a_n \to +\infty$

$a_n > \bar{\theta}^n \nearrow +\infty$

$$\boxed{a_n = \theta^n \qquad \frac{a_{n+1}}{a_n} = \theta}$$

ES 20 $\quad \sum_1^\infty x^n \log x^n = (\log x) \sum_1^\infty n x^n$ $\quad$ $x>0$

serie a termini positivi

$\sum_1^\infty x^n = \frac{x}{1-x}$

$|x|<1$

C.R. $\quad \sqrt[n]{n x^n} \to x$

$\sqrt[n]{n} \cdot x$

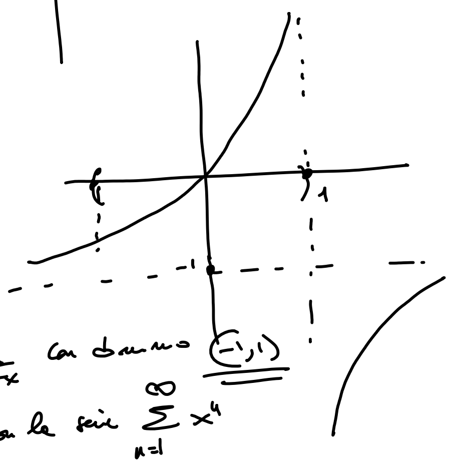
$\downarrow$

1

se $x<1$ la serie converge

$x>1$ la serie diverge

se $x=1$ la serie converge



$f(x) = \frac{x}{1-x}$ con dominio $(-1,1)$

coincide con la serie $\sum_{n=1}^{\infty} x^n$

$\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\quad$ conv ass $\quad$ $\sum_{k=0}^{\infty} \frac{|x|^k}{k!}$

$$\frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \frac{|x|}{k+1} \to 0$$

la serie converge assolutamente $\forall x \in \mathbb{K}$.

(questa serie converge più rapidamente di una serie geometrica

Stirling

$$\frac{x^k}{k!} \sim \frac{x^k}{\left(\frac{k}{e}\right)^k \sqrt{2\pi k}} = \left(\frac{ex}{k}\right)^k \frac{1}{\sqrt{2\pi k}}$$
)

Teorema (bellissimo)

$$\boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x}$$

In particolare $\boxed{\sum_{k=0}^{\infty} \frac{1}{k!} = e}$ $\Longmapsto$ e è irrazionale

facile

Es. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

Valore assoluto $\displaystyle\sum \frac{x^{2k}}{(2k)!}$ Crit. rap

$$\frac{x^{2(k+1)}}{(2(k+1))!} \frac{(2k)!}{x^{2k}} = \frac{x^2}{(2k+2)(2k+1)} \to 0$$

quindi converge assolutamente $\forall x \in \mathbb{R}$

e definisce su $\mathbb{R}$ una funzione pari infatti

Teo $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x$

In particolare $\uparrow$ è periodica di periodo 2π.

Es 34 $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n}$

Studiamo la convergenza assoluta

$$\left(\frac{|x|^{n^2}}{n}\right)^{\frac{1}{n}} = \frac{|x|^n}{\sqrt[n]{n}} \longrightarrow \begin{cases} 0 & \text{se } |x|<1 \\ +\infty & \text{se } |x|>1 \end{cases}$$

$\Rightarrow$ la serie converge se $|x|<1$ e non converge se $|x|>1$

se $x=1$ diverge

se $x=-1$ $\quad \sum \frac{(-1)^{n^2}}{n} = \sum \frac{(-1)^n}{n} \qquad (-1)^{n^2} = (-1)^n$!

converge per Leibniz

Es 38 $\quad \displaystyle\sum_{n=0}^{\infty} x^{x^n}$, $n \geq 1$, x^x ← def solo per $x>0$

$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x \log x} = 1$.

$\lim_{x\to 0+} x^{x^n} = \lim e^{x^n \log x} = 1$

$x>0$

$x \geq 1$ diverge

se $0<x<1$ diverge

$\left(\frac{1}{2}\right)^{\frac{1}{2^n}} \to$