Lezione 26 29-11-21

lunedì 29 novembre 2021 08:15

1191

$$\int \frac{dx}{x\sqrt{x^2-2}}\,dx \underset{x=\frac{1}{t}}{=} -\frac{1}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}}{x}\right)$$

$$\longrightarrow \int f(x)\,dx \underset{x=u(t)}{=} \int \underbrace{f(u(t))\,u'(t)}\,dt$$

$$\longrightarrow \int \underbrace{f(u(t))\,u'(t)}_{g(t)}\,dt = \int f(x)\,dx \Big|_{x=u(t)}$$

Calcoliamo l'integrale in altra maniera

$$\int \frac{dx}{x\sqrt{x^2-2}} \underset{\substack{y=\frac{x}{\sqrt{2}}\\ x=\sqrt{2}y}}{=} \cancel{\sqrt{2}}\int \frac{dy}{\cancel{\sqrt{2}}y\sqrt{2y^2-2}} = \frac{1}{\sqrt{2}}\left(\int \frac{dy}{y\sqrt{y^2-1}}\right) =$$

$|y|>1$, $y>0$

$$\int \frac{dy}{y\sqrt{y^2-1}} \underset{y=\text{ch}\,t}{=}$$

$\sqrt{y^2-1}$

$\text{ch}^2 t - \text{sh}^2 t = 1$

$\longrightarrow \text{sh}^2 t = \text{ch}^2 t - 1$

suggerisce $y = \text{ch}\,t$

" $dy = \text{sh}\,t\,dt$ "

$\frac{dy}{dt} = \text{sh}\,t$

$$= \int \frac{\cancel{\text{sh}\,t}\,dt}{\text{ch}\,t\,\cancel{\text{sh}\,t}}$$

$$= 2\int \frac{dt}{e^t+e^{-t}}$$

$\text{ch}\,t := \frac{e^t+e^{-t}}{2}$

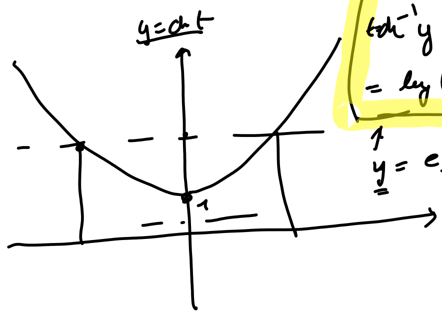
$$= 2\int \frac{e^t\,dt}{e^{2t}+1} \qquad u'(t)\,dt = du$$

$$\underset{u=e^t}{=} 2\int \frac{du}{u^2+1} = 2\arctg u \underset{u=e^t}{=} 2\arctg e^t = 2\arctg e^{\text{ch}^{-1}y}$$

$$\boxed{\text{ch}^{-1}y = \log\left(y+\sqrt{y^2-1}\right)}$$

$y = \frac{e^t+e^{-t}}{2}$, $2ye^t = e^{2t}+1$

$u=e^t>0$

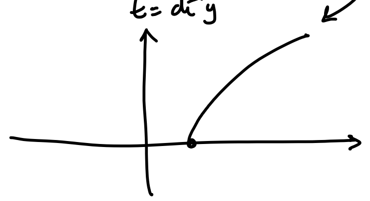


$$= 2\arctg\left(y+\sqrt{y^2-1}\right)$$

$$= 2\arctg\left(\frac{x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}-1}\right)$$

$$= 2\arctg\left(\frac{x+\sqrt{x^2-2}}{\sqrt{2}}\right)$$

$\text{ch}^{-1}y$ è il ramo principale del ch^{-1}



Ris di prima era $-\frac{1}{\sqrt{2}}\arcsin\frac{\sqrt{2}}{x}$

Rivediamo le Tavole di int e

$$\boxed{2\,\text{Arctan}\left(x+\sqrt{x^2-1}\right) = \pi - \text{Arcsen}\,\frac{1}{x} \quad \forall\, x\geq 1} \quad (*)$$

NB $\lim_{x\to+\infty} = \lim_{x\to+\infty}$ ✓

$\text{ch}^{-1}: [1,+\infty) \mapsto [0,+\infty)$

l'altro ramo è

$\text{ch}^{-1}: [1,+\infty) \mapsto [-\infty, 0]$

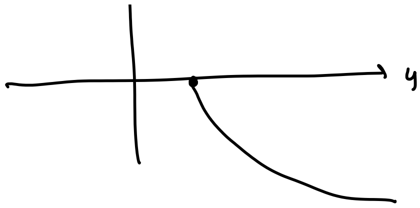
$\text{ch}^{-1}y$

Es dimostrare $(*)$ su $[1,+\infty)$

Sugg.: usare Lagrange ($f'(t)=0$ su I intervallo

$\Rightarrow f \equiv$ cost.

$f = \lim_{t\to+\infty} f(t)$ se $I=[1,+\infty)$



Quindi $2\arctg\left(y+\sqrt{y^2-1}\right) = \pi - \operatorname{Arcsen}\frac{1}{y}$ $\quad y=$

$= \pi - \operatorname{Arcsen}\frac{\sqrt2}{x}$ dividendo per $\sqrt2$ ottengo la richiesta di forma ecc.

$\Rightarrow$

$\frac{\pi}{\sqrt2}$

$\int f(x)\,dx = F(x) + c$, $\forall c\in\mathbb{R}$

$= F(x)$

INTEGRAZIONE PER PARTI

$$\int f'g = fg - \int f g' \quad \left(\text{deriva della } (fg)' = f'g + fg'\right)$$

[D] 1211 - 1235

1211 $\int \log x = \int (x)' \lg x \, dx = x\log x - \int x \frac{1}{x} = x\lg x - x$

$f(x)=x$, $g(x)=\lg x$ $\qquad = x(\lg x - 1)$

$$\boxed{\int \log x = x(\lg x - 1)}$$

1212 $\int \arctg x = \int x' \arctg x = x\arctg x - \int \frac{x}{1+x^2}\,dx$

$y=x^2 \quad dy = 2x\,dx$

$= x\arctg x - \frac12 \int \frac{dy}{1+y}$

$= x\arctg x - \frac12 \log(1+x^2)$

1214 $\int x \operatorname{sen} x = -\int x(\cos x)' = -x\cos x + \int \cos x = -x\cos x + \operatorname{sen} x$

1218 $\int x^2 e^{3x} = \int x^2\left(\frac{e^{3x}}{3}\right)' =$ etc.

1232 $\int e^x \sin x = e^x \sin x - \int e^x \cos x = e^x \sin x - e^x \cos x - \int e^x \sin x$

$f' = e^x$, $g = \sin x$

$\Rightarrow \quad 2\int e^x \sin x = e^x \sin x - e^x \cos x \quad \Rightarrow \quad \int e^x \sin x = \frac{e^x(\sin x - \cos x)}{2}$

$\int\sqrt{1-x^2}$, $\int\sqrt{x^2-1}$, $\int\sqrt{x^2+1}$

NB $a \neq 0$, $\sqrt{ax^2+bx+c}$

$ax^2+bx+c =$

$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$= a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right)$

Completamento del quadrato

$$\int\sqrt{1-x^2}\,dx = \int x'\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\,\frac{1}{2}\,\frac{1}{\sqrt{1-x^2}}(-2x)$$

$$= x\sqrt{1-x^2} + \int\frac{x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int\frac{1-x^2}{\sqrt{1-x^2}} + \int\frac{1}{\sqrt{1-x^2}}$$

$$= x\sqrt{1-x^2} + \arcsin x - \int\sqrt{1-x^2}$$

$$\Rightarrow \boxed{\int\sqrt{1-x^2} = \frac{x\sqrt{1-x^2}+\arcsin x}{2}}$$

Altro metodo

$$\int\sqrt{1-x^2}\,dx = \int|\cos t|\cos t\,dt = \int\cos^2 t\,dt = \int\frac{1+\cos 2t}{2}\,dt$$

$$= \frac{t}{2} + \frac{\sin 2t}{4}$$

$x = \sin t$

$-1 \leq x \leq 1$ $\quad x > 0$, $0 \leq t \leq \frac{\pi}{2}$

PARENTESI TRIGONOMETRICA.

$\cos^2 t + \sin^2 t = 1$

$\cos 2t = \cos^2 t - \sin^2 t$

$= \cos^2 t - (1-\cos^2 t)$

$= 2\cos^2 t - 1$

$\Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$

$$\int\sqrt{1-x^2} = \frac{\arcsin x}{2} + \frac{\cancel{2}\sin t\cos t}{\cancel{4}\,2}$$

$t = \arcsin x$, $x = \sin t$

$$= \frac{\arccos x}{2} + \frac{x\sqrt{1-x^2}}{2} \quad ✓$$

$$\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$$

e $\sin^2 t = 1 - \cos^2 t = 1 - \frac{1+\cos 2t}{2}$

$$= \frac{1-\cos 2t}{2}$$

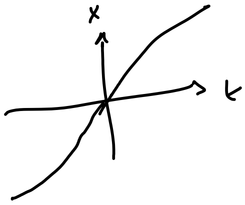
$$\boxed{\begin{aligned} \cos^2 t &= \frac{1+\cos 2t}{2} \\ \sin^2 t &= \frac{1-\cos 2t}{2} \end{aligned}}$$

$$\int \sqrt{1+x^2} \underset{\text{per parti}}{=} \int x' \sqrt{1+x^2} = x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}$$

$$= x\sqrt{1+x^2} - \int \frac{1+x^2}{\sqrt{1+x^2}} + \int \frac{1}{\sqrt{1+x^2}}$$

$$= x\sqrt{1+x^2} + sh^{-1} x - \int \sqrt{1+x^2}$$

$$\int \sqrt{1+x^2} = \frac{x\sqrt{1+x^2} + sh^{-1} x}{2}$$



Altro metodo.

$$\int \underbrace{\sqrt{1+x^2}}_{ch\, t}\, dx \underset{x = sh\, t}{=} \int ch^2 t \, dt$$

$$dx = ch\, t \, dt$$

$$ch^2 t - sh^2 t = 1$$
$$ch^2 t = 1 + sh^2 t$$

PARENTESI IPERBOLICA (ES!!)

$$ch^2 x = \frac{ch 2x + 1}{2}$$
$$sh^2 x = \frac{ch 2x - 1}{2}$$
$$sh 2x = 2\, sh x\, ch x$$

$$= \int \frac{ch 2t + 1}{2}\, dt$$

$t = sh^{-1} x \qquad x = sh\, t \qquad ch\, t = \sqrt{1 + sh^2 t}$

$$= \frac{sh 2t}{4} + \frac{t}{2} = \frac{sh\, t\, ch\, t + sh^{-1} x}{2} = \frac{x\sqrt{1+x^2} + sh^{-1} x}{2}$$

ES Calcolare $\int \sqrt{x^2-1}$ sia per parti sia con sostituzione iperbolica