[D] §4 Int del tipo $\boxed{\int \frac{mx+n}{\sqrt{ax^2+bx+c}}}$

Abbiamo già studiato:
$\int \sqrt{1-x^2}$
$\int \sqrt{x^2-1}$
$\int \sqrt{1+x^2}$

1262 $\int \frac{dx}{\sqrt{2+3x-2x^2}}$ ← $m=0,\ n=1$

Studiamo $\varphi(x) = -2x^2+3x+2 = -2\left(x^2-\frac{3}{2}x-1\right)$

$= -2\left(\left(x-\frac{3}{4}\right)^2-\frac{9}{16}-1\right) = -2\left(\left(x-\frac{3}{4}\right)^2-\frac{9+16}{16}\right)$

Completiamo il quadrato (!)

$= -2\left(\left(x-\frac{3}{4}\right)^2-\frac{25}{16}\right)$

$= 2\left(\frac{25}{16}-\left(x-\frac{3}{4}\right)^2\right)$

$$\int \frac{dx}{\sqrt{2+3x-2x^2}} = \frac{1}{\sqrt{2}}\cdot\frac{4}{5}\int \frac{dx}{\sqrt{1-\left(\frac{4}{5}\left(x-\frac{3}{4}\right)\right)^2}} = \frac{1}{\sqrt{2}}\int \frac{d\left(\frac{4}{5}\left(x-\frac{3}{4}\right)\right)}{\sqrt{1-\left(\frac{4}{5}\left(x-\frac{3}{4}\right)\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{4}{5}\left(x-\frac{3}{4}\right)\right) = \frac{1}{\sqrt{2}}\arcsin\left(\frac{4}{5}x-\frac{3}{5}\right) = \frac{1}{\sqrt{2}}\arcsin\frac{4x-3}{5}$$

$u'(x)\,dx = du$ $\quad u = \frac{4}{5}\left(x-\frac{3}{4}\right)$

$u'(x) = \frac{4}{5}$

[D] 1265 $\int \frac{3x-6}{\sqrt{x^2-4x+5}}\,dx$ $\quad (m=3,\ n=-6)$

$\varphi(x) = x^2-4x+5 \qquad \varphi' = 2x-4$

$$= \frac{3}{2}\int \frac{2x-4}{\sqrt{x^2-4x+5}}\,dx = \frac{3}{2}\int \frac{\varphi'}{\sqrt{\varphi}}\,dx = \frac{3}{2}\int \frac{d\varphi}{\sqrt{\varphi}} = 3\sqrt{\varphi} = 3\sqrt{x^2-4x+5}$$ ✓

Se fosse stato $\int \frac{3x-5}{\sqrt{x^2-4x+5}} = \frac{3}{2}\int \frac{2x-\frac{10}{3}}{\sqrt{x^2-4x+5}}$

$4+\frac{10}{3}$

$$= \frac{3}{2}\int\left(\frac{2x-4}{\sqrt{\varphi}} + \frac{22/3}{\sqrt{x^2-4x+5}}\right)dx$$

$$= \frac{3}{2}\left(2\sqrt{p} + \frac{22}{3}\int \frac{1}{x^2-4x+5}\right) \quad \underline{\text{etc.}}$$

2° tipo $\boxed{\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}, \quad t = \frac{1}{mx+n}}$

1271 $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$ $\quad t = \frac{1}{x+1}, \quad x+1 = \frac{1}{t}, \quad x = \frac{1}{t} - 1$

$dx = -\frac{1}{t^2}\,dt$

$$= -\int \frac{t}{t^2}\frac{1}{\sqrt{x^2+2x}}\,dt$$

qui $x = x(t) = \frac{1}{t} - 1 = \frac{1-t}{t}$

$$x^2+2x = \left(\frac{1-t}{t}\right)^2 + 2\,\frac{1-t}{t} = \frac{1+t^2-\cancel{2t}+\cancel{2t}-2t^2}{t^2} = \frac{1-t^2}{t^2}$$

$$= -\int \frac{1}{t}\frac{1}{\sqrt{\frac{1-t^2}{t^2}}}\,dt = -\int \frac{1}{\sqrt{1-t^2}} = -\arcsin\frac{1}{x+1}$$

---

§6 $\boxed{\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \dots\right)}$ sostituzione da fare è

$\boxed{t^n = \frac{ax+b}{cx+d} \text{ dove } n = \text{m.c.m}(q_1, q_2, \dots)}$

$R(x,y,z)$ = funzione razionale di x,y,z ossia rapporto di due polinomi in x,y,z $\quad R(x,y,z) = \frac{P(x,y,z)}{Q(x,y,z)}$

Es. 1318 (Att. la risp di [D] è sbagliata)

$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ $\quad \underline{\frac{ax+b}{cx+d} = x} \quad a=1, \; b=c=d=0$

$R(y,z) = \frac{1}{y+z}$ $\quad y = x^{\frac{1}{2}}, \quad z = x^{\frac{1}{3}}$ $\quad q_1 = 2 \quad q_2 = 3$

m c m$(2,3) = 6$

$\underline{x = t^6}$ $\quad dx = 6t^5\,dt$

$$6\int \frac{t^5}{t^3+t^2}\,dt = 6\int \frac{t^3}{t+1}$$

T... funz. razionale di t

abbiamo ricondotto il problema al calcolo di una primitiva di questo tipo

$\frac{P(t)}{Q(t)}$ e vogliamo ridurci al caso di $\deg P < \deg Q$

Dividiamo tra polinomi:

$$
\begin{array}{l|l}
t^3 & t+1 \\ \hline
t^3+t^2 & t^2-t+1 \\
\hline
-t^2 & \\
-t^2-t & \\
\hline
t & \\
t+1 & \\
\hline
-1 &
\end{array}
$$

$t^3 = (t+1)\cdot t^2 - t^2$

$t^3 = (t+1)(t^2-t+1) - 1$

↑ resto.

$$= 6 \int \frac{(t+1)(t^2-t+1)-1}{t+1} = 6\left(\int t^2-t+1 - \int \frac{1}{t+1}\right)$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log|1+t|\right) \qquad t = x^{\frac{1}{6}}$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - \log\left(1+\sqrt[6]{x}\right)$$

questo [D] mette + (sbagliato)

1323 $\int x \sqrt{\frac{x-1}{x+1}}\, dx \qquad R(x,y) = xy \qquad y = \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}$

$p_1 = 1, \; q_1 = 2$

$a = 1 \quad b = -1$

$c = 1 \quad d = 1$

$t^2 = \frac{x-1}{x+1} \qquad (x+1)t^2 = x-1 \qquad xt^2 + t^2 = x - 1$

$n = 2$

$t^2 + 1 = x(1-t^2)$

$x = \frac{1+t^2}{1-t^2}$

$$x' = \frac{2t(1-\cancel{t^2}) + (1+\cancel{t^2})2t}{(1-t^2)^2} = \frac{4t}{(1-t^2)^2}$$

$$\int \frac{1+t^2}{1-t^2}\, t\, \frac{4t}{(1-t^2)^2}\, dt = 4\int \frac{(1+t^2)\,t^2}{(1-t^2)^3}\, dt$$

$$= 4 \int \frac{(1+t^2)\,t^2}{(1-t)^3\,(1+t)^3} = -4 \int \frac{(1+t^2)\,t^2}{\underbrace{(t+1)^3(t-1)^3}_{Q(t)}}\, dt = \text{attenzione!}$$

$Q(t)$ ha radici ± 1 con molteplicità 3.

Formula di Hermite - Ostrogradskij per le primitive di funzioni razionali $\frac{P(x)}{Q(x)}$ con radici multiple di $Q(x)$.

Con $\deg P < \deg Q$

Dal Teorema Fondamentale dell'algebra segue che

$$Q(x) = \underbrace{(x-x_1)^{n_1} \cdots (x-x_r)^{n_r}}_{r \text{ radici reali } x_h \in \mathbb{R} \text{ con molt. } (n_h)} \left((x-u_1)^2+v_1^2\right)^{m_1} \cdots \left((x-u_j)^2+v_j^2\right)^{m_j}$$

$v_k \neq 0$
$m_k \geq 1$
$0 \leq j$ se $j=0$ non ci sono i termini quadratici

$u_k + i v_k$ è radice complessa di Q ($v_k \neq 0$)

$0 \leq r$ 0 $\Leftrightarrow$ nessuna radice reale

$\underline{1+x^2}$, $n_1 \geq 1$

radici $\pm i$ $i = \sqrt{-1}$

$(x-z_1)(x-\bar{z}_1)$ se z_1 è radice complessa di Q

$Q(z_1) = 0$

$Q(\bar{z}_1) = \overline{Q(z_1)} = 0$

$z = x+iy$
$\bar{z} = x-iy$

$(x-u)^2+v^2$ è un polinomio di 2° grado irriducibile su $\mathbb{R}$
$u, v \in \mathbb{R}$

Formula.

$\hat{Q}$ = come Q con molt ridotte di 1

$$= (x-x_1)^{n_1-1} \cdots (x-x_r)^{n_r-1}\left((x-u_1)^2+v_1^2\right)^{m_1-1} \cdots$$

Secondo Hermite e Ostrogradskij

$$\boxed{\frac{P}{Q} = D\left(\frac{\hat{P}}{\hat{Q}}\right) + \frac{c_1}{x-x_1} + \cdots + \frac{c_r}{x-x_r} + \frac{2a_1(x-u_1)+b_1}{(x-u_1)^2+v_1^2} + \cdots}$$

$\hat{P}$ da determinare polinomio di grado $< \deg \hat{Q}$

e c_h e a_i, b_i da determinare

$$\int \frac{P}{Q} = \frac{\hat{P}}{\hat{Q}} + \int$$

$$\int \frac{c}{x-\bar{x}} = c \log|x-\bar{x}|$$

$$\int \frac{2a(x-\bar{x})+b}{(x-\bar{x})^2+\bar{y}^2} = a\int \underbrace{\frac{d\left((x-\bar{x})^2+\bar{y}^2\right)}{(x-\bar{x})^2+\bar{y}^2}}_{\log} + \underbrace{\int \frac{b}{(x-\bar{x})^2+\bar{y}^2}}_{\text{arctg}}$$

$P(x) = 1$ $Q(x) = x(x+1)^2$

$x_1 = -1$, $r=2$, $n_1 = 2$
$x_2 = 0$ $n_2 = 1$
$j = 0$ (no radici complesse)

Esempi

1285 $$\int \frac{1}{x(x+1)^2}\,dx$$

$Q(x) = x(x+1)^2$

$\hat{Q}(x) = x+1$ $\underline{\hat{P} = c}$ $\deg \hat{P} < \deg \hat{Q}$

$$\frac{1}{x(x+1)^2} = D\left(\frac{c}{x+1}\right) + \frac{a}{x} + \frac{b}{x+1}$$

$$= -\frac{c}{(x+1)^2} + \frac{a}{x} + \frac{b}{x+1} = \frac{-cx + a(x+1)^2 + b\,x(x+1)}{x(x+1)^2} = \frac{x^2(a+b) + x(-c+2a+b) + a}{x(x+1)^2}$$

$a=1 \qquad a+b=0\,, \quad b=-1\,, \qquad c=2a+b=1$

$$\frac{1}{x(x+1)^2} = D\left(\frac{1}{x+1}\right) + \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x(x+1)^2} = \frac{1}{x+1} + \lg|x| - \lg|x+1| = \frac{1}{x+1} + \lg\left|\frac{x}{x+1}\right|$$

Calcolare $I_n = \int x^n e^{-x}$, $\quad I_4(x) \qquad \forall\, n \geq 0$

$$I_0 = \int e^{-x} = -e^{-x}$$

$$I_n = \int x^n e^{-x} = -e^{-x}x^n + n\int x^{n-1}e^{-x}\,, \quad n\geq 1$$

$$= -x^n e^{-x} + n\, I_{n-1}$$

Calcoliamo ad esempio I_4

$$I_4 = e^{-x}\left(-x^4 + 4e^{x} I_3\right) \qquad I_3 = -e^{-x}x^3 + 3I_2$$

$$I_2 = -x^2e^{-x} + 2I_1$$

$$I_1 = -xe^{-x} + I_0 = -xe^{-x} - e^{-x}$$

$$I_4 = e^{-x}\left(Q(x)\right) \qquad \text{deg } Q = 4 \quad = -x^4 + \cancel{4x^3}$$

$$J_n = \int \frac{dx}{(x^2+a^2)^n} \qquad J_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} \qquad a>0$$

$$= \frac{1}{a}\,\text{arctg}\,\frac{x}{a} \qquad \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)^2+1}$$

$n\geq 2$

$$\underbrace{\int \frac{dx}{(x^2+a^2)^n}}_{J_n} = \int \frac{x'\,dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} + n2\int \frac{x^2}{(x^2+a^2)^{n+1}}$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\,\underbrace{\int \frac{\cancel{x^2+a^2}}{(x^2+a^2)^{\cancel{n+1}}}}_{J_n} - 2na^2\underbrace{\int\frac{1}{(x^2+a^2)^{n+1}}}_{J_{n+1}}$$

$$(2n-1)J_n = 2na^2\, J_{n+1} - \frac{x}{(x^2+a^2)^n}$$

Quindi

$$J_m = \frac{1}{2(m-1)a^2}\left(\frac{x}{(x^2+a^2)^{m-1}} + (2m-3)J_{m-1}\right) \qquad \text{Controllare!}$$

$m = n+1$

$2n-1 = 2(m-1)-1 = 2m-3$