Lezione 30 6/12/21

lunedì 6 dicembre 2021 08:19

ES* Studiare il grafico di $f(x) = \frac{2}{1-x} + \sin\frac{1}{1-x}$, $0 \le x < 1$

[In particolare $f'(x) > 0$, ha un asintoto verticale in $x=1$, ma ha infiniti flessi]

Giusti Es. Cap. 7° §8

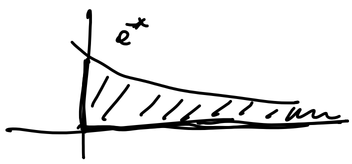
113 Calcolare $\int_0^{\infty} x^n e^{-x}\,dx$ ovviamente converge. $\frac{e^{-x}}{x^{n+2}} \xrightarrow[x\to+\infty]{} 0$

$\Rightarrow \exists M>1, \ \frac{e^{-x}}{x^{n+2}} < 1$ $\quad \int_M^{\infty} x^n e^{-x} < \int_M^{\infty} \frac{1}{x^2} < \infty$ per confronto conv..

$$n \ge 0 \quad \underbrace{\int_0^{\infty} x^n e^{-x}}_{I_n} = \underbrace{\left[-e^{-x} x^n\right]_0^{\infty}}_{\uparrow} + n \underline{\int_0^{\infty} e^{-x} x^{n-1}} = \begin{cases} n \underline{\int_0^{\infty} e^{-x} x^{n-1}} & n \ge 1 \\ 1 & n=0 \end{cases}$$

$\int_0^{\infty} e^{-x} = 1$ $\quad [F]_0^{\infty} := \lim_{M\to+\infty} [F]_0^M$

$F \in C([0,\infty))$



per $n \ge 1$ $\begin{cases} I_n = n I_{n-1} \\ I_0 = 1 \end{cases}$

ovvia $I_n = n!$

ES Studiare il grafico di $x^n e^{-x}$ in $[0,+\infty)$

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104. $\int_2^{\infty} \underbrace{\frac{\log x}{(1+x)^2}}_{f(x)}\,dx$ ovviamente converge perché

Parentesi sulla integrabilità delle potenze in 0 e in ∞

Per quale $\alpha \in \mathbb{R}$, $\int_1^\infty \frac{1}{x^\alpha} < \infty \Longleftrightarrow$ ... $\int \frac{}{x^\alpha}$ ... $\frac{}{-\alpha+1}$ ... $(\alpha-1)\,x^{\alpha-1}$

$$\boxed{\begin{array}{l} \int x^{-\alpha} = -\frac{1}{\alpha-1}\frac{1}{x^{\alpha-1}} \quad \alpha \neq 1 \\ \alpha = 1 \\ \int \frac{1}{x} = \log x, \end{array}}$$

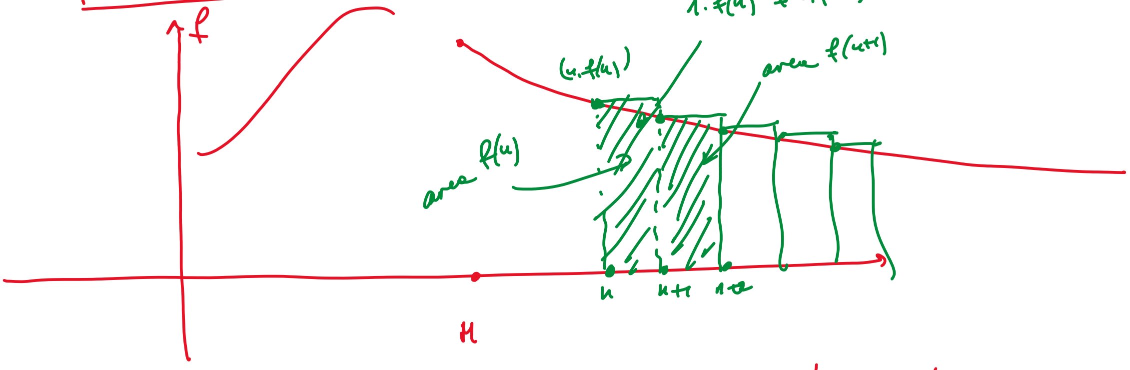
Per $\alpha \le 0$ ovviamente diverge: $\alpha \le 0 \Rightarrow \frac{1}{x^\alpha} \ge 1$ e $\int_1^\infty 1 = +\infty$

Sia $\alpha > 0$

$$\int_1^\infty \frac{1}{x^\alpha} = \lim_{M\to\infty}\left[-\frac{1}{\alpha-1}\frac{1}{x^{\alpha-1}}\right] = \begin{cases} \frac{1}{\alpha-1} & \text{se } \alpha>1 \\ +\infty & \text{se } \alpha \le 1 \end{cases}$$

$$\lim_{M\to\infty} \underbrace{M-1}_{[x]_1^M} = +\infty$$

Ricorda che $\boxed{\sum_1^\infty \frac{1}{n^\alpha} < \infty \Longleftrightarrow \alpha > 1}$

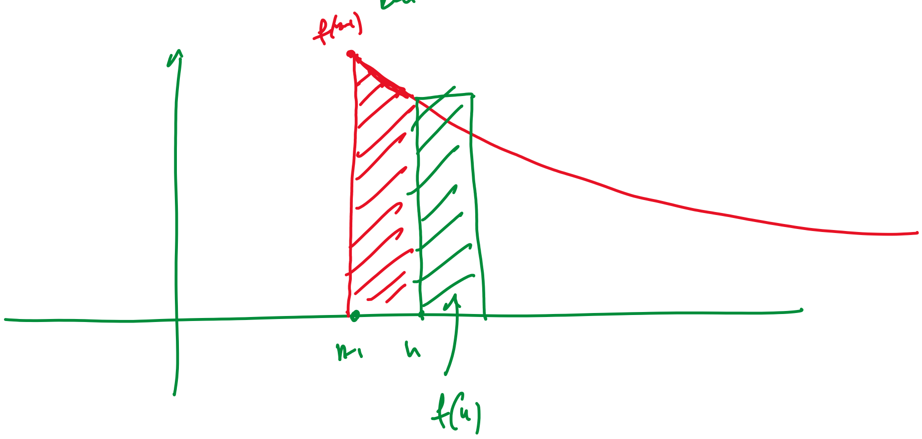


f definitivamente decrescente ossia $\exists M > 1 \mid f$ è decres. su $[M, +\infty)$

$$\int_M^\infty f(x)dx \qquad \sum_{n=M}^\infty f(n)$$

$$\sum_{k=n}^\infty f(k) \ge \int_n^\infty f(x)\,dx$$

Se $\sum_{k=n}^\infty f(k)$ converge $\Rightarrow \int_n^\infty f(x)\,dx < \infty$



Os: se f è decrescente su $(n-1, \infty)$

l'area rossa è $\ge$ dell'area verde

ossia $\int_{n-1}^{n} f(x)dx \ge \int_{n-1}^{n} f(n)\,dx = f(n)$

$$\underbrace{\int_{n-1}^{n}}\ f(x) \ge f(n) \quad \forall \quad x \in [n-1,n]$$

$$\underbrace{\int_{n-1}^{\infty} f(x)\,dx} \ge \sum_{k=n}^{\infty} f(k)$$

& $\int_{n-1}^{\infty} f(x)\,dx < \infty \implies \sum_{k=n}^{\infty} f(k)$ è finita

ora se $\int_{\underline{\underline{M}}}^{\infty} f$ converge allora converge $\sum_{k=\underline{\underline{M}}}^{\infty} f(k)$

Per quali α $\int_0^1 \frac{1}{x^\alpha} < \infty$ ovviamente se $\alpha \le 0$ $\frac{1}{x^\alpha} \in C([0,1])$ quindi conv

$$\int_0^1 \frac{1}{x^\alpha} = \lim_{\varepsilon \to 0^+} \underline{\left[-\frac{1}{\alpha-1}\frac{1}{x^{\alpha-1}}\right]_\varepsilon^1} \qquad \alpha \ne 1$$

$$\lim_{\varepsilon \to 0^+} \left[\log x\right]_\varepsilon^1 = +\infty \qquad \alpha = 1$$

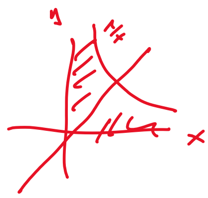
$$\lim_{\varepsilon \to 0^+} \left[-\frac{1}{\alpha-1} + \underbrace{\frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}\right] \qquad \underline{\alpha-1>0}$$

$$= \begin{cases} +\infty & \alpha > 1 \\ \frac{1}{1-\alpha} & \alpha < 1 \end{cases}$$

$$\boxed{\int_0^1 \frac{1}{x^\alpha} = \begin{cases} +\infty & \text{se } \alpha \ge 1 \\ \frac{1}{1-\alpha} & \alpha < 1 \end{cases}}$$

area $= 2$

$$\int_0^\infty \frac{1}{x^\alpha} = \underset{\substack{\uparrow \\ \text{conv. } \alpha<1}}{\int_0^1 \frac{1}{x^\alpha}} + \underset{\substack{\uparrow \\ \alpha>1}}{\int_1^\infty \frac{1}{x^\alpha}} = +\infty, \ \forall \alpha$$



$$\int_2^\infty \frac{\log x}{(1+x)^2} = \int_2^\infty \frac{1}{(1+x)^{3/2}} \cdot \underbrace{\left[\frac{\log x}{(1+x)^{1/2}}\right]}_{\downarrow\, 0}$$

$$\exists\, M>1 \quad \frac{\log x}{\sqrt{(1+x)}} < 1 \quad \forall\, x > M$$

$$\int_M^\infty \frac{\log x}{(1+x)^2} \le \int_M^\infty \frac{1}{(1+x)^{3/2}} < \infty$$

Calcoliamo
$$\int_2^\infty \frac{\log x}{(1+x)^2} = \int_2^\infty \left(-\frac{1}{1+x}\right)' \log x = \left[-\frac{1}{1+x}\log x\right]_2^\infty + \int_2^\infty \frac{1}{1+x}\,\frac{1}{x}$$

PER PARTI!

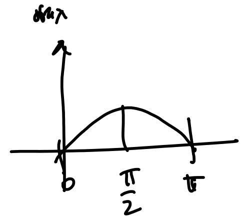
$$= \frac{\log 2}{3} + \int_2^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) = \frac{\log 2}{3} + \Big[\underbrace{\log x - \log (x+1)}_{=\log\frac{x}{x+1}}\Big]_2^\infty$$

$$= \frac{\log 2}{3} - \log\frac{2}{3} = \frac{\log 2}{3} - \log 2 + \log 3 = -\frac{2}{3}\log 2 + \log 3$$

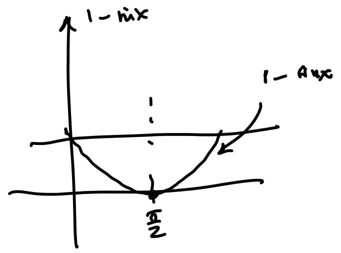
E 107
$$\int_0^\pi \frac{dx}{\sqrt{1-\sin x}}$$

Prima di calcolare cerchiamo di capire se l'integrale converge!

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\sin x}} + \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{\sqrt{1-\sin x}}$$



Prima oss
$$\int_0^\pi \frac{dx}{\sqrt{1-\sin x}} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\sin x}}$$
perché $\sqrt{1-\sin(\frac{\pi}{2}-x)} = \sqrt{1-\sin(\frac{\pi}{2}+x)}$

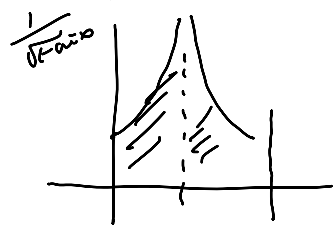


$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\sin x}} = \lim_{\varepsilon\to 0^+} \int_0^{\frac{\pi}{2}-\varepsilon} \frac{dx}{\sqrt{1-\sin x}}$$

$$\parallel$$

$x = \frac{\pi}{2} - y$

$$\underline{y = \tfrac{\pi}{2} - x} \qquad \int_{\frac{\pi}{2}}^{0} \frac{-dy}{\sqrt{1-\sin(\frac{\pi}{2}-y)}} = \int_0^{\frac{\pi}{2}} \frac{dy}{\sqrt{1-\cos y}}$$



$\sin(\frac{\pi}{2}-y) =$

$$1-\cos y \sim \frac{y^2}{2} \overset{def}{\iff} \lim \frac{1-\cos y}{\frac{y^2}{2}} = 1$$

$$\lim_{x\to x_0} f(x) = \lim_{h\to 0} f(x_0+h)$$

$$\sqrt{1-\cos y} \sim \frac{y}{\sqrt{2}} \quad \text{per } y \to 0$$

per confronto asintotico $\int_0^{\pi/2} \frac{dy}{\sqrt{1-\cos y}}$ conv. $\iff \int_0^{?} \frac{1}{y} = +\infty$

$\parallel$

$+\infty$

Es da 116 a 134 Dire se convergono i seguenti integrali:

120 $\int_0^1 \frac{\sin\sqrt{x}}{x}\,dx$ converge. (risultato per $t\to 0$.)

perché: in 1 è regolare e in 0 $\sin\sqrt{x} \sim \sqrt{x}$ $\quad \frac{\sin\sqrt{x}}{x} \sim \frac{1}{\sqrt{x}}$

e $\frac{1}{\sqrt{x}}$ è int. e quindi per confronto asint. converge

(è converg.)

123 $\int_2^\infty \frac{\sin x}{\log x}\,dx$

Prov. crit. conv. ass.: $\int \frac{|\sin x|}{\log x} \leq \int \frac{1}{\log x} = +\infty$

per confronto con $\frac{1}{x}$: $\frac{1}{\log x} \geq \frac{1}{x}$ $\forall x \geq 1$

(Cfr Es* 134) criterio analogo al criterio di Leibniz per serie

$(-1)^n$ è periodica di periodo 2, con max $=1$, e min $=-1$

$-1 \leq (-1)^n \leq 1$.

$$\int_2^\infty \frac{\sin x}{\log x} = \left[-\frac{\cos x}{\log x}\right]_2^\infty - \int_2^\infty \frac{\cos x}{\log^2 x}\frac{1}{x}$$

$$= \frac{\cos 2}{\log 2} - \int_2^\infty \frac{\cos x}{\log^2 x}\frac{1}{x} \qquad \left(\frac{1}{\log x}\right)' = -\frac{1}{(\log x)^2}\frac{1}{x}$$

è convergente!

$\sin x$ è periodica di periodo 2π, con max $=1$ e min $=-1$, e $-1 \leq \sin x \leq 1$

perché $\int_2^\infty \left|\frac{\cos x}{x\log^2 x}\right| \leq \int_2^\infty \frac{dx}{x\log^2 x}$ $\leadsto$ $\sum \frac{1}{n\log^2 n}$

$$= \int_2^\infty \frac{dy}{y^2} = \frac{1}{2}$$