Lezione 8 8/10/21
venerdì 8 ottobre 2021 08:12

[GE] Cap 3 — Calcolare il limite di a_n giustificando i passaggi

Es 24 $a_n = \sqrt[n]{2n^5+1}$

$$a_n = (2n^5+1)^{\frac{1}{n}}$$
$$= (2n^5)^{\frac{1}{n}}\left(1+\frac{1}{2n^5}\right)^{\frac{1}{n}}$$
per la proprietà delle potenze

$$= 2^{\frac{1}{n}}\left(n^{\frac{1}{n}}\right)^5\left(1+\frac{1}{2n^5}\right)^{\frac{1}{n}}$$

Argomento intuitivo
$$2n^5+1 \sim 2n^5$$
$$(2n+1)^{\frac{1}{n}} \sim (2n^5)^{\frac{1}{n}}$$
$$= 2^{\frac{1}{n}} \quad n^{\frac{5}{n}}$$

$2^{\frac{1}{n}} \to 1$ per limite notevole

$n^{\frac{1}{n}} \to$ " "

$$\left(n^{\frac{1}{n}}\right)^5 = \underbrace{n^{\frac{1}{n}} \cdots n^{\frac{1}{n}}}_{5} \to 1 \quad \text{per l'Alg. d. limiti}$$

$$1 < \left(1+\frac{1}{2n^5}\right)^{\frac{1}{n}} \le 2^{\frac{1}{n}}$$
$\downarrow$ 1, $\downarrow$ 1

$\Rightarrow$ per Teorema del confronto $\boxed{a_n \le b_n \le c_n \text{ e } a_n, c_n \to \alpha \Rightarrow b_n \to \alpha}$

$$\left(1+\frac{1}{2n^5}\right)^{\frac{1}{n}} \to 1$$

Per A.d.L. $a_n \to 1$

Es.31 $a_n = (2^n+3^n)^{\frac{1}{n}}$

intuitivamente
$2^n+3^n \sim 3^n \Rightarrow a_n \to 3$

$$= 3\left(1+\left(\tfrac{2}{3}\right)^n\right)^{\frac{1}{n}}$$
prop potenze

(come prima) $1 < \left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} < 2^{\frac{1}{n}}$ e per confronto
$$\left(1+\left(\tfrac{2}{3}\right)^n\right)^{\frac{1}{n}} \to 1$$

Per AdL $\lim a_n = 3$

Es 38 $a_n = \dfrac{n^2}{n+1} - \dfrac{n^2+1}{n}$

$$a_n = \frac{n^3-(n^2+1)(n+1)}{n(n+1)}$$

$$= \frac{-n^2+\cdots}{}$$
termini di grado < 2

Se $P(x)$ è un polinomio il suo grado $m = \deg(P)$ è l'esponente del monomio di grado massimo

$$P(x) := a_m x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$$

n^2+n

$\longrightarrow -1$ per algebra dei limiti

$\Big($ $\underline{\underline{a_m \neq 0}}$ $m = \deg(P)$

$\underline{a_k \in \mathbb{R}}$, $x \in \mathbb{R}$

dati variabile

In generale se $P(x)$ e $Q(x)$ sono polinomi di grado rispettivamente k e h.

termini di grado inferiore a k

$$\frac{P(n)}{Q(n)} = \frac{a_k n^k + t.g.i}{b_h n^h + t.g.i}$$

$$= \frac{a_k}{b_h}\frac{n^k}{n^h} \cdot \frac{1 + c\frac{1}{n} + \dots}{1 + d\frac{1}{n} + \dots}$$

$$= \frac{a_k}{b_h} n^{k-h} \alpha(n) \quad \text{con } \alpha(n) \to 1 \text{ per } n \to +\infty$$

$$\longrightarrow \begin{cases} 0 & k<h \\ \frac{a_k}{b_k} & k=h \\ +\infty, \text{ se } a_k b_h > 0 & k>h \\ -\infty, \text{ se } a_k b_h < 0 & k>h \end{cases}$$

Es 51 $\quad a_n = n\left(\sqrt[3]{8+\sin 2^{\frac{1}{n}}} - 2\right)$ $\quad \sqrt[3]{8}$

Ricordo

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a - b = \dfrac{a^3-b^3}{a^2+ab+b^2}$

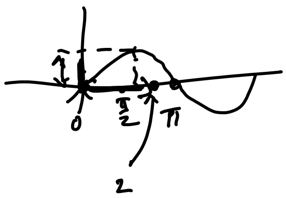
razionalizzo con $a = \sqrt[3]{8+\sin 2^{\frac{1}{n}}}$ e $b=2$

$$a_n = n \frac{\cancel{8} + \sin 2^{\frac{1}{n}} - \cancel{8}}{\left(8+\sin 2^{\frac{1}{n}}\right)^{\frac{2}{3}} + \left(8+\sin 2^{\frac{1}{n}}\right)^{\frac{1}{3}}\cdot 2 + 4}$$

$$\geq \frac{n \sin 2^{\frac{1}{n}}}{4} > \frac{n \cdot \sin 1}{4}, \quad (n \geq 2)$$

$\sin 2^{\frac{1}{n}} > 0$

$1 < 2^{\frac{1}{n}} \leq 2$



$n \geq 2$, $\underline{1 < 2^{\frac{1}{n}}} < \frac{\pi}{2}$

e $x \in (0, \frac{\pi}{2}) \to \sin x$ è strett. crescente

$\Rightarrow \sin 2^{\frac{1}{n}} > \sin 1 > 0$, $\forall n \geq 2$

Quindi $a_n > c\, n$ $\quad c>0$ $\Rightarrow$ $a_n \to +\infty$ per confronto.

Più semplicemente $\quad a_n = n\left(\sqrt[3]{8+\sin 2^{\frac{1}{n}}} - 2\right)$

$\sin 2^{\frac{1}{n}} > \sin 1$ per $n \geq 2$.

Con le stesse motivazioni

$$a_n = n\left(\sqrt[3]{9+\sin 2^{\frac{1}{n}}}-2\right) \underset{n\geq 2}{>} n\left(\underbrace{\sqrt[3]{8+\sin 1}-2}_{a}\right)$$

$$9+\sin 1 > 8 \Rightarrow a>0 \Rightarrow n\cdot a \to +\infty$$

Es $$a_n = n\left(\underbrace{\sqrt[3]{8+\frac{1}{n^2}}-2}\right)$$

rallo ultimo $\quad n\cdot \dfrac{\left(\frac{1}{n^2}\right)}{c} = \qquad c>2$

$$\frac{1}{n}c \Rightarrow 0$$

Es 47 $$\frac{\log n!}{n\log n} = \frac{\log n!}{\log n^n}$$

$$\boxed{e\left(\frac{n}{e}\right)^n \leq n! \leq e\cdot n\left(\frac{n}{e}\right)^n}$$

(si dimostra per induzione)

[Es. $\dfrac{n!}{n^n} \to 0$ è vero

$$\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\frac{1}{n}, \quad n\geq 3$$

$$\underset{n\geq 3}{<} \frac{1}{n} \to 0.$$]

Intuitivamente, in prima approssimazione $n! \approx \left(\frac{n}{e}\right)^n$

$$\frac{\log n!}{n\log n} \approx \frac{\log\left(\frac{n}{e}\right)^n}{n\log n} = \frac{n(\log n-1)}{n\log n} \to 1$$

$$\log\left(e\left(\frac{n}{e}\right)^n\right) \leq \log n! \leq \log\left(en\left(\frac{n}{e}\right)^n\right)$$

$$\frac{1+n(\log n-1)}{n\log n} \leq \frac{\log n!}{n\log n} \leq \frac{1+\log n+n(\log n-1)}{n\log n}$$

$\downarrow$ alg. di lim. $\qquad\qquad \downarrow$

$1 \qquad\qquad 1$

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

dove

$a_n, b_n > 0$

$a_n \sim b_n \overset{\text{def}}{\Longleftrightarrow} \lim \frac{a_n}{b_n} = 1$

Es 68 $\quad n^2\, 2^{-\sqrt{n}} = \dfrac{n^2}{2^{\sqrt{n}}}$

$= e^{2\log n}\, e^{-\sqrt{n}\log 2} = e^{(2\log n - \sqrt{n}\log 2)}$

$n^2 = e^{2\log n}$ $\quad = e^{-\sqrt{n}\log 2\left(1 - \frac{2\log n}{\log 2\sqrt{n}}\right)}$

$\underline{-\sqrt{n}\log 2}\left(1 - \dfrac{2\log n}{\log 2\cdot\sqrt{n}}\right) \to -\infty$

per A d L estesa

$\log 2 > 0$

$1 - \dfrac{2\log n}{\log 2\cdot\sqrt{n}} \to 1.$

A d L estesa

$a_n \to +\infty \quad e\ b_n \to \underline{\alpha > 0}$

$\Rightarrow \quad a_n b_n \to +\infty$

$a_n \to -\infty \quad e\ b_n \to \alpha > 0$

$\Rightarrow \quad a_n \cdot b_n \to -\infty$

ho usato che se $\quad b_n \to \infty \Rightarrow e^{-b_n} \to 0$

$e^{-b_n} = \dfrac{1}{e^{b_n}}$ $\quad e^{b_n} \to +\infty$

$\boxed{a_n \to +\infty \Rightarrow \frac{1}{a_n} \to 0}$

$\Rightarrow e^{-b_n} \to 0$

Attenzione: il viceversa NON è vero

$a_n \to 0 \not\Rightarrow \dfrac{1}{a_n} \to +\infty$

NO

$a_n \to 0$ e $a_n > 0 \Rightarrow \dfrac{1}{a_n} \to +\infty$

$a_n \to 0$ e $a_n < 0 \not\Rightarrow \dfrac{1}{a_n} \to -\infty$

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67 $\quad \left(1 + \dfrac{7}{n}\right)^{2n}$ $\quad$ useremo $\quad \underline{\left(1 + \frac{x}{n}\right)^n \to e^x}$

$\quad \shortparallel$

$\left(\left(1 + \dfrac{7}{n}\right)^n\right)^2 \to (e^7)^2 = e^{14}$

A.d.L e

$a_n \to \alpha \quad a_n^p \to \alpha^p \quad \forall\, p \in \mathbb{N}$

Es 72 $\quad a_n = \dbinom{2n}{n}$

COEFFICIENTE BINOMIALE

$n \geq m \geq 0$ numeri interi

$\dbinom{n}{m} := \dfrac{n!}{m!\,(n-m)!} \in \mathbb{N}$

"n sopra m"

$\dbinom{m}{\ } = \dbinom{m}{\ }$ ovvia

$$\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}, \quad m \geq 1$$

$$a_n = \frac{(2n)!}{n!\ n!} = \frac{2n\ (2n-1) \cdot (2n-2) \cdots (n+1)\ \cancel{n!}}{n!\ \cancel{n!}}$$

$(n=4 \qquad 8! = \underline{8\ 7\ 6 \cdot 5}\ 4!)$ ✓ $\qquad 2n-n$

$$= \underbrace{\frac{2\cancel{n}}{\cancel{n}}}_{k=0} \cdot \underbrace{\frac{(2n-1)}{n-1}}_{k=1} \cdot \underbrace{\frac{(2n-2)}{n-2}}_{k=2} \cdots \underbrace{\frac{n+1}{1}}_{k=n-1} > 2^n \to +\infty.$$

$$\frac{2n-k}{n-k} \geq 2 \qquad \cancel{2n}-k \geq \cancel{2n}-2k, \quad 2k > k \quad \underline{k \geq 1}$$

$\underline{\text{Es}}$ $\lim \binom{2n}{n}^{\frac{1}{n}}$ (sappiamo dall'esercizio precedente che $\lim \binom{2n}{n}^{\frac{1}{n}} \geq 2$

$$\binom{2n}{n} = \frac{(2n)!}{n!\ n!}$$

$$\boxed{\text{Teorema} \quad a_n > 0 \ \text{e} \ \lim \frac{a_{n+1}}{a_n} = \alpha \ \text{con} \ \alpha \geq 0 \ \text{o} \ \alpha = +\infty \ \Rightarrow \lim \sqrt[n]{a_n} = \alpha}$$

$$a_n = \binom{2n}{n}$$

$$\frac{a_{n+1}}{a_n} = \binom{2(n+1)}{n+1} \frac{1}{\binom{2n}{n}} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{n!}\ ((n+1)\cancel{n!})} \frac{\cancel{n!}\ \cancel{n!}}{\cancel{(2n)!}}$$

$$\to 4. \quad \Rightarrow a_n \to 4$$
