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Since the term in outer parenthesis on the left of (7) has 0 at position $(n, -n)$ and below, and the term in parenthesis on the right has 0 at position $(n, -n-1)$ and below, we may now apply induction on n and the fact that $X^{i,-i}=0$ for $i<0$ to deduce that $x \in B^0(X)$.

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THE JORDAN CURVE THEOREM FOR PIECEWISE SMOOTH CURVES

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1. Introduction. It is the purpose of this note to provide an elementary proof of the Jordan Curve Theorem for the class of piecewise smooth curves. The only tools which we require are the notions of compactness, continuity, and the concept of the index of a closed curve relative to a point. Since these topics are included in a standard advanced undergraduate or beginning graduate course in complex analysis, it is our hope that the proof will fit in well with such a course.

We begin with an informal outline of the proof as it would apply to a polygon. In order to prove the Jordan Arc Theorem for a simple polygon, it suffices to demonstrate that its complement is arcwise connected. Suppose this is true for all simple polygons having at most n segments. A simple polygon P_{n+1} having $n+1$ segments is obtained by adjoining a single segment σ to a simple polygon P_n having n-segments. Any two points in the complement of P_{n+1} can be joined by a polygonal arc C in the complement of P_n. If C does not intersect σ, then it clearly lies in the complement of P_{n+1}. If it does intersect σ, then by drawing parallel lines on either side of σ, it is easily seen that C may be replaced by a polygon which does not intersect P_{n+1}. Hence, the complement of P_{n+1} is connected. In order to obtain a valid induction proof, it suffices to note that the complement of a single segment is indeed connected.

Now let P be a simple closed polygon. The Jordan Curve Theorem for P asserts that the complement of P is comprised of two nonempty components E and I. Let Γ be the simple polygon obtained by removing from P a segment σ. Choose ζ to be a point which lies outside of a disk containing P in its interior. Denote by E the set of points in the complement of P which can be joined, in the complement of Γ to ζ by a polygonal arc which does not cross σ. Let I consist of all other points in the complement of P. Evidently E is connected, since any two of its points can be joined in E to ζ, and hence to each other. In order to show that I is connected, we observe that any point $z\in I$ can be joined in the complement of Γ to ζ by a polygonal arc C_z which necessarily crosses σ. Since $E\cap I=\emptyset$, it follows that C_z must emerge from σ on the opposite side from which it approached σ. Otherwise, by drawing a line parallel to σ, we could show that $z\in E$. By similar reasoning, it can be shown that if $w\in I$, then C_w approaches σ from the same side as C_z. Hence z can be joined to w by an arc which does not cross P. It follows that I is connected. However, it is not immediately evident that it is nonempty. This possibility is excluded by showing that two points on opposite sides of σ have different indices with respect to P, and hence are in different components.

The objective of this paper is accomplished if we can show that the preceding argument remains valid with straight line segments replaced by simple smooth arcs. A moment of reflection shows that the only property of a straight line which was used is that it is not intersected by a line parallel to it. In Lemma 1, it is shown that to each simple smooth arc Γ there can be associated a system of "parallel" (not necessarily simple) arcs which do not intersect it. Lemma 2 is devoted to proving that any point sufficiently close to an interior point of Γ lies on at least one of these parallel arcs. It then follows immediately from Lemmas 1 and 2 that any two points of the complement Γ which are sufficiently close to Γ and on "the same side" of Γ can be joined by an arc not crossing Γ. Lemma 3 provides the step needed to draw the same conclusion concerning points on "opposite sides" of Γ. Lemma 4 provides the tool needed to deduce the Jordan Arc Theorem for arcs consisting of $n+1$ smooth arcs from its truth for those having n smooth arcs.

2. Definitions and notation. An arc is said to be *smooth* if it has a C^1 parametrization. A *piecewise smooth arc* is one which is obtained by joining end to end a finite number of smooth arcs. If an arc C is parametrized by $z=\Phi(t)$, $a\leqq t\leqq b$, and $S\subset[a, b]$, we shall denote by $C\{S\}$ the image of S under Φ. At other times, if $z, \zeta\in C$, we shall use $C[z, \zeta]$ to denote a portion of C joining z to ζ. By C' we mean the complement of the arc C with respect to the plane.

3. Preliminaries. We begin by introducing, for each smooth arc C, a class of arcs C_ϵ which plays the roll of the lines parallel to a given segment. These arcs will be used to connect points close to C by an arc which does not intersect C.

LEMMA 1. *Let* $C: z=\Phi(t)$, $0\leqq t\leqq L$, *be a simple smooth arc parametrized by arc-length. Define, for each real* ϵ, C_ϵ *to be that arc parametrized by* $z=\Phi_\epsilon(t)\equiv\Phi(t)+i\epsilon\,\Phi'(t)$, $0\leqq t\leqq L$. *There exists a* $d>0$ *such that* $C_\epsilon\cap C=\emptyset$ *when* $0<|\epsilon|<d$.

Proof. We begin by showing that portions of C and C_ϵ, corresponding to sufficiently small neighborhoods of the parametric interval $[0, L]$, are disjoint. Let

$t, \tau \in [0, L]$. We then have, after some manipulation,

$$\Phi_\epsilon(t) - \Phi(\tau) = (t - \tau + i\epsilon)\Phi'(t) + \int_\tau^t [\Phi'(s) - \Phi'(t)]ds.$$

By uniform continuity, there exists a $\delta > 0$ such that $|\Phi'(s) - \Phi'(t)| < 1/2$ if $|s-t| < \delta$. Hence, if $|t-\tau| < \delta$, we have

$$|\Phi_\epsilon(t) - \Phi(\tau)| > |t - \tau + i\epsilon||\Phi'(t)| - |t - \tau|/2.$$

Now $|\Phi'(t)| = 1$ since t represents arc-length. Consequently

$$\Phi_\epsilon(t) \neq \Phi(\tau) \quad \text{if } |t - \tau| < \delta \quad \text{and} \quad \epsilon \neq 0. \tag{1}$$

We next prove that for ϵ sufficiently small, each point on C has a neighborhood which is disjoint from C_ϵ. To this end we choose points $0 = t_0 < t_1 < \cdots < t_n = L$ such that $|t_k - t_{k-1}| < \delta/4$. It is then a consequence of (1) that

$$C\{|t - t_k| \leqq \delta/4\} \cap C_\epsilon\{|t - t_k| \leqq \delta/2\} = \varnothing. \tag{2}$$

The point sets $C\{|t-t_k| \leqq \delta/4\}$ and $C\{|t-t_k| \geqq \delta/2\}$ are disjoint and compact since C is simple and the continuous image of a compact set is compact. Hence they have a positive distance d_k. The fact that the portions of C and C_ϵ, corresponding to the set $|t-t_k| \geqq \delta/2$, have a distance at most $|\epsilon|$ then shows that

$$C_\epsilon\{|t - t_k| \geqq \delta/2\} \cap C\{|t - t_k| \leqq \delta/4\} = \varnothing \quad \text{if } |\epsilon| < d_k. \tag{3}$$

By combining (2) and (3) it is easily seen that $C\{|t-t_k| \leqq \delta/4\} \cap C_\epsilon = \varnothing$ if $|\epsilon| < d_k$. It follows that $C \cap C_\epsilon = \varnothing$ if $|\epsilon| < d = \min\{d_k\}$. This completes the proof.

We next use a standard variational argument to show that any point sufficiently close to an interior point of C lies on one of the arcs C_ϵ.

LEMMA 2. *Let C and C_ϵ be as defined in Lemma 1. If $z \notin C$ is closer to C than it is to either end point of C, then there exists a $t_0 \in (0, L)$ and an $\epsilon_0 \neq 0$ such that $z = \Phi(t_0) + i\epsilon_0\Phi'(t_0)$, that is $z \in C_{\epsilon_0}$.*

Proof. Since z is closer to C than it is to $\Phi(0)$ or $\Phi(L)$, there exists a $t_0 \in (0, L)$ such that $|z - \Phi(t_0)| = \operatorname{dist}\{z, C\}$. Using the definition of distance and the identity $z - \Phi(t) = z - \Phi(t_0) + \Phi'(t_0)(t - t_0) + o(t - t_0)$, we have

$$|z - \Phi(t_0)|^2 \leqq |z - \Phi(t)|^2 = |z - \Phi(t_0)|^2 + 2\operatorname{Re}[z - \Phi(t_0)]\overline{\Phi'(t_0)}(t - t_0) + o(t - t_0).$$

It follows from the fact that $t - t_0$ can be either positive or negative that $2\operatorname{Re}[z - \Phi(t_0)]\overline{\Phi'(t_0)} = 0$. But this is equivalent to $z - \Phi(t_0) = i\epsilon_0\Phi'(t_0)$ for some real $\epsilon_0 \neq 0$. This completes the proof.

The previous two lemmas allow us to say that two points $z \in C_\epsilon$ and $\zeta \in C_\eta$ are on the same or opposite sides of C according to whether ϵ and η have the same or opposite signs. Note that we have not excluded the possibility of a point being on both sides of C. Fortunately, this is not important for our purpose. Once the Jordan Curve Theorem has been proved, it is an easy exercise to show that if $z\in C_\epsilon$, $0<|\epsilon|<d$, then z is only on one side of C.

The previous two lemmas will be used to show that any two points sufficiently close to and on the same side of C can be joined by an arc in C'. In order to prove the same result for points on opposite sides of C we shall need another lemma.

LEMMA 3. *Let C be as in Lemma 1. There exists a $d>0$ such that the 'half neighborhood' $z=\Phi(L)+\epsilon\,\Phi'(L)e^{i\theta}$, $0<\epsilon<d$, $-\pi/2\leqq\theta\leqq\pi/2$, is disjoint from C.*

Proof. There exists a δ such that $|\Phi'(s)-\Phi'(L)|<1/2$ if $|s-L|<\delta$. We have

$$\Phi(L)+\epsilon\Phi'(L)e^{i\theta}-\Phi(t)=\int_t^L[\Phi'(s)-\Phi'(L)]ds+\Phi'(L)[(L-t)+\epsilon e^{i\theta}].$$

It follows from the triangle inequality and the fact that $|\Phi'(L)|=1$ that the right side of the above equality is greater in absolute value than

$$|L-t+\epsilon\, e^{i\theta}|-(L-t)/2$$

if $L-t<\delta$. The above expression is easily seen to be positive for $\epsilon>0$ and $-\pi/2\leqq\theta\leqq\pi/2$. Hence the 'half neighborhood' is disjoint from $C\{L-\delta\leqq t\leqq L\}$. Let d be the distance from $\Phi(L)$ to $C\{0\leqq t\leqq L-\delta\}$. If $0<\epsilon<d$, then the 'half neighborhood' is disjoint from C. This completes the proof.

LEMMA 4. *Let C be as in Lemma 1, A a compact set and z, ζ two points of $(C\cup A)'$, each of which is closer to an interior point of C than it is to A or to an endpoint of C. If* (i) *$C\cap A=\Phi(0)$ or* (ii) *$C\cap A=\Phi(0)\cup\Phi(L)$ and z, ζ are on the same side of C, then z can be joined to ζ by an arc in $(A\cup C)'$.*

Proof. Let $z_1=\Phi(t_1)$ and $\zeta_1=\Phi(\tau_1)$ be points on C which minimize the respective distances from z and ζ to C. The segments $[z, z_1]$ and $[\zeta, \zeta_1]$ then intersect C only at the points z_1 and ζ_1 respectively. As a consequence of Lemma 2, we have $z=\Phi(t_1)+i\epsilon_0\Phi'(t_1)$ and $\zeta=\Phi(\tau_1)+i\eta_0\Phi'(\tau_1)$ where ϵ_0 and η_0 are nonzero. It follows that z and ζ can be connected in $(A\cup C)'$ to points $z_2=\Phi(t_1)+i\epsilon\Phi'(t_1)$ and $\zeta_2=\Phi(\tau_1)+i\eta\Phi'(\tau_1)$ for all small ϵ, η which have the same signs as ϵ_0 and η_0. Assume that $\eta=\pm\epsilon$ and that $|\epsilon|$ is less than the δ of Lemma 1 and the d of Lemma 3. If case (i) holds we suppose that $t_1<\tau_1$. The arc $C[\Phi(t_1), \Phi(L)]$ is disjoint from A and hence has a positive distance δ from it. Let $|\epsilon|$ be less than δ and the d's of Lemmas 1 and 3. If $\eta=\epsilon$ then, by Lemma 1, the arc $C_\epsilon\{t_1\leqq t\leqq\tau_1\}$ serves to join z_2 and ζ_2 in $(C\cup A)'$. If $\eta=-\epsilon$ we may join z_2 and ζ_2 to the points $\Phi(L)+i\epsilon\Phi'(L)$ and $\Phi(L)-i\epsilon\Phi'(L)$ by the arcs $C_\epsilon\{t_1\leqq t\leqq L\}$ and $C_{-\epsilon}\{\tau_1\leqq\tau\leqq L\}$. It then follows from Lemma 3 that these two points can be joined by an arc in $(C\cup A)'$. The proof of (ii) is similar and will be omitted.

4. **The Jordan Arc Theorem.** Once one has proved either the Jordan Curve

Theorem or its companion the Jordan Arc Theorem, the proof of the other one is relatively simple. Lemmas 1, 2, and 3 are now used to give a simple proof of the Jordan Arc Theorem for our special class of curves.

THEOREM 1. *The complement of a simple piecewise smooth arc C is an open connected set having C as its boundary.*

Proof. It follows from the fact that C is compact that C' is open and that the boundary of C' is contained in C. Lemma 1 shows that each smooth point of C is a boundary point of C'. Since the boundary of any set is closed, the "corner points" of C are also in the boundary of C'.

It remains to show that C' is connected. We proceed by induction on the number of smooth segments of C to show that C' is arcwise connected. Suppose then that C is a simple smooth arc. If $z, \zeta \in C'$ we may join them by a smooth arc Γ which does not pass through either end point of C. If Γ does intersect C we may, because of the continuity of the parametrization of Γ, join z and ζ in C' to points z_1 and ζ_1 which are arbitrarily close to interior points of C. By Lemma 4, z_1 can be connected to ζ_1 by an arc in C'. Thus any two points in C' can be connected by an arc in C'. Hence C' is arcwise connected.

Suppose now that Theorem 1 is true for arcs having n smooth segments. If C_{n+1} has $n+1$ smooth segments, let C_n denote the first n segments and C the last. If $z, \zeta \in C'_{n+1}$, then by our induction hypothesis, z and ζ can be joined by an arc Γ in C'_n. We may assume that Γ does not pass through an end point of C, since removing a point from an open connected set does not disconnect it. We may then join z and ζ in C'_{n+1} to points z_1 and ζ_1 which are arbitrarily close to interior points of C. It follows from Lemma 4, with $A = C_n$, that z_1 can be connected to ζ_1 by an arc in C'_{n+1}. This completes the proof.

5. The Jordan Curve Theorem. We are now in a position to state and prove the main theorem of this note.

THEOREM 2. *The complement of a simple closed piecewise smooth curve C consists of two components, E and I, each having C as its boundary. Moreover, the index of C is equal to zero in E and, if C is oriented properly, is equal to one in I.*

Proof. We first show that C' consists of at most two components. Since C is compact, there exists a point ζ which lies outside of a disk containing C in its interior. Let E denote the set of points which can be joined to ζ by an arc in C'. E is clearly connected since any two of its points can be connected to ζ by an arc in E. Let $I = C' - E$. If $I \neq \emptyset$, let Γ be the simple piecewise smooth arc obtained by removing an open smooth segment γ from C. By Theorem 1, any point $z_1 \in I$ can be joined to ζ by an arc $\Gamma_{z_1} \subset \Gamma'$. This curve necessarily crosses γ for otherwise z would be in E. As in the proof of Theorem 1, we now choose points z_1', ζ' arbitrarily close to interior points of γ such that $\Gamma_{z_1}[z_1, z_1']$ and $\Gamma_{z_1}[\zeta', \zeta]$ are in C'. We claim that z_1' and ζ' are on opposite sides of γ for otherwise by Lemma 4 z could be joined to ζ by an arc in C'. Let z_2 be another point in I and let z_2' play the role analogous to z_1'. The point z_2' must be on the same side of γ as z_1' for otherwise z_2 could be connected to ζ by an arc in C'. Since z_1' and z_2' are on the same side of γ, it follows from Lemma 4 that z_1 and z_2 can be connected by an arc in C'. Hence I is connected.

We next consider the difference between the index of C at two points on opposite sides of a smooth portion of C. If $z_0=\Phi(t_0)$ is such a point, it follows from Lemma 1 that, as long as ϵ retains its sign, $z_\epsilon=\Phi(t_0)+i\epsilon\Phi(t_0)$ is in the same component of C'. It follows that

$$\Delta = n(C, z_0 + i\epsilon\Phi'(t_0)) - n(C, z_0 - i\epsilon\Phi'(t_0))$$

is constant for all small $\epsilon>0$. By using the continuity of Φ' at t_0, it is easily shown that

$$\left(\frac{1}{\Phi(t) - z_0 - i\epsilon\Phi'(t_0)} - \frac{1}{\Phi(t) - z_0 + i\epsilon\Phi'(t_0)}\right)\Phi'(t) = \frac{2i\epsilon}{(t - t_0)^2 + \epsilon^2} + \mathcal{E},$$

where given an $\eta>0$ there exists a $\delta>0$ such that

$$|\mathcal{E}| < \eta\frac{\epsilon}{[(t - t_0)^2 + \epsilon^2]}$$

if $|t-t_0|<\delta$. If $\tilde{C}$ denotes the portion of C corresponding to $|t-t_0|>\delta$, we then have

$$\Delta = \frac{1}{2\pi i}\int_{\tilde{C}}\left(\frac{1}{z - z_0 - i\epsilon\Phi'(t_0)} - \frac{1}{z - z_0 + i\epsilon\Phi'(t_0)}\right)dz$$
$$+ \frac{1}{2\pi i}\int_{t_0-\delta}^{t_0+\delta}\left(\frac{2i\epsilon}{(t - t_0)^2 + \epsilon^2} + \mathcal{E}\right)dt.$$

The first integral tends to zero as $\epsilon\to 0$ since its integrand is continuous at $\epsilon=0$. In the second integral we substitute $t-t_0=\epsilon s$ and then let $\epsilon\to 0$. We then obtain

$$|\Delta - 1| < \eta.$$

But since Δ is an integer we must have $\Delta=1$. It follows that C' has at least two components. But we already know that C' has at most two components; hence I is not empty. The above argument also shows that each smooth point of C is a boundary point of both E and I. That the 'corners' are boundary points follows from the fact that the boundary is a closed set. Since in E (the unbounded component of C') the index of C is zero, it follows that in I it is ± 1. Hence by reorienting if necessary, we can arrange that it is 1. This completes the proof.

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Reference

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