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EXISTENCE OF QUASI-PERIODIC ORBITS FOR TWIST HOMEOMORPHISMS OF THE ANNULUS

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WE WILL PROVE that any area preserving "twist" homeomorphism f on the annulus has quasi-periodic orbits of all frequencies ω in an interval $[\rho(f_0), \rho(f_1)]$. It is easy to see that there are no quasi-periodic orbits of frequency ω when ω is not in this interval. In stating this result, we give a liberal interpretation of what it means for an orbit to be quasi-periodic: the closure of such an orbit may be a Cantor set, not a circle.

The method used in this paper is closely related to a method Percival has previously used to find quasi-periodic orbits numerically [3, 4]. However, to the best of my knowledge, Percival has not proved an existence theorem using his method.

To state our theorem, it is easier to work with the universal cover A of the annulus than with the annulus itself. Let $A = \{(x, y) \in \mathbb{R}^2: 0 \le y \le 1\}$. Let $T: A \to A$ be the translation $T(x, y) = (x + 1, y)$. Let f be an area perserving, orientation preserving, and boundary component preserving homeomorphism of A such that $fT = Tf$. In addition suppose that $f(x, y)_1 > f(x, z)_1$, when $y > z$, where $p_1 = x$ if $p = (x, y) \in A$. This is the "twist condition".

Let $f_i = f|\mathbb{R} \times i$, $i = 0, 1$. Let $B = \{(x, x') \in \mathbb{R}^2: f_0(x) \le x' \le f_1(x)\}$. From the twist condition, it follows that for each $(x, x') \in B$, there exists a unique $y = g(x, x') \in [0, 1]$ and $y' = g'(x, x') \in [0, 1]$ such that $f(x, y) = (x', y')$. The functions g and g' are continuous functions on B.

For any homeomorphism $h: \mathbb{R} \to \mathbb{R}$ such that $h(x + 1) = h(x) + 1$, let

$$\rho(h) = \lim_{n \to \pm\infty} \frac{h^n(x)}{n}.$$

A well known theorem of Poincaré states that this limit exists and is independent of x.

The following is our main result.

THEOREM. *Suppose $\rho(f_0) \le \omega \le \rho(f_1)$. Then there exists a weakly order preserving mapping $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(t + 1) = \phi(t) + 1$ and*

$$f(\phi(t), \eta(t)) = (\phi(t + \omega), \eta(t + \omega)) \tag{1}$$

where $\eta(t) = g(\phi(t), \phi(t + \omega))$.

The mapping ϕ is not necessarily continuous. However, we will show:

ADDENDUM 1. *If t is a point of continuity of ϕ, then so are $t + \omega$ and $t - \omega$.*

The meaning of this theorem depends on whether ω is rational or irrational. If ω is rational, say $\omega = p/q$ in lowest terms, then the theorem implies the existence of a point (x, y) such that $f^q(x, y) = (x + p, y)$. For, if ϕ satisfies the conditions of the

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theorem, and $(x, y) = (\phi(t), \eta(t))$, for some $t \in \mathbb{R}$, then $f^q(x, y) = (x + p, y)$. This case is a consequence of a famous theorem of Birkhoff [1].

In the case ω is irrational, we have:

ADDENDUM 2. *If $\omega \notin \mathbb{Q}$, then ϕ is not constant on any interval.*

When ω is irrational, we let M_ϕ be the closure of the set of $(\phi(t), \eta(t))$ such that t is a point of continuity of ϕ. Since ϕ is weakly order preserving, the set of points of discontinuity of t is at most countable. Then M_ϕ is the same as the union of all limits from below $(\phi(t-), \eta(t-))$ and all limits from above $(\phi(t+), \eta(t+))$. We set $\Sigma_\phi = M_\phi/T$.

In the case ϕ is continuous, it is clear that M_ϕ is homeomorphic to R and Σ_ϕ is homeomorphic to a circle. Moreover, letting $\bar{f}$ be the homeomorphism of the annulus A/T induced by f, we have that $\bar{f}/\Sigma_\phi$ is conjugate to a rotation with rotation number $\equiv \omega$ (mod. 1).

In the case that ϕ is not continuous, it follows from Addenda 1 and 2 that Σ_ϕ is a Cantor set invariant under $\bar{f}$. It is easily checked from the conditions imposed on ϕ in the conclusion of theorem 1 and addenda 1 and 2 that $\bar{f}|\Sigma_\phi$ is topologically semi-conjugate to the rotation of a circle with rotation number $\equiv \omega$(mod 1). In fact, identifying $(\phi(t-), \eta(t-))$ with $(\phi(t+), \eta(t+))$ and then identifying Tp with p, gives a circle, on which the homeomorphism induced by f is topologically conjugate to a rotation. These identifications may also be described purely in terms of the topological dynamics of $\bar{f}|\Sigma_\phi$ points of Σ_ϕ which approach each other under indefinite forward and backward iteration under $\bar{f}$ are identified. A known, and not very difficult, argument then shows that $\bar{f}|\Sigma_\phi$ is topologically conjugate to one of the well known Denjoy minimal systems: $\bar{f}|\Sigma_\phi$ is minimal, and Σ_ϕ can be embedded in the circle so that $\bar{f}|\Sigma_\phi$ extends to an orientation preserving homeomorphism of the circle.

From the fact that ϕ is weakly order preserving, it follows that $t \to \phi(t) - t$ has bounded variation. Hence the Fourier expansion $\sum_{n=-\infty}^{\infty} a_n \exp(2\pi\, \text{int})$ of $\phi(t) - t$ converges pointwise everywhere, and converges to $\phi(t) - t$, whenever t is a point of continuity of ϕ [5].

In view of the definition of $\eta(t)$, its Fourier expansion $\sum_{n=-\infty}^{\infty} b_n \exp(2\pi\, \text{int})$ is Césaro summable everywhere, and sums to $\eta(t)$, whenever t is a point of continuity of ϕ. Moreover, if f is C^1 and $\frac{\partial f(x, y)_1}{\partial y} > 0$, then $\eta(t)$ has bounded variation, so its Fourier series converges pointwise, and converges to $\eta(t)$ when t is a point of continuity of ϕ.

Consider a point of continuity t_0 of ϕ and define

$$x_k = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi\, \text{in}(t_0 + k\omega))$$

$$y_k = \sum_{n=-\infty}^{\infty} b_n \exp(2\pi\, \text{in}(t_0 + k\omega)),$$

where the second sum is understood in the sense of Césaro summability. By Addendum 1, $t_0 + k\omega$ is a point of continuity of ϕ for all k. Hence

$$x_k = \phi(t_0 + k\omega),\ y_k = \eta(t_0 + k\omega),$$

so $f(x_k, y_k) = (x_{k+1}, y_{k+1})$. Thus, we have found a quasi-periodic orbit of frequency ω.

§1. OUTLINE OF THE PROOF

Let Y_ω denote the set of all weakly order preserving mappings $\phi:\mathbb{R}\to\mathbb{R}$ such that $\phi(t+1)=\phi(t)+1$, $f_0(\phi(t))\le\phi(t+\omega)\le f_1(\phi(t))$, and ϕ is continuous from the left, i.e. $\phi(t-)=\phi(t)$. Let X_ω denote the set of all $\phi\in Y_\omega$ such that $\phi(t)\ge 0$ for $t>0$ and $\phi(t)\le 0$ for $t\le 0$.

From the fact that f is area preserving, it follows that $g(x,x')\,\mathrm{d}x - g'(x,x')\,\mathrm{d}x'$ is a closed 1-form on B. Hence there exists a C^1 function $h(x,x')$ on B such that

$$\mathrm{d}h(x,x') = g(x,x')\,\mathrm{d}x - g'(x,x')\,\mathrm{d}x'. \tag{1.1}$$

For $\phi\in Y_\omega$, we define

$$F_\omega(\phi)=\int_{t=0}^{1} h(\phi(t),\phi(t+\omega))\,\mathrm{d}t. \tag{1.2}$$

In §2, we will show that $X_\omega\ne\emptyset$ if and only if $\rho(f_0)\le\omega\le\rho(f_1)$.

In §4, we will define a metric on Y_ω. In §5, we will show that X_ω is compact and in §6 that F_ω is continuous, with respect to that metric. Hence there exists $\phi\in X_\omega$ where F_ω takes its maximum.

For $a\in\mathbb{R}$, let $T_a:\mathbb{R}\to\mathbb{R}$ be the translation $T_a(x)=x+a$. In §3, we will show that $F_\omega(\phi T_a)=F_\omega(\phi)$ for any $a\in\mathbb{R}$. If $\phi\in Y_\omega$, we have $\phi T_a\in X_\omega$ where $a=\sup\phi^{-1}(-\infty,0)$. Hence, if F_ω takes its maximum on X_ω at ϕ, it also takes its maximum on Y_ω at ϕ.

For $\phi\in Y_\omega$ and $t\in\mathbb{R}$, we define

$$V(\phi,t)=\frac{\partial}{\partial x}[h(\bar{x},x)+h(x,x')], \tag{1.3}$$

evaluated at

$$\bar{x}=\phi(t-\omega),\ x=\phi(t),\ x'=\phi(t+\omega). \tag{1.4}$$

In §7–10, we will show that if F_ω takes its maximum on Y_ω at ϕ, then we have the following "Euler–Lagrange equation:"

$$V(\phi,t)=0,\ \text{for all } t\in\mathbb{R}. \tag{1.5}$$

This is essentially a special case of "Euler–Lagrange equation" due to Percival[3]. However, Percival gives no proof. In fact, the usual argument applies as long as ϕ is C^1, with non-vanishing derivative, and $f_0(\phi(t))<\phi(t+\omega)<f_1(\phi(t))$, for all $t\in\mathbb{R}$. However, to prove our existence theorem, we need an extension of the usual argument which involves some (unfortunately lengthy) reasoning of the type which is familiar from elementary theory of functions of one real variable.

Equation (1) follows easily from the "Euler–Lagrange equation". From the definition (1.1) of h and the definition (1.3) of $V(\phi,t)$, we get

$$V(\phi,t)=-g'(\bar{x},x)+g(x,x'), \tag{1.6}$$

where $\bar{x}$, x, x' are given by (1.4). Applying this with $t+\omega$ in place of t, and using the "Euler–Lagrange equation" (1.5), we obtain

$$g(\phi(t+\omega),\phi(t+2\omega))=g'(\phi(t),\phi(t+\omega)).$$

In view of our definition of η, this gives

$$\eta(t+\omega) = g'(\phi(t), \phi(t+\omega)).$$

Hence

$$\begin{aligned} f(\phi(t), \eta(t)) &= f(\phi(t), g(\phi(t), \phi(t+\omega))) \\ &= (\phi(t+\omega), g'(\phi(t), \phi(t+\omega))) \\ &= (\phi(t+\omega), \eta(t+\omega)), \end{aligned}$$

where the second equation is a consequence of the definition of g and g'.

Thus, once we have shown that F_ω takes a maximum on Y_ω, and satisfies the Euler–Lagrange equation whenever F_ω takes its maximum at ϕ, we obtain the Theorem of the introduction.

We will prove the Addenda in §11–12.

§2. $X_\omega \neq \emptyset$ IF AND ONLY IF $\rho(f_0) \leq \omega \leq \rho(f_1)$.

Proof. "Only if." Suppose $\phi \in Y_\omega$. If $n > 0$, then $f_0{}^n(\phi(t)) \leq \phi(t+n\omega) \leq f_1{}^n(\phi(t))$, so

$$\lim_{n\to\infty} \frac{f_0{}^n(\phi(t))}{n} \leq \lim_{n\to\infty} \frac{\phi(t+n\omega)}{n} \leq \lim_{n\to\infty} \frac{f_1{}^n(\phi(t))}{n},$$

or

$$\rho(f_0) \leq \omega \leq \rho(f_1). \tag{2.1}$$

Thus, (2.1) holds if $Y_\omega \neq \emptyset$.

"If." For $0 \leq s \leq 1$, let $g_s : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_s(t) = sf_1(t) + (1-s)f_0(t).$$

Obviously, g_s is a homeomorphism of $\mathbb{R}$ and $g_s(t+1) = g_s(t) + 1$. The quantity $\rho(g_s)$ is a non-decreasing function of s, so there is at least one value s_0 of s for which $\rho(g_{s(0)}) = \omega$, since

$$\rho(g_0) = \rho(f_0) \leq \omega \leq \rho(f_1) = \rho(g_1).$$

Set $g = g_{s(0)}$ and let $\bar{g} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the induced homeomorphism.

We will construct $\phi \in X_\omega$ in two different ways, according to whether ω is rational or irrational. First, suppose ω is rational, say $\omega = p/q$, p, $q \in \mathbb{Z}$, with p relatively prime to q. A theorem of Poincaré asserts that the set P of periodic points of $\bar{g}$ is non-empty. Let $\tilde{P} = \pi^{-1}P$, where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ denotes the projection. We define $\phi(0)$ to be the greatest non-positive element of $\tilde{P}$. Given $t \in \mathbb{R}$, we can write it in the form

$$t = n\left(\frac{p}{q}\right) + m + r$$

where n, $m \in \mathbb{Z}$, $-\frac{1}{q} < r \leq 0$. We define

$$\phi(t) = g^n(\phi(0)) + m.$$

Since $\phi(0) \in \tilde{P}$, $g^q(\phi(0)) = \phi(0) + p$. It follows that ϕ is well defined.

Using $\rho(g)=\frac{p}{q}$, we see that ϕ is weakly order preserving. For, suppose $n\left(\frac{p}{q}\right)+m>n'\left(\frac{p}{q}\right)+m'$, but $g^n(\phi(0))+m\le g^{n'}(\phi(0))+m'$. Then $g^{n-n'}(\phi(0))\le m'-m$. In the case $n-n'>0$, we get

$$\rho(g)\le\frac{m'-m}{n-n'}<\frac{p}{q}.$$

In the case $n-n'<0$, we get

$$\rho(g)\ge\frac{m'-m}{n-n'}>\frac{p}{q},$$

so either way, we have a contradiction.

By definition, $\phi(0)\le 0$, so $\phi(t)\le 0$, for $t\le 0$, since ϕ is weakly order preserving. For $t>0$, $\phi(t)\in\tilde{P}$ and $\phi(t)\neq\phi(0)$. Since ϕ is weakly order preserving and $\phi(0)$ is the greatest non-positive element of $\tilde{P}$, we get $\phi(t)>0$.

It follows immediately from the definition of ϕ that $g(\phi(t))=\phi\left(t+\frac{p}{q}\right)$. Hence, $f_0(\phi(t))\le\phi\left(t+\frac{p}{q}\right)\le f_1(\phi(t))$. We have defined ϕ so that it is continuous from the left. Thus, $\phi\in X_{p/q}$.

In the case ω is irrational, there is a weakly cyclic order preserving continuous mapping $h:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ such that $h\bar{g}(\theta)=h(\theta)+\omega$ (mod. 1.), for $\theta\in\mathbb{R}/\mathbb{Z}$, where $\bar{g}:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is the mapping induced by g. Let $\tilde{h}:\mathbb{R}\to\mathbb{R}$ be a lifting of h, i.e. a continuous mapping such that $\pi\tilde{h}=h\pi$. There is some choice of h and also of $\tilde{h}$: altogether, we may add any constant to $\tilde{h}$. We make $\tilde{h}$ unique by specifying $\tilde{h}(0)=0$. Since h is weakly cyclic order preserving, $\tilde{h}$ is weakly order preserving. Let $\phi(t)=\inf\tilde{h}^{-1}(t)$. Obviously, ϕ is weakly order preserving. Clearly, h has degree 1, so $\tilde{h}(t+1)=\tilde{h}(t)+1$, and $\phi(t+1)=\phi(t)+1$. By definition, $\phi(0)\le 0$, and 0 is the greatest number t such that $\phi(t)\le 0$. So, $\phi(t)\le 0$, for $t\le 0$ and $\phi(t)\ge 0$ for $t>0$. We have $\tilde{h}(g(t))=\tilde{h}(t)+\omega$, so the definition of ϕ gives $g(\phi(t))=\phi(t+\omega)$. The definition of g then gives

$$f_0(\phi(t))\le g(\phi(t))=\phi(t+\omega)\le f_1(\phi(t)).$$

The definition of ϕ implies that it is continuous from the left. Hence, $\phi\in X_\omega$. □

§3. TRANSLATION INVARIANCE OF F_ω

It is obvious from the definitions of g and g' that

$$\begin{aligned}g(x+1,x'+1)&=g(x,x')\\ g'(x+1,x'+1)&=g'(x,x')\end{aligned}\tag{3.1}$$

Hence, $h(x+1,x'+1)-h(x,x')$ is a constant C. We have

$$C=\int_\gamma g(x,x')\,dx-g(x,x')\,dx'$$

where γ is any path in B connecting any poing (x_0, x_0') in B with $(x_0+1, x_0'+1)$. But along a path of the form $\gamma(t) = (t, f_0(t))$, the 1-form under the integral sign vanishes identically. Hence $C = 0$, i.e.

$$h(x+1, x'+1) = h(x, x'). \tag{3.2}$$

From this formula, the definition of F_ω, and $\phi(t+1) = \phi(t)+1$, it follows that F_ω is translation invariant, i.e.

$$F_\omega(\phi T_a) = F_\omega(\phi). \tag{3.3}$$

§4. METRIC ON Y_ω

For any weakly order preserving mapping $\phi : \mathbb{R} \to \mathbb{R}$, we define

$$\text{graph } \phi = \{(x, y) \in \mathbb{R}^2 : \phi(x-) \le y \le \phi(x+)\}.$$

If $\psi : \mathbb{R} \to \mathbb{R}$ is a second weakly order preserving mapping, we set

$$\mathrm{d}(\phi, \psi) = \max\{\sup_\xi \inf_\eta |\xi - \eta|, \sup_\eta \inf_\xi |\xi - \eta|\},$$

where ξ ranges over graph ϕ, η ranges over graph ψ, and $| \ |$ denotes the Euclidean norm on $\mathbb{R}^2$. This may be infinite.

If $\phi \subset X_\omega$, then $(0,0)$, $(1,1) \in$ graph ϕ, and $\phi(t+1) = \phi(t)+1$. Consequently, for ϕ, $\psi \in X_\omega$, $\mathrm{d}(\phi, \psi)$ is given by (4.1), where now ξ ranges over $[0,1]^2 \cap$ graph ϕ and η ranges over $[0,1]^2 \cap$ graph ψ. We then obtain $\mathrm{d}(\phi, \psi) \le 1$ for ϕ, $\psi \in X_\omega$.

Obviously, $\mathrm{d}(\phi T_a, \phi) \le a$, for any $a \in \mathbb{R}$. Since for any $\phi \in Y_\omega$, there exists $a \in \mathbb{R}$ such that $\phi T_a \in X_\omega$, we obtain from the triangle inequality for d that $\mathrm{d}(\phi, \psi) < \infty$, for ϕ, $\psi \in Y_\omega$. It may be verified that d is a metric on Y_ω. One point to observe is that since every element of Y_ω is continuous from the left, $d(\phi, \psi) = 0 \Leftrightarrow \phi = \psi$.

§5. X_ω IS COMPACT, WITH RESPECT TO d.

Proof. Let S be the set of closed subsets of $[0,1]^2$ and let d' be the Hausdorff metric. (S, d') is compact ([2], 3.16, problem 3). The mapping $\phi \to$ graph $\phi \cap [0,1]^2$ embeds X_ω isometrically as a closed subset of S, so X_ω is compact.

§6. $F : Y_\omega \to \mathbb{R}$ IS CONTINUOUS

Proof. Let

$$M = \sup_{(x,x') \in B} \max\{1, |g(x,x')|, |g'(x,x')|\}.$$

From (3.1), it follows that $M < \infty$. From the definition of F_ω and the mean value theorem, it follows that

$$|F_\omega(\phi) - F_\omega(\psi)| \le M \int_{t=0}^{1} (|\phi(t) - \psi(t)| + |\phi(t+\omega) - \psi(t+\omega)|)\, dt,$$

since $\partial h/\partial x = g$ and $\partial h/\partial x' = -g'$.

Let $1 \geq \epsilon > 0$. Let $\delta = \epsilon^2/1000\, M^2$. Suppose $d(\phi, \psi) < \delta$. We will show $|F_\omega(\phi) - F_\omega(\psi) < \epsilon$.

From $d(\phi, \psi) < \delta < 10^{-3}$, the periodicity property $\phi(t+1) = \phi(t) + 1$ and $\psi(t+1) = \psi(t) + 1$, and the fact that ϕ and ψ are weakly increasing, it follows easily that $|\phi(t) - \psi(t)| < \frac{1001}{1000} < 2$ for all $t \in \mathbb{R}$.

Suppose $a \in \mathbb{R}$. Let π_a denote the set of all $t \in (a, a+1)$ such that $|\phi(t) - \psi(t)| \geq \frac{\epsilon}{5M}$. From the assumption that $d(\phi, \psi) < \delta$, we obtain

$$\phi(t+\delta) \geq \phi(t) + \frac{199\epsilon}{1000M} \tag{6.2}$$

in the case $\psi(t) \geq \phi(t) + \epsilon/5M$ and

$$\phi(t-\delta) \leq \phi(t) - \frac{199\epsilon}{1000\, M} \tag{6.3}$$

in the case $\psi(t) \leq \phi(t) - \epsilon/5M$.

Let π'_a (resp. π''_a) denote the set of $t \epsilon (a, a+1)$ where (6.2) (resp. 6.3) holds. Then

$$\pi_a \subset \pi'_a \cup \pi''_a.$$

At any point $t \in \pi'_a$ the variation of ϕ over the interval $[t, t+\delta]$ is $\geq \frac{199\epsilon}{1000M}$. Since the total variation of ϕ over $(a, a+1)$ is ≤ 1, it follows that π'_a can be covered by at most $\left[\frac{1000M}{199\epsilon}\right] + 1 \leq 7\frac{M}{\epsilon}$ intervals of length $\delta = \epsilon^2/1000M^2$. Hence the measure $\mu(\pi'_a)$ of π'_a is at most $7M\delta/\epsilon < \epsilon/100M$. Similarly, $\mu(\pi''_a) \leq \epsilon/100M$. Hence

$$\mu(\pi_a) \leq \mu(\pi'_a) + \mu(\pi''_a) \leq \epsilon/50\, M.$$

Since $|\phi(t) - \psi(t)| \leq 2$ for all $t \in \mathbb{R}$ and $|\phi(t) - \psi(t)| \leq \epsilon/5M$ for $t \epsilon (0,1)\text{-}\pi_0$ and for $t \epsilon (\omega, \omega+1) - \pi_\omega$, we obtain from (6.1) that

$$\begin{aligned} |F_\omega(\phi) - F_\omega(\psi)| &\leq M\left(2\mu(\pi_0) + 2\mu(\pi_\omega) + \frac{2\epsilon}{5M}\right) \\ &\leq M\left(\frac{4\epsilon}{50M} + \frac{2\epsilon}{5M}\right) < \epsilon. \end{aligned}$$

COROLLARY. *F_ω takes a maximum value on Y_ω at a point which lies in X_ω.*

Proof. Since F_ω is a continuous function on the compact space X_ω, it takes a maximum value on X_ω. Since F_ω is translation invariant and $Y_\omega = \bigcup_{a \in \mathbb{R}} T_a X_\omega$, the maximum value for F_ω on X_ω is also a maximum value for F_ω on Y_ω. □

§7. COMPUTATION OF THE VARIATION OF F_ω

LEMMA. *Suppose $a \leq 0 \leq b$ and $a < b$. Suppose an element ϕ_s of Y_ω is given for $a \leq s \leq b$, $\phi_s(t)$ is a C^2 function of s for each fixed t, and $\frac{\partial \phi_s(t)}{\partial s}$, $\frac{\partial^2 \phi_s(t)}{\partial s^2}$ are uniformly* *bounded and measurable for $a \leq s \leq b$, $t \in \mathbb{R}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\phi_s)|_{s=0} = \int_{t=0}^{1} V(\phi, t)\dot{\phi}(t), \tag{7.1}$$

where

$$\dot{\phi}_s(t) = \frac{\partial \phi_s(t)}{\partial s}, \dot{\phi} = \dot{\phi}_0, \phi = \phi_0.$$

Proof. From the definition of F_ω and the assumption that $\phi_s(t)$ is a C^2 function of s for each fixed t, we obtain

$$\begin{aligned} &\frac{F_\omega(\phi_{\Delta s}) - F_\omega(\phi)}{\Delta s} \\ &= \int_{t=0}^{1} \int_{u=0}^{1} \left[\frac{\partial h}{\partial x}(\phi_{u\Delta s}(t), \phi_{u\Delta s}(t+\omega))\dot{\phi}_{u\Delta s}(t)\right. \\ &\quad \left. + \frac{\partial h}{\partial x'}(\phi_{u\Delta s}(t), \phi_{u\Delta s}(t+\omega))\dot{\phi}_{u\Delta s}(t+\omega)\right] \mathrm{d}u\, \mathrm{d}t. \end{aligned}$$

Since $\dfrac{\partial h}{\partial x}$ and $\dfrac{\partial h}{\partial x'}$ are uniformly continuous on B, and $\dfrac{\partial \phi_s}{\partial s}$, $\dfrac{\partial^2 \phi_s}{\partial s^2}$ are uniformly bounded, it follows that the quantity under the integral sign converges uniformly, as $\Delta s \to 0$. Going to the limit $\Delta s = 0$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\phi_s)|_{s=0} &= \int_{t=0}^{1} \left[\frac{\partial h}{\partial x}(\phi(t), \phi(t+\omega))\dot{\phi}(t)\right. \\ &\quad \left. + \frac{\partial h}{\partial x'}(\phi(t), \phi(t+\omega))\dot{\phi}(t+\omega)\right] \mathrm{d}t \\ &= \int_0^1 \left[\frac{\partial h}{\partial x}(\phi(t), \phi(t+\omega)) + \frac{\partial h}{\partial x'}(\phi(t-\omega), \phi(t))\right]\dot{\phi}(t)\, \mathrm{d}t \\ &= \int_0^1 V(\phi, t)\, \dot{\phi}(t)\, \mathrm{d}t. \qquad \square \end{aligned}$$

§8. ONE PARAMETER FAMILIES

We fix $t_0 \in \mathbb{R}$, $\phi \in Y_\omega$ and we will construct three 1-parameter families ϕ_s, ψ_s, ξ_s in this section. The constructions depend on a choice of a C^∞ function on ρ on $\mathbb{R}/Z$ with values in $[0, 1]$. We will suppose ρ is identically 1 in a neighborhood of $\pi(t_0)$, where $\pi : \mathbb{R} \to \mathbb{R}/Z$ denotes the projection.

We let $u_s : \mathbb{R} \to \mathbb{R}$, be the unique family of diffeomorphisms, defined for $s \in \mathbb{R}$, and depending smoothly on $s \in \mathbb{R}$ and $t \in \mathbb{R}$ such that $u_0 = id$, $\dfrac{\partial u(t)}{\partial s} = \rho\pi u_s(t)$. Such a family exists and is unique by the fundamental existence and uniqueness theorem for ordinary differential equations.

We define $\phi_s = u_s\phi$. It is not necessarily the case that $\phi_s \in Y_\omega$ for $|s|$ sufficiently small. However, if for some $a \leq 0 \leq b$, and $a < b$, we have $\phi_s \in Y_\omega$ for $a \leq s \leq b$ then the other hypotheses of the Lemma in §7 are satisfied. Formula (7.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\phi_s)|_{s=0} = \int_{t=0}^{1} V(\phi, t)\rho\pi\phi(t)\,\mathrm{d}t. \tag{8.1}$$

We let $t_1 = \sup \phi^{-1}(\phi(t_0)+\frac{1}{2})$. We define

$$\begin{aligned}\psi_s(t) &= u_s\phi(t), \text{if } \exists n \in \mathbb{Z},\ t_0+n \le t < t_1+n\\ &= \phi(t), \text{ otherwise,}\end{aligned}$$

and

$$\begin{aligned}\xi_s(t) &= \phi(t), \text{if } \exists n \in \mathbb{Z},\ t_0+n \le t < t_1+n\\ &= u_s\phi(t), \text{otherwise.}\end{aligned}$$

Again, it is not necessarily the case that ψ_s and ξ_s are in Y_ω for $|s|$ small. But, if for some $a \le 0 \le b$ and $a < b$, we have ψ_s (resp. ξ_s) $\in Y_\omega$ for $a \le s \le b$, then the other hypotheses of the Lemma in §7 are satisfied. Formula (7.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\psi_s)|_{s=0} = \int_{t_0}^{t_1} V(\phi, t)\rho\pi\phi(t)\,\mathrm{d}t \tag{8.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\xi_s)|_{s=0} = \int_{t_1-1}^{t_0} V(\phi, t)\rho\pi\phi(t)\,\mathrm{d}t. \tag{8.3}$$

§9. CONDITIONS WHICH CANNOT BE SATISFIED AT A MAXIMUM

In this section, we will show that if any one of the conditions (9.1–9.4) is satisfied at $t = t_0$ and $t_0 - \omega$, t_0, and $t_0 + \omega$ are all points of continuity of ϕ, then F_ω does not take its maximum at ϕ.

$$\phi(t) = f_0\phi(t-\omega) \text{ and } \phi(t+\omega) > f_0\phi(t), \tag{9.1}$$

or

$$\phi(t) < f_1\phi(t-\omega) \text{ and } \phi(t+\omega) = f_1\phi(t), \tag{9.2}$$

or

$$\phi(t) > f_0\phi(t-\omega) \text{ and } \phi(t+\omega) = f_0\phi(t), \tag{9.3}$$

or

$$\phi(t) = f_1\phi(t-\omega) \text{ and } \phi(t+\omega) < f_1\phi(t). \tag{9.4}$$

LEMMA. *If* (9.1) *or* (9.2) *is satisfied then* $V(\phi, t) > 0$. *If* (9.3) *or* (9.4) *is satisfied, then* $V(\phi, t) < 0$.

Proof. From the definition of g and g', we see

$$x' = f_0(x) \Leftrightarrow g(x, x') = 0 \Leftrightarrow g'(x, x') = 0$$
$$x' = f_1(x) \Leftrightarrow g(x, x') = 1 \Leftrightarrow g'(x, x') = 1.$$

From (1.6), these equivalences, and the fact that $0 \le g(x, x') \le 1$, $0 \le g'(x, x') \le 1$, the conclusions of the Lemma follow immediately. □

Suppose $t_0 = \phi^{-1}\phi(t_0)$. Provided that ρ has support in a sufficiently small interval about $\pi\phi(t_0)$, we have that $\phi_s \in Y_\omega$ for $s \ge 0$ sufficiently small (resp. $s \le 0$ of sufficiently small absolute value) and, by (8.1) and the above Lemma, $\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\phi_s)|_{s=0} > 0$ (resp. <0), if (9.1) or (9.2) (resp. (9.3) or (9.4)) is satisfied for $t = t_0$. Hence F does not take its maximum at ϕ.

If $t_0 \neq \phi^{-1}\phi(t_0)$, then $\phi^{-1}\phi(t_0)$ is an interval. Let α, β be its endpoints, where $\alpha < \beta$. If (9.1) or (9.2) is satisfied for $t = t_0$, it is satisfied for all $t \in [t_0, \beta)$. Moreover, $V(\phi, t)$ is an increasing function of t in (α, β), by (1.6), the fact that $g(x, x')$ is an increasing function of x' and the fact that $g'(\bar{x}, x)$ is a decreasing function of $\bar{x}$. It is easily seen that if ρ has support in a sufficiently small neighborhood of $\pi\phi(t_0)$, then $\psi_s \in Y_\omega$ for $s \ge 0$ sufficiently small, and by (8.2) and the above Lemma, $\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\psi_s)|_{s=0} > 0$. Hence F_ω does not take its maximum at $\phi = \psi_0$.

If (9.3) or (9.4) is satisfied for $t = t_0$, similar reasoning shows that if ρ has support in a sufficiently small neighborhood of $\pi\phi(t_0)$, then $\xi_s \in Y_\omega$ for $s \le 0$ of sufficiently small absolute value, and $\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\xi_s)|_{s=0} < 0$. Hence F_ω does not take its maximum at $\phi = \xi_0$. □

§10. PROOF OF THE EULER-LAGRANGE EQUATION

In this section we will prove (1.5), under the assumption that F_ω takes its maximum at ϕ. It is enough to prove (1.5) when $t - \omega$, t, and $t + \omega$ are the points of continuity of ϕ, since this is the case for all but at most countably many $t \in \mathbb{R}$ and $V(\phi, t)$ is continuous from the left.

From §9, we know that none of the conditions (9.1–9.4) can be satisfied (when $t - \omega$, t, and $t + \omega$ are points of continuity of ϕ). This means that $\phi(t) = f_0\phi(t - \omega) \Leftrightarrow \phi(t + \omega) = f_0\phi(t)$ and $\phi(t) = f_1\phi(t - \omega) \Leftrightarrow \phi(t + \omega) = f_1\phi(t)$. If either of these conditions holds, $V(\phi, t) = 0$ by the reasoning used in the proof of the Lemma in §9.

Hence, it is enough to consider a point $t_0 \in \mathbb{R}$ such that $f_0\phi(t_0 - \omega) < \phi(t_0) < f_1\phi(t_0 - \omega)$ and $f_0\phi(t_0) < \phi(t_0 + \omega) < f_1\phi(t_0)$, and $t_0 - \omega$, t_0, and $t_0 + \omega$ are points of continuity of ϕ.

Suppose $t_0 = \phi^{-1}\phi(t_0)$. Then, if ρ has support in a sufficiently small neighborhood of $\pi\phi(t_0)$, we have $\phi_s \in Y_\omega$ for s sufficiently small, and (8.1) holds. The hypothesis that F takes its maximum at $\phi = \phi_0$ implies $\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\phi_s)|_{s=0} = 0$. Since $V(\phi, t)$ is continuous at $t = t_0$ (by the hypothesis that $t - \omega$, t, and $t + \omega$ are continuous at $t = t_0$), $\phi(t)$ is continuous at t_0, and $t_0 = \phi^{-1}\phi(t_0)$, the fact that

$$\int_{t=0}^{1} V(\phi, t)\rho\pi\phi(t)\,\mathrm{d}t = 0,$$

for all ρ of the type we consider, implies $V(\phi, t_0) = 0$.

If $t_0 \neq \phi^{-1}\phi(t_0)$, then $\phi^{-1}\phi(t_0)$ is an interval. Let α and β be its endpoints with $\alpha < \beta$. Then $V(\phi, t)$ is an increasing function of t in (α, β), by (1.6), the fact that $g(x, x')$ is an increasing function of x' and the fact that $g'(\bar{x}, x)$ is a decreasing function of $\bar{x}$. It is easily seen that if ρ has support in a sufficiently small neighborhood of $\pi\phi(t_0)$, then $\psi_s \in Y_\omega$ for $s \geq 0$ sufficiently small and $\xi_s \in Y_\omega$ for $s \leq 0$ sufficiently small. Hence (8.2) and (8.3) hold. The assumption that F_ω takes its maximum at $\phi = \psi_0 = \xi_0$ implies

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\psi_s)|_{s=0} \leq 0$$

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\omega(\xi_s)|_{s=0} \geq 0.$$

In view of the fact that $V(\phi, t)$ is increasing on (α, β), (8.2) gives $V(\phi, t_0) \leq 0$ and (8.3) gives $V(\phi, t_0) \geq 0$. Hence $V(\phi, t_0) = 0$. □

This completes the proof of the Theorem stated in the introduction.

§11. PROOF OF ADDENDUM 1

In view of (1.6), the fact that $g(x, x')$ is an increasing function of x', and the fact that $g(\bar{x}, x)$ is a decreasing function of $\bar{x}$, it follows that if ϕ is continuous at t, then $V(\phi, t+) \geq V(\phi, t-)$, and we have equality if and only if ϕ is continuous at both $t - \omega$ and $t + \omega$. Since (1) is equivalent to the Euler–Lagrange equation $V(\phi, t) = 0$, we have equality, and ϕ is continuous at $t - \omega$ and $t + \omega$. □

§12. PROOF OF ADDENDUM 2

We have already seen that if ϕ is constant in an interval (α, β), then $V(\phi, t)$ is increasing in that interval. Moreover, the argument which proved that (§9) also shows that $V(\phi, t)$ is constant if and only ϕ is constant on $(\alpha - \omega, \beta - \omega)$ and on $(\alpha + \omega, \beta + \omega)$. Since $V(\phi, t) = 0$ identically, we have that ϕ is constant on $(\alpha - \omega, \beta - \omega)$ and $(\alpha + \omega, \beta + \omega)$. By iterating this argument and using $\phi(t + 1) = \phi(t) + 1$, we get that ϕ is constant on the interval $(\alpha + n\omega + m, \beta + n\omega + m)$ for any $n, m \in \mathbb{Z}$. Since $\omega \notin \mathbb{Q}$, this implies ϕ is constant on $\mathbb{R}$, whic contradicts $\phi(t + 1) = \phi(t) + 1$.

Hence ϕ is not constant in any interval. □

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