Injectivity of the Laplace transform

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The goal of this short note is to give a simple proof of the injectivity of the Laplace transform. We begin with two lemmas.

Lemma 1 (Weierstrass Approximation Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then for any $\varepsilon > 0$ there exists a real polynomial p such that* $\max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$.

We will not prove this theorem here. There are many different ways to prove it. For a relatively self-contained proof, see e.g. `http://www.math.sc.edu/~schep/weierstrass.pdf`.

Lemma 2. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. If*

$$\int_a^b x^n f(x)\,dx = 0$$

for any integer $n \geq 0$, then $f(x) \equiv 0$.

Proof. From the above assumption, it follows that

$$\int_a^b p(x)f(x)\,dx = 0$$

for any polynomial p. For any $\varepsilon > 0$ we can find a polynomial p such that $\max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$ by Lemma 1. Then

$$\begin{aligned}\int_a^b f^2(x)\,dx &= \int_a^b (f(x)-p(x))f(x)\,dx + \underbrace{\int_a^b p(x)f(x)\,dx}_{=0}\\ &\leq \int_a^b |f(x)-p(x)||f(x)|\,dx\\ &\leq \varepsilon(b-a)\max_{x\in[a,b]}|f(x)|.\end{aligned}$$

Since ε is arbitrary, this means that

$$\int_a^b f^2(x)\,dx = 0.$$

Hence, $f(x) \equiv 0$ since f is continuous (otherwise we could find some interval (c,d), $a < c < d < b$, where $f^2(x) > \frac{1}{2}\max_{x\in[a,b]} f^2(x) > 0$, and the integral would be greater than $\frac{d-c}{2}\max_{x\in[a,b]} f^2(x) > 0$). $\square$

Theorem 1. *Assume that* $f, g\colon [0,\infty) \to \mathbb{R}$ *are continuous and of exponential order. If* $L[f] = L[g]$, *then* $f = g$.

Proof. Set $u = f - g$. Then $L[u] = 0$ and we need to show that $u = 0$. Note that u is of exponential order, $|u(x)| \le Me^{cx}$ for some $M, c \in \mathbb{R}$ and

$$\int_0^\infty e^{-px} u(x)\, dx = 0, \qquad p > c.$$

Fix $b > c$, take $p = b + n + 1$, $n = 0, 1, 2, \ldots$, and make the change of variable $y = e^{-x}$ ($x = -\ln y$, $dy = -e^{-x}\, dx$). Then

$$\begin{aligned} 0 &= \int_0^\infty e^{-px} u(x)\, dx \\ &= \int_0^\infty e^{-bx} e^{-nx} e^{-x} u(x)\, dx \\ &= \int_0^1 y^b y^n u(-\ln y)\, dy \\ &= \int_0^1 y^n v(y)\, dy, \end{aligned}$$

where $v(y) = y^b u(-\ln y)$. v is clearly continuous on $(0, 1]$ (note that $\ln 1 = 0$). Setting $v(0) = 0$ it follows that v is also continuous at 0, since

$$|v(y)| = |y^b u(-\ln y)| = e^{-bx}|u(x)| \le Me^{-(b-c)x} \to 0 \text{ as } y \to 0$$

(note that $x = -\ln y \to \infty$ as $y \to 0$). It now follows from Lemma 2 that $v(y) \equiv 0$ on $[0, 1]$. Hence, $u(x) \equiv 0$ on $[0, \infty)$. $\square$

Remark. By splitting into real and imaginary parts, it is not difficult to see that the result holds also for complex-valued functions.

Remark. The theorem can also be proved by completely different methods, e.g. complex analysis or Fourier integrals. Using these methods one can also find explicit inversion formulas.