WEIERSTRASS' PROOF OF THE WEIERSTRASS APPROXIMATION THEOREM

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At age 70 Weierstrass published the proof of his well-known Approximation Theorem. In this note we will present a self-contained version, which is essentially his proof. For a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ define for $h > 0$

$$S_h f(x) = \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) e^{-(\frac{u-x}{h})^2}\, du.$$

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded uniformly continuous function. Then $S_h f$ converges uniformly to f as $h \downarrow 0$.*

Proof. Let $\epsilon > 0$. then there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Assume $|f(x)| \le M$ on $\mathbb{R}$. Using that $\int_{-\infty}^{\infty} e^{-v^2}\, dv = \sqrt{\pi}$, one also verifies easily that

$$\frac{1}{h\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\frac{u-x}{h})^2}\, du = 1.$$

This implies that we can write

$$f(x) = \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-(\frac{u-x}{h})^2}\, du.$$

Now let $h_0 > 0$ such that $h_0 < \frac{\epsilon\delta\sqrt{\pi}}{2M}$, then

$$\begin{aligned}
|S_h f(x) - f(x)| &\le \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{\infty} |f(u) - f(x)| e^{-(\frac{u-x}{h})^2}\, du \\
&\le \frac{\epsilon}{2} + \frac{1}{h\sqrt{\pi}} \int_{|x-u|\ge\delta} |f(u) - f(x)| e^{-(\frac{u-x}{h})^2}\, du \\
&\le \frac{\epsilon}{2} + \frac{2M}{h\sqrt{\pi}} \int_{|x-u|\ge\delta} e^{-(\frac{u-x}{h})^2}\, du \\
&= \frac{\epsilon}{2} + \frac{2M}{\sqrt{\pi}} \int_{|v|\ge\frac{\delta}{h}} e^{-v^2}\, dv \le \frac{2Mh}{\delta\sqrt{\pi}} \int_{|v|\ge\frac{\delta}{h}} |v| e^{-v^2}\, dv \\
&\le \frac{\epsilon}{2} + \frac{4Mh}{\delta\sqrt{\pi}} \int_0^{\infty} v e^{-v^2}\, dv = \frac{\epsilon}{2} + \frac{2hM}{\delta\sqrt{\pi}} < \epsilon
\end{aligned}$$

for all $0 < h \le h_0$ and all $x \in \mathbb{R}$. □

Theorem 2 (Weierstrass Approximation Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is on $[a, b]$ a uniform limit of polynomials.*

Proof. We begin by extending f to a bounded uniformly continuous function on $\mathbb{R}$ by defining $f(x) = f(a)(x - a + 1)$ on $[a-1, a)$, $f(x) = -f(b)(x - b - 1)$ on $(b, b+1]$, and $f(x) = 0$ on $\mathbb{R} \setminus [a-1, b+1]$. In particular there exists $R > 0$ such that $f(x) = 0$

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for $|x| > R$. Let $\epsilon > 0$ and M such that $|f(x)| \leq M$ for all x. Then by the above theorem there exists $h_0 > 0$ such that for all $x \in \mathbb{R}$ we have $|S_{h_0} f(x) - f(x)| < \frac{\epsilon}{2}$. Since $f(u) = 0$ for $|u| > R$, we can write

$$S_{h_0} f(x) = \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) e^{-(\frac{u-x}{h_0})^2}\, du.$$

On $[\frac{-2R}{h_0}, \frac{2R}{h_0}]$ the power series of e^{-v^2} converges uniformly, so there exists N such that

$$\left| \frac{1}{h_0\sqrt{\pi}} e^{-(\frac{u-x}{h_0})^2} - \frac{1}{h_0\sqrt{\pi}} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0} \right)^{2k} \right| < \frac{\epsilon}{4RM}$$

for all $|x| \leq R$ and all $|u| \leq R$, since in that case $|u - x| \leq 2R$. This implies that

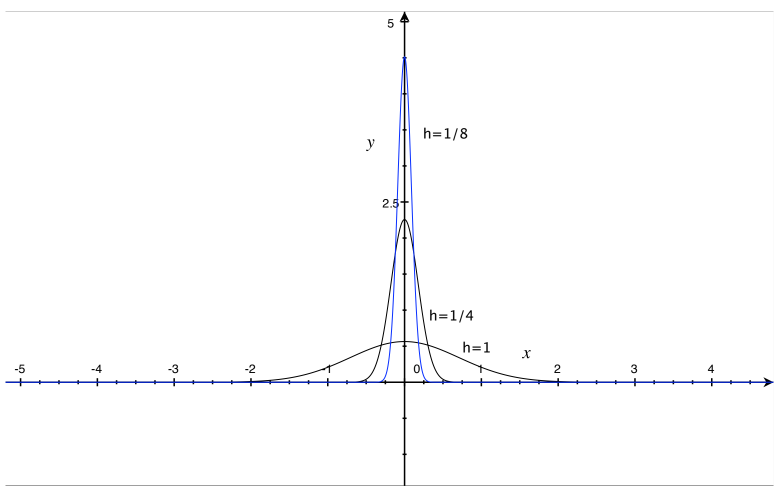
$$\left| S_{h_0} f(x) - \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0} \right)^{2k} du \right| < \frac{\epsilon}{2}$$

for all $|x| \leq R$. If we put

$$P(x) = \frac{1}{h_0\sqrt{\pi}} \int_{-R}^{R} f(u) \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{u-x}{h_0} \right)^{2k} du,$$

then $P(x)$ is a polynomial in x of degree at most $2N$ such that $|S_{h_0} f(x) - P(x)| < \frac{\epsilon}{2}$ for all $|x| \leq R$. This implies that $|f(x) - P(x)| < \epsilon$ for all $x \in [a, b]$. $\square$

Remark. The function $S_h f$ is the convolution of f with a Gaussian heat kernel. These heat kernels form an approximate identity. The following figure shows the kernel for the values $h = 1, h = \frac{1}{4}$ and $h = \frac{1}{8}$.



References

[1] K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen, *Verl. d. Kgl. Akad. d. Wiss. Berlin* **2**(1885) 633–639.