venerdì 11 dicembre 2020 15:30

CRITERIO DI INSTABILITA' (PER LINEARIZZAZIONE).

$\dot{x} = f(x)$ $\quad f \in C^1(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^n$

$f(0) = 0 \iff 0$ è un equilibrio

1. Sia $A = f'(0)$. Se $\forall \lambda \in \sigma(A)$, $\mathrm{Re}\,\lambda < 0$

$\Rightarrow$ 0 è asint. stabile

Infatti, se $|x_0| \le \delta \le \delta_0$ (δ_0 opportuno).

$$|\Phi(t, x)| \le e^{\alpha t} \delta_0 \quad \text{(esponenzialmente stabile)}$$

$\max_{\lambda \in \sigma(A)} \mathrm{Re}\,\lambda < \alpha < 0$.

$$\boxed{\begin{array}{l} 2.\ \exists\, \lambda \in \sigma(A) \text{ con } \mathrm{Re}\,\lambda > 0. \\ \text{allora } 0 \text{ è instabile.} \end{array}}$$

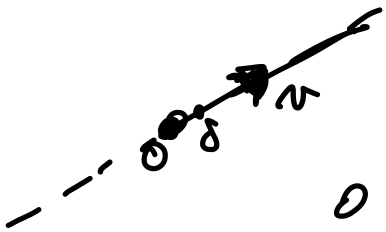
[Ricordiamo il caso lineare:

$\dot{x} = Ax$, $\quad \lambda \in \sigma(A)$ con $\mathrm{Re}\,\lambda > 0$.

$\exists\, v \in \mathbb{R}^n,\ |v|=1,\ Av = \lambda v$

$\underline{x(t) = \delta e^{\lambda t} v}$ è soluzione $\forall\, \delta > 0.$

$\dot{x} = \delta \lambda e^{\lambda t} v = \delta e^{\lambda t} A v = A(\delta e^{\lambda t} v)$
$= Ax$



$\underline{0 \text{ è instabile}}.$

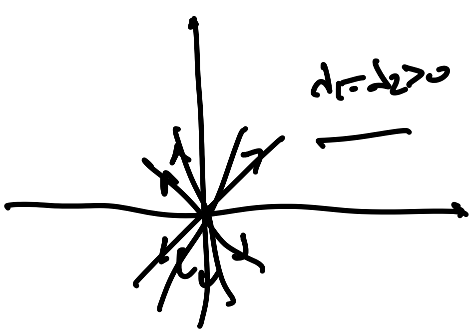
[ovvero $|x(t)| = \delta e^{(\mathrm{Re}\,\lambda)t} \xrightarrow[t\to+\infty]{} +\infty$]

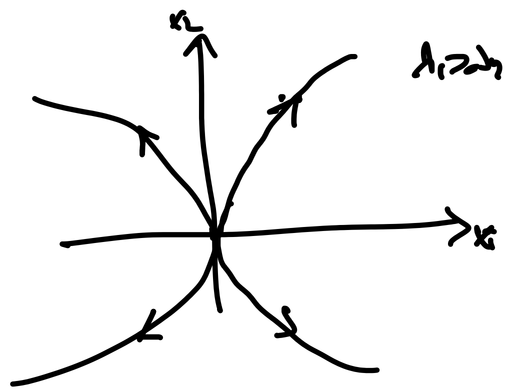
Riassunto della stabilità di un sistema lineare 2×2.

$A \in \mathrm{Mat}(2\times 2)$, $\{\lambda_1, \lambda_2\} = \sigma(A)$ ($\dot{x} = Ax$, stabilità di $x=0$)

1) $\underline{\lambda_i \text{ reali}}$

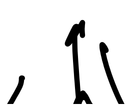
$\lambda_1 \cdot \lambda_2 > 0 \iff$ NODO $\begin{cases} \lambda_1, \lambda_2 < 0 & \text{stabile} \\ \lambda_1, \lambda_2 > 0 & \text{instabile} \end{cases}$

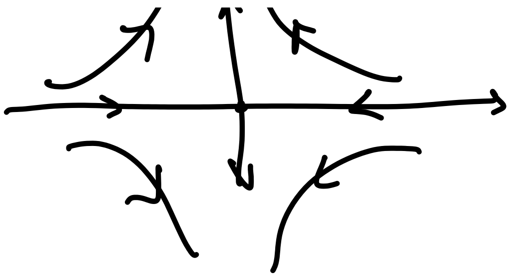




$\lambda_1 \cdot \lambda_2 < 0 \iff$ SELLA instabile

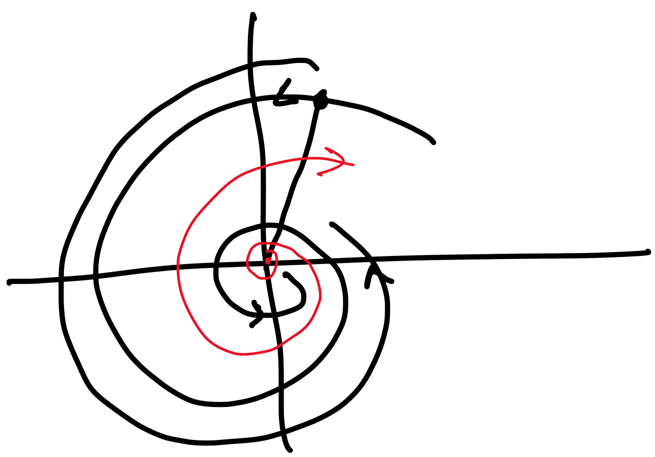
$\lambda_1 < 0 < \lambda_2$





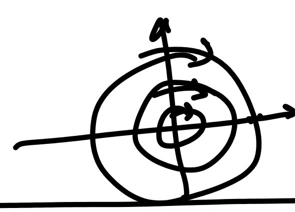
2) $\lambda_\pm = \alpha \pm i\beta$ (A reale)

$\alpha \neq 0 \iff$ FUOCO $\begin{cases} \alpha < 0 & \text{asint. stabile} \\ \alpha > 0 & \text{instabile.} \end{cases}$



$\underline{\alpha = 0} \iff$ CENTRO STABILE

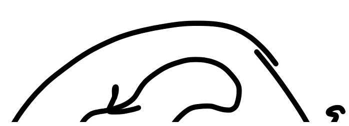
$e^{i\beta t}$

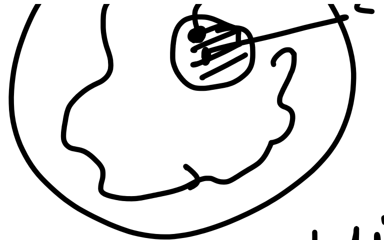


(1) x_0 è stabile per il sistema $\dot{x} = f(x)$, $f(x_0) = 0$. se $\forall\ \varepsilon > 0\ \exists\, \delta \mid$

$|\Phi(t, x) - x_0| \leq \varepsilon$

$\forall\ t \geq 0\ \ \forall\ x \mid |x - x_0| < \delta.$





(2) x_0 è asint. stabile se $\underline{x_0 \text{ è stabile}}$ e $\exists\, \delta > 0$

$$(*) \qquad \lim_{t \to +\infty} |\phi(t,x) - x| = 0 \quad \forall\, |x| < \delta.$$

Può succedere che valga (*) ma che x_0 non sia stabile.

Esempio

$f(0,0) = g(0,0) := 0$

$$\begin{cases} \dot{x} = x - y - x(x^2+y^2) + \dfrac{xy}{\sqrt{x^2+y^2}} =: f \\ \dot{y} = \underset{1}{x} + \underset{0}{y} - \underset{0}{y}(x^2+y^2) - \dfrac{x^2}{\sqrt{x^2+y^2}} =: g \end{cases}$$

$\underline{(x_0, y_0) = (1, 0)}$.

$f(1,0) = 0$
$g(1,0) = 0$.
$f(0,0) = 0 = g(0,0)$
PUNTI DI EQUILIBRIO

$F = (f, g)$

$$F \in C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{\overset{(0,0)}{0}\})$$

F **non** è C^1 vicino a $(0,0)$.

Osservazione $\quad z = (x,y), \quad |z| = \sqrt{x^2+y^2}$

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \le \frac{|z|^2}{|z|} = |z| \qquad \Big| \quad \dot{x} = x^{2/3}$$

$$\partial \frac{xy}{\ldots} = \frac{y}{\ldots} + xy\, \partial_x (x^2+y^2)^{-\frac{1}{2}}$$

$\partial_x \quad \overline{\sqrt{x^2+y^2}} \qquad \overline{\sqrt{x^2+y^2}}$

$$= \frac{y}{\sqrt{x^2+y^2}} + xy\,(-1)\,\frac{x}{(x^2+y^2)^{3/2}}$$

$$= \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 y}{(x^2+y^2)^{3/2}}$$ = funz. omog. di grado 0

non è cont. nell'origine.

$$x = t\bar{x} \qquad y = t\bar{y} \qquad t \to 0$$

$$h(t\bar{x}, t\bar{y}) = h(\bar{x}, \bar{y})$$

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introduciamo coordinate polari

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

N.B. è un cambio di variabili regolare su $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r > 0 \quad \text{per } r > 0.$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

$$c := \cos\theta \qquad s := \sin\theta$$

$$\dot{r}c - rs\dot{\theta} = f(rc, rs)$$

$$\dot{r}s + rc\dot{\theta} = g(rc, rs)$$

$$\begin{cases} \dot{r}c - rs\dot{\theta} = rc - rs - rcr^2 + \dfrac{r^2 cs}{r} \\ \end{cases}$$

s

$$= rc - rs - r^3 c + rcs$$

$$\dot{r}s + rc\dot{\theta} = rc + rs - r^3 s - rc^2$$

c

$$\begin{cases} \dot{r}c^2 - rsc\dot{\theta} = rc^2 - rcs - r^3 c^2 + rc^2 s \\ \dot{r}s^2 + rcs\dot{\theta} = rsc + rs^2 - r^3 s^2 - rsc^2 \end{cases}$$

sommando

$$\dot{r} = r - r^3 = r(1-r^2)$$

$$\boxed{\dot{r} = r(1-r^2)}$$

$$\dot{r}cs - rs^2\dot{\theta} = rsc - rs^2 - r^3 cs + rcs^2$$

$$\dot{r}cs + rc^2\dot{\theta} = rc^2 + rcs - r^3 cs - rc^3$$

sottraggo alla seconda relazione la prima:

$$r\dot{\theta} = r - rc(c^2+s^2) = r(1-c)$$

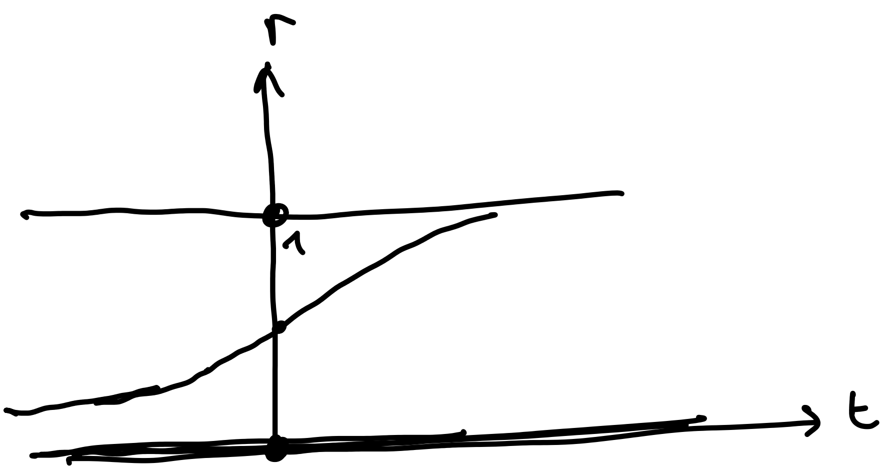
$$\boxed{\dot{\theta} = 1 - \cos\theta}$$

$$= 1 - \cos 2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}$$

$$= 2\sin^2\frac{\theta}{2}$$

$$\begin{cases} \dot\theta = 2\sin^2\frac{\theta}{2} \\ \dot r = r(1-r^2) \end{cases}, \quad r>0,$$

Oss: $(0,0)$ e $(1,0)$ in coord. cartesiane sono gli unici pti di equilibrio.



$r\geq 0$

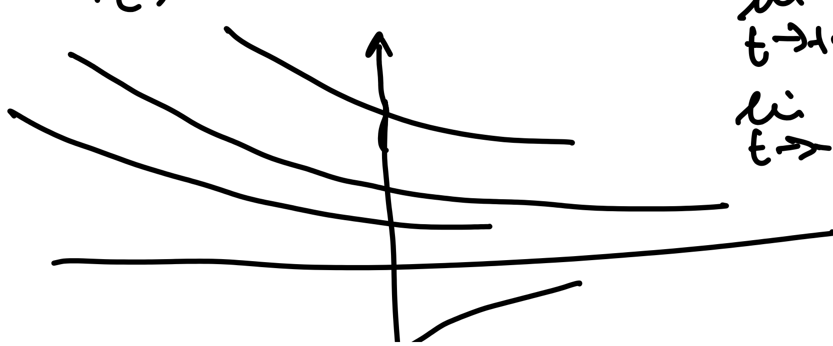
$0<r<1$, $\dot r<0$.

$$\lim_{t\to+\infty} r(t)=1, \quad \lim_{t\to-\infty} r(t)=0$$

$$\ddot r = \dot r(1-r^2) - 2r^2\dot r = \dot r(1-3r^2)$$

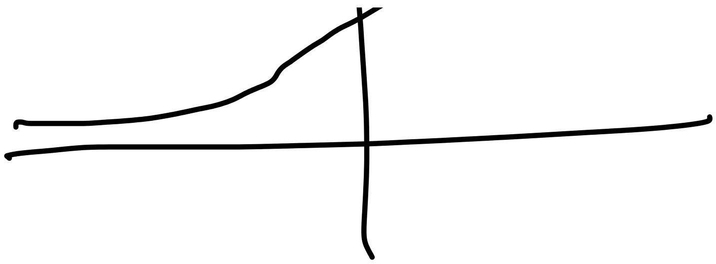
$3r^2=1 \quad r=\frac{1}{\sqrt3} \qquad \frac{1}{\sqrt3}>\frac12$

$r(0)>1$. $\quad \dot r<0$.



$$\lim_{t\to+\infty} r(t)=1$$

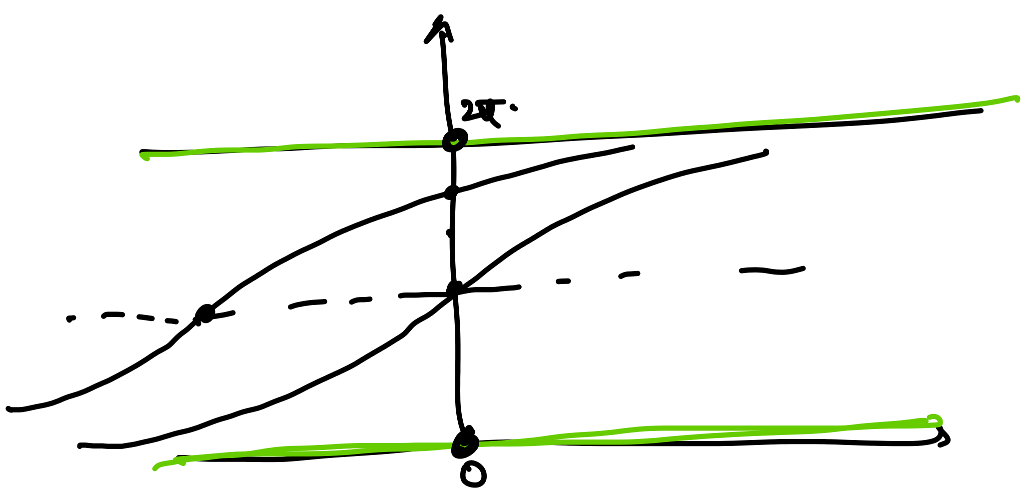
$$\lim_{t\to-\infty} r(t)=+\infty.$$



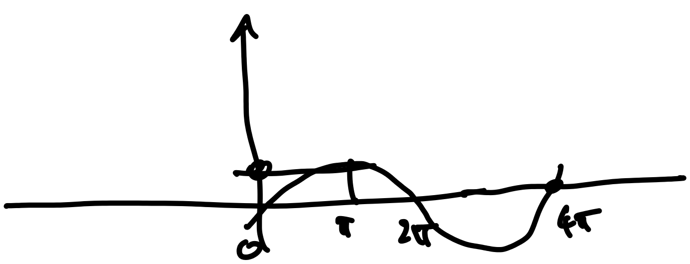
$\lim_{t \to +\infty} r(t) = 1 \quad \forall \ r(0) > 0.$

$\dot\theta = 2 \sin^2 \frac{\theta}{2}$, θ mod 2π.

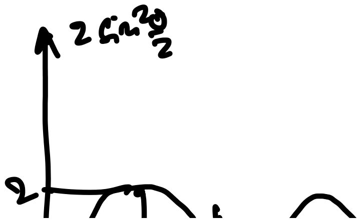
$= 1 - \cos\theta$

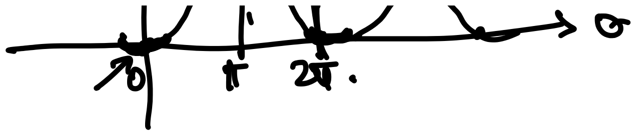


$\dot\theta = 0 \qquad \sin^2 \frac{\theta}{2} = 0$



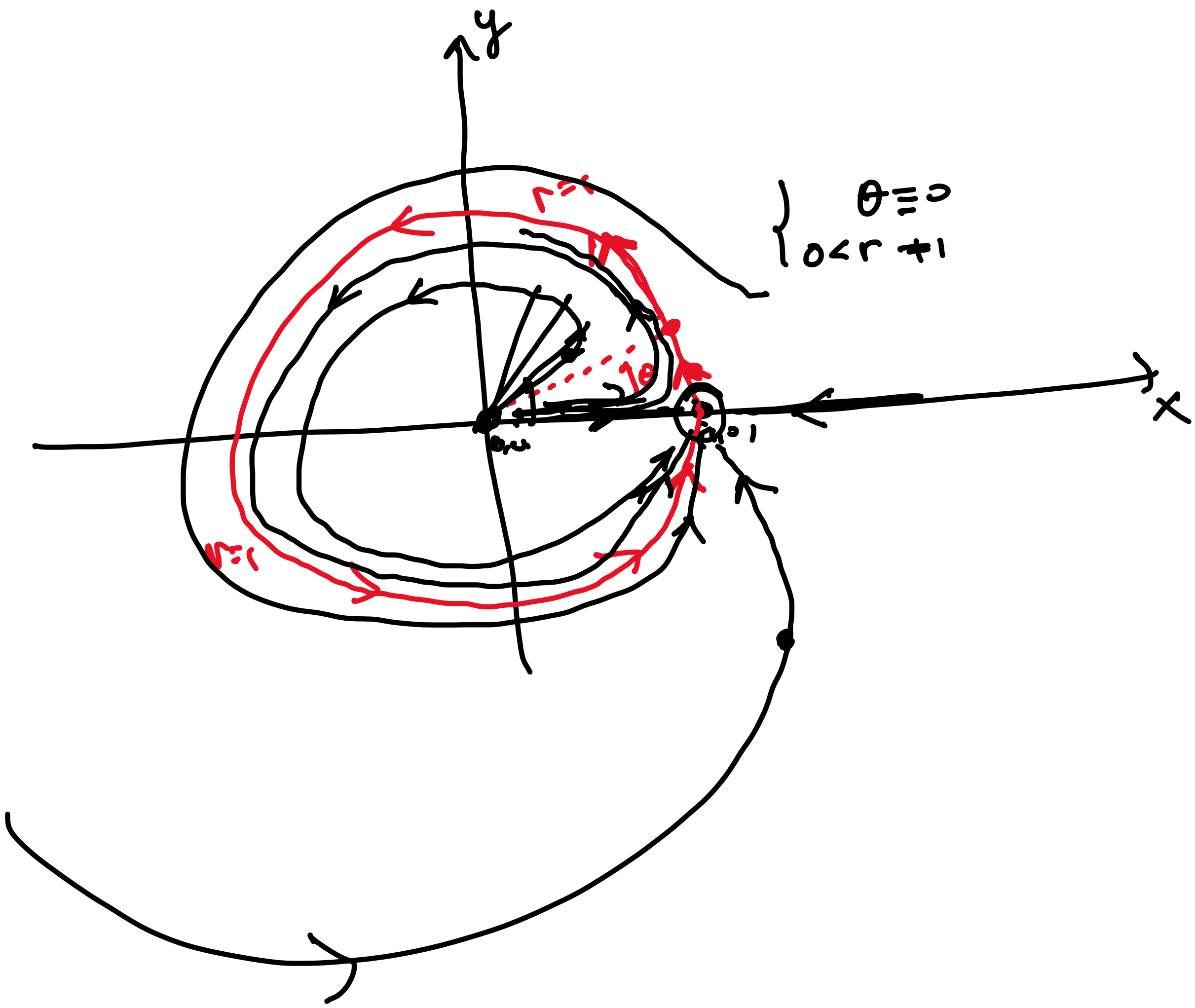
$\sin\theta \sim \theta$





$\dot\theta > 0$ per $0 < \theta(0) < 2\pi$.

$\ddot\theta = \sin\theta \ \dot\theta$



$$\lim \left| \Phi(t,(x_0,y_0)) - (1,0) \right| = 0 \quad \forall (x_0,y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

ma $(1,0)$ non è stabile.

Appelli scritti: 21/1/21 (14-17), 1/2/21 (9-12), 21/6/21 (9-12), 8/9/21 (14-17)

Acute MI