venerdì 23 ottobre 2020 16:17

Es. 17 [AMS]

$$(1) \begin{cases} \dot{x} = |x-t| \\ x(0) = a \end{cases}$$

Discutere intervallo di $\exists$ e limiti.

$$y := x - t$$

$$(2) \begin{cases} \dot{y} = \dot{x} - 1 = |x-t| - 1 = |y| - 1 \\ y(0) = a \end{cases}$$

Se y è soluzione di (2) $\Rightarrow$ $x(t) = y(t) + t$ è soluzione di (1).

OSS. $f(x,t) := |x-t|$ è Lip. in x

c'è $\exists !$ locale.

$$(2) \boxed{\begin{cases} \dot{y} = |y| - 1 \\ y(0) = a \end{cases}} \longleftarrow \begin{cases} \dot{y} = f(y) \\ y(0) = a \end{cases} \quad f(y) = |y| - 1$$

Se $y \geq 0$, $\dot{y} = y - 1$

Sol. generale di $\dot{y} = y - 1$

$\dot{y} - y = -1$ $\qquad \underbrace{\dot{y} e^{-t} - y e^{-t}}_{(ye^{-t})^{\cdot}} = -e^{-t}$

$$y e^{-t} - y_0 e^{-t_0} = -\int_{t_0}^{t} e^{-s}\, ds$$

$$= e^{-t} - e^{-t_0}$$

$$y = y_0 e^{(t-t_0)} + 1 - e^{t-t_0}$$

$$= (y_0-1)\, e^{t-t_0} + 1$$

La soluzione di (3) $\begin{cases} \dot y = y-1 \\ y(t_0)=y_0. \end{cases}$

Se $y \le 0$ (4) $\begin{cases} \dot y = -y-1 \\ y(t_0)=y_0 \end{cases}$

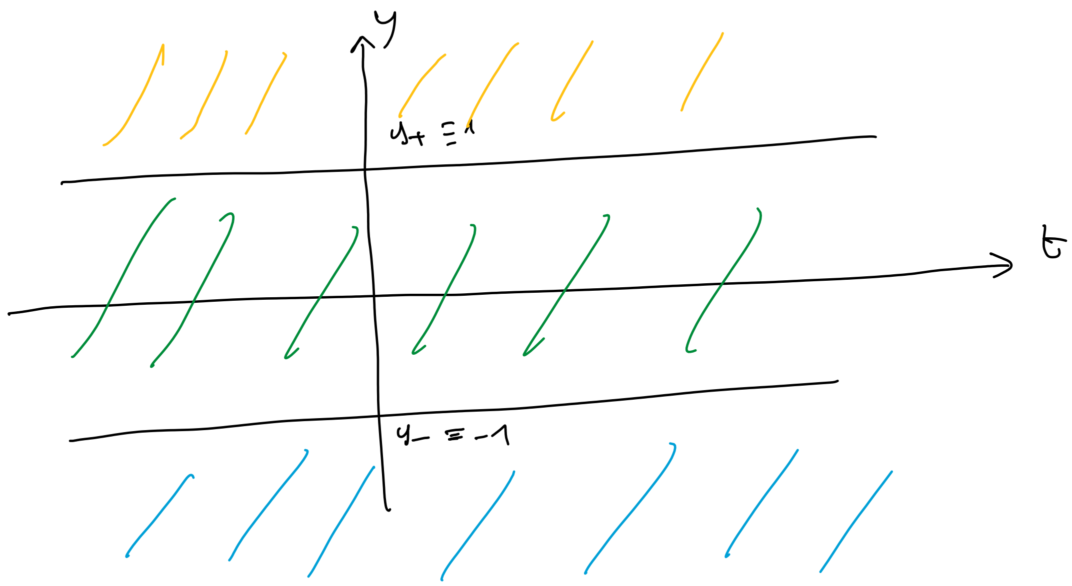
$\dot y + y = -1$ $\qquad (e^t y)' = -e^t$

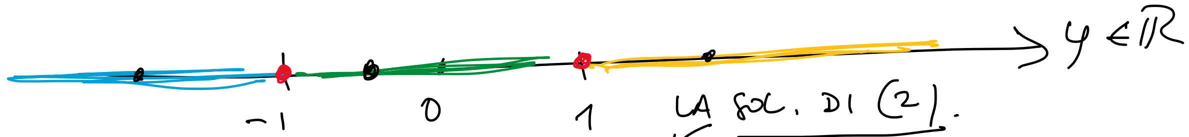
$$e^t y - e^{t_0} y_0 = -(e^t - e^{t_0})$$

$$\longrightarrow \quad y = e^{t_0-t} y_0 + e^{t_0-t} - 1 = e^{t_0-t}(y_0+1) - 1$$

Ricerca equilibri $\iff f(y)=0 = |y|-1$

$\iff y = \pm 1$. $\qquad y_\pm(t) \equiv \pm 1$



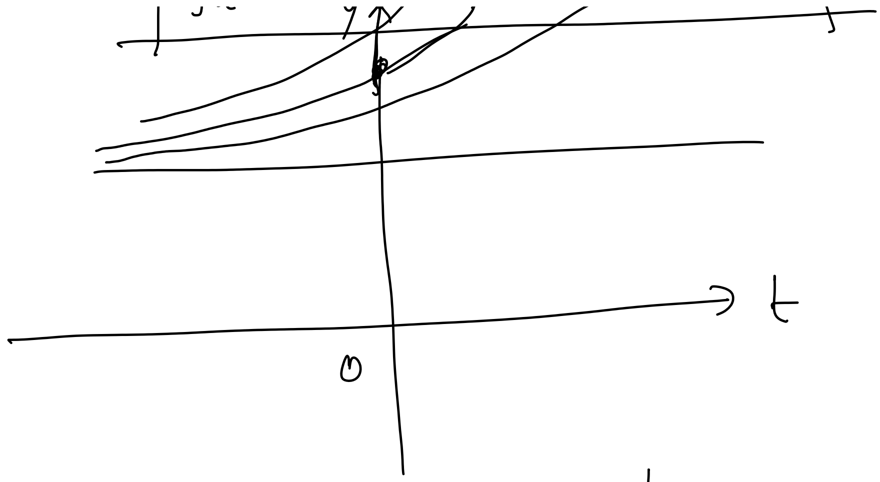


LA SOL. DI (2).

Quindi, se $a>1 \Rightarrow y_a(t) > 1, \forall t$

(per unicità) $\Rightarrow |y_a(t)| = y_a(t)$

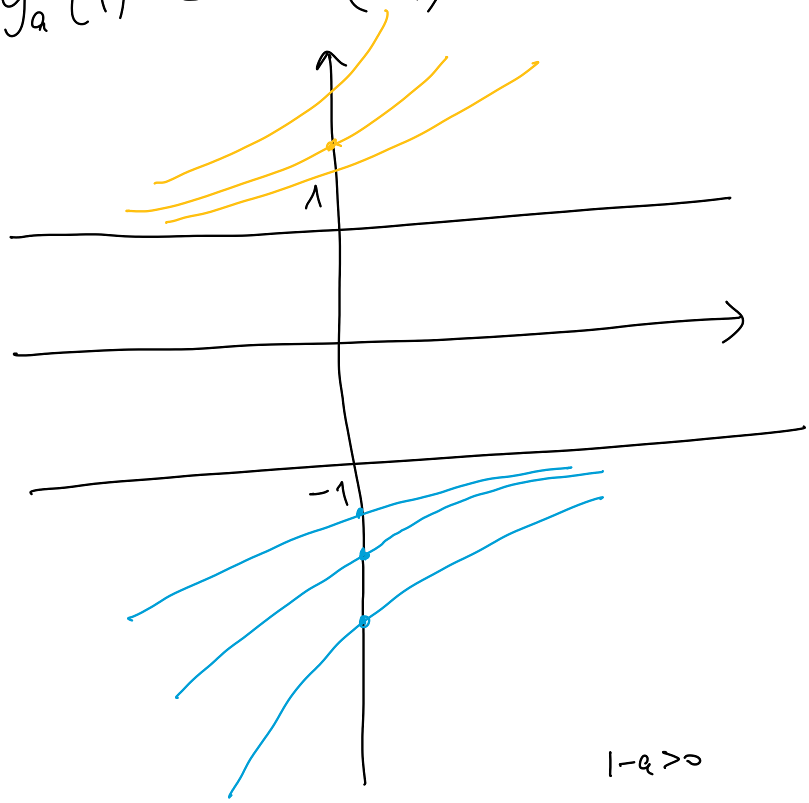
e quindi $\boxed{y_a(t) = (a-1)\, e^t + 1}$



$a > 1$, $y_a(t) = (a-1)e^t + 1$

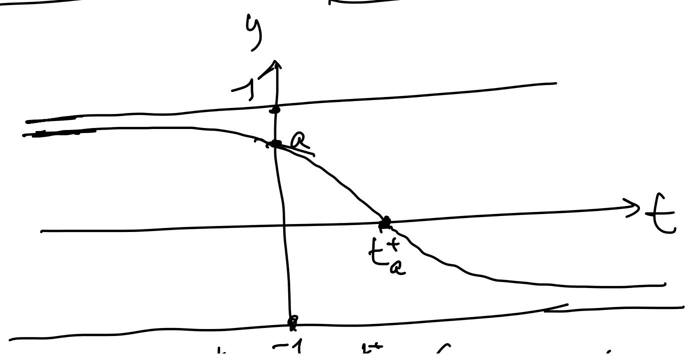
$a < -1$, $y_a(t) < -1$, $\forall t$

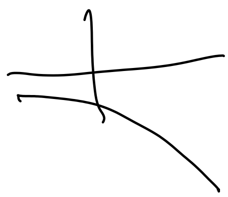
$y_a(t) = e^{-t}(a+1) - 1$



$1 - a > 0$

$0 < a < 1$ $\qquad y_a(t) = -e^t(1-a) + 1$





$t_a^+ = (1-a)^{-1} > 1$

$t_a^+ : \; e^{t_a}(1-a) = 1$

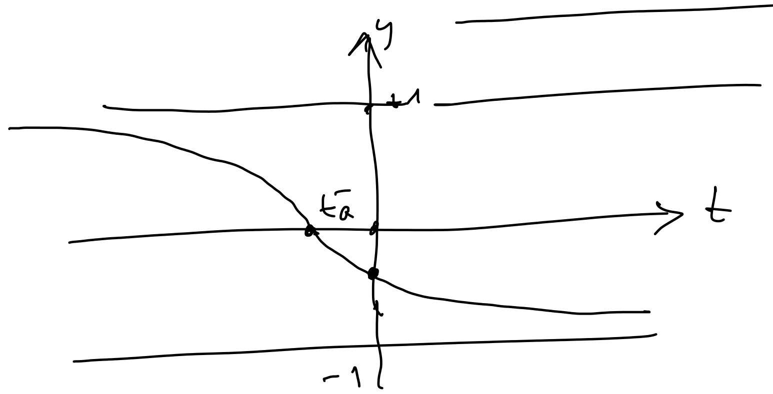
$t_a^+ = \log(1-a)^{-1} > 0.$

$t \geq t_a^+ \qquad y_a(t) = \underline{(1-a)^{-1} e^{-t} - 1}$

$0 < a < 1 \qquad y_a(t) = \begin{cases} 1 - e^t(1-a), & t < t_a^+ \\ (1-a)^{-1} e^{-t} - 1 & t \geq t_a^+ \end{cases}$

$(T_-, T_+) = \mathbb{R}.$

$-1 < a < 0 \qquad \underline{y_a(t) = e^{-t}(a+1) - 1}, \quad \underline{t \geq t_a^-}$



$t_a^-, \quad a+1 = e^{t_a^-}, \quad t_a^- = -\log(a+1)^{-1}$

$0 < a+1 < 1$

$t \leq t_a^- \qquad y_a = -e^{-t_a^-} e^t + 1 = 1 - (a+1)^{-1} e^t$

$y_a(t) = \begin{cases} e^{-t}(a+1) - 1, & t \geq t_a^- \\ 1 - (a+1)^{-1} e^t, & t < t_a^- \end{cases}$

$y_0(t) = \begin{cases} e^{-t} - 1, & t \geq 0 \\ 1 - e^t, & t < 0. \end{cases}$ controllo diretto

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[AA, § 3.5]

12 Se $f(x)$ è dispari e $x(t)$ è soluzione di ...

$\dot{x} = f(x) \Rightarrow -x(t)$ anche è soluzione.

$$(-x)^{\cdot} = -\dot{x} = -f(x) = f(-x).$$

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13. f pari e $C^1(\mathbb{R})$ e sia $x_0(t)$ sol. di

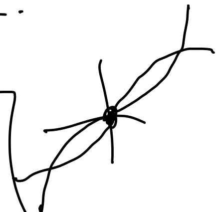
$$(*)\begin{cases} \dot{x} = f(x) \\ x(0) = 0 \end{cases}$$

Dimostra che x_0 è dispari.

→ Essendo $f \in C^1$, è Lip. $\Rightarrow$ unicità loc.

→ x_0 dispari $\overset{\text{def.}}{\iff}$ $x_0(-t) = -x_0(t)$

$\iff$ $\boxed{-x_0(-t) \overset{(\cdot\cdot)}{=} x_0(t)}$.



Ma $\boxed{y(t) := -x_0(-t)}$ è soluzione di $(*)$.

$\Rightarrow$ $y(t) \equiv x_0(t)$ ossia $-x_0(-t) = x_0(t)$

$\iff$ $x_0(t)$ dispari.

$$y(0) = -x_0(0) = 0.$$

$$\dot{y}(t) = \dot{x}_0(\underbrace{-t}_{s}) = f(x_0(\underbrace{-t}_{s}))$$

$$= f(-\underbrace{x_0(-t)})$$

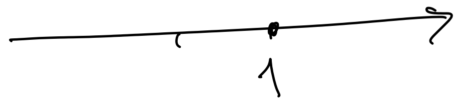
$$= f(y(t)),$$

$\Rightarrow$ $y(t)$ verifica $(*)$ $\Rightarrow$ $y(t) = x_0(t)$. ■

Es. 21 Studiare la convessità delle soluzioni

$$\dot{x} = x^3 - 1$$

Equilibri: $x^3 = 1 \iff x = 1$



$$\ddot{x} = 3x^2 \cdot \dot{x} = 3x^2(x^3-1)$$

$x(0) = x_0 > 1$ $\quad \ddot{x} > 0$, st. convessa.

Se $x_0 < 1$ $\quad \ddot{x} < 0$, st. concava

$$\dot{x} = x^2 \Leftarrow (T_-, T_+) \neq \mathbb{R}.$$

Es. 19.. $\begin{cases} \dot{x} = \log(1+x^2) \\ x(0) = 0 \end{cases}$

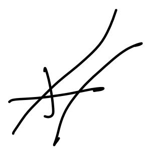
Dim che $\lim_{t\to+\infty} x(t) = +\infty$ $\quad \lim_{t\to-\infty} x(t) = -\infty$

N.B. $|\log(1+x^2)| \le c|x| \overset{\text{crit. max}}{\Longrightarrow} (T_-, T_+) = \mathbb{R}$.

$$\log(1+x^2) \le \begin{cases} \log 2 & |x| \le 1 \\ \log 2x^2 & |x| > 1 \end{cases}$$

$= \log 2 + 2\log|x|$

$< \log 2 + 2|x|$.



$$\log(1+x^2) \le \log 2 + 2|x|$$