lunedì 23 novembre 2020 16:12

ES. Trovare la soluzione generale del sistema (usando algebra lineare).

$$\begin{cases} \underline{\dot{x} = Ax} \end{cases}, \quad A := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \iff \begin{cases} \dot{x}_1 = x_1 + x_3 \\ \dot{x}_2 = x_1 + x_2 \\ \dot{x}_3 = x_3 \end{cases}$$

$$\sigma(A) = \{ \lambda \mid \det(A - \lambda I) = 0 \}.$$

$$\det \begin{pmatrix} \boxed{\begin{matrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{matrix}} & \begin{matrix} 1 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1-\lambda \end{pmatrix} \star$$

$$= (1-\lambda)^3. \star$$

$$\sigma(A) = \{1\}$$

$a(1) :=$ molt. alg. dell'autov. $1 = 3$

La forma canonica di Jordan può essere

$$\cdot \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \leftarrow \text{3 autov. indip.}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{pmatrix} \; I + N_2 \leftarrow \text{2 autov. indip.}$$

$$I + N_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \text{1 autov. indip.}$$

Ricerca autovettori:

$$Au = u$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$u_1 + u_3 = u_1 \Rightarrow u_3 = 0.$$

$$\ldots = u \ldots \Rightarrow u_1 = 0$$

$u_1 + u_2 = u_2 \Rightarrow \alpha_1 = 0$

$u_3 = u_3 \leftarrow$

$\Rightarrow \exists\,!$ autovettore (a meno di moltiplic. per $\alpha \in \mathbb{C}\setminus\{0\}$)

$u = (0,1,0)$, $\quad \underline{g(1) = 1}$

Verifica che $(A-I)^3 = 0$

$d(1)$ = indice di nilpotenza di $\lambda = 1$ $= 3$

$$B := (A-I) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$$

$$B^2 = (A-I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^3 = (A-I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

$P_{min}(z) = (z-1)^3$, $\quad z \in \mathbb{C}$,

$z \to A$, $\quad c \to cI$

$P_{min}(A) = (A-I)^3 = 0$.

$x_1 := u e^t = (0, e^t, 0)$ $\quad u := \boxed{u^{(1)} = (0,1,0)}$ $\quad \lambda = 1$

SCHEMA Dato $u^{(1)}$ autov. [illegible]

$\checkmark\ (A-\lambda I)\,u^{(1)} = 0 \longrightarrow x_1 = u^{(1)} e^t \checkmark$

$(A-\lambda I)\,u^{(2)} = u^{(1)} \longrightarrow x_2 = (u^{(2)} + t\,u^{(1)})\,e^t \checkmark$

$(A-\lambda I)\,u^{(3)} = u^{(2)} \longrightarrow x_3 = \left(u^{(3)} + t\,u^{(2)} + \frac{t^2}{2!}u^{(1)}\right)e^t$

$Au^{(1)} = u^{(1)}$

$Au^{(2)} = u^{(2)} + u^{(1)}$

$Au^{(3)} = u^{(3)} + u^{(2)}$

$3 = d(1)$ $\quad x_1, x_2, x_3$ sono soluzioni di $\dot{x} = Ax$

e sono indipendenti.

$$(A-I) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x_3 = 0, \quad x_1 = 1, \quad 0 = 0 \implies \boxed{u^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3 = 1, \quad x_1 = 0, \quad 0 = 0 \qquad \boxed{u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$x^{(1)}(t) = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \qquad x^{(2)}(t) = \left(\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{u^{(2)}} + t \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{u^{(1)}} \right) e^t = \begin{pmatrix} e^t \\ te^t \\ 0 \end{pmatrix}$$

$$x^{(3)}(t) = \left(\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{u^{(3)}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{u^{(2)}} + \frac{t^2}{2} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{u^{(1)}} \right) e^t = \begin{pmatrix} te^t \\ \frac{t^2}{2} e^t \\ e^t \end{pmatrix}$$

Controllo:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_1 + x_3 \\ \dot{x}_2 &= x_1 + x_2 \\ \dot{x}_3 &= x_3 \end{aligned}} \quad \checkmark \checkmark \checkmark \quad x^{(1)} \text{ ok.}$$

$$x^{(2)} = (e^t, te^t, 0) \quad \text{ok.}$$

$$e^t = e^t \quad \checkmark$$

$$e^t(1+t) = e^t + te^t \quad \checkmark$$

$$(1+t)e^t = te^t + e^t \quad \checkmark$$

$$te^t + \frac{t^2}{2}e^t = e^t + \frac{t}{2}e^t \quad \checkmark \qquad \text{ok.}$$

Rsp. $\boxed{c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{3}}$

Risoluzione diretta

$$\dot{x}_3 = x_3 \Rightarrow x_3 = e^t$$

$$\longrightarrow \boxed{\dot{x}_1 - x_1 = e^t} \quad \dots$$

$$A = I + B, \quad B := A - I = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B^2 \neq 0 \quad B^3 = 0$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Una soluzione fondamentale è data da:

$$e^{At}$$

$$\left([C,D] = 0 \Rightarrow \exp(C+D) = \exp(C) \cdot \exp(D) \right)$$

$$e^{At} = e^{tI} \cdot e^{Bt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \cdot e^{Bt}$$

$$e^{Bt} = I + Bt + \frac{(Bt)^2}{2} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & t \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

N.B. Vale $\forall$ matrice nilpotente B di ordine $d \geq 2$

$$\exp B = \sum_{k=0}^{d} \frac{B^k}{k!}$$

$$\boxed{e^{At} = \begin{pmatrix} e^t & 0 & te^t \\ te^t & e^t & \frac{t^2}{2} e^t \\ 0 & 0 & e^t \end{pmatrix}}$$

$$[u^{(1)}, u^{(2)}, u^{(3)}]\, c = \sum c_i u^{(i)}$$

$$e^{At} c = c_1 \underbrace{\begin{pmatrix} e^t \\ te^t \\ 0 \end{pmatrix}}_{x^{(2)}} + c_2 \underbrace{\begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}}_{x^{(1)}} + c_3 \underbrace{\begin{pmatrix} te^t \\ t^2e^t \\ e^t \end{pmatrix}}_{x^{(3)}}$$

$$U^{-1}AU \stackrel{?}{=} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = [u^{(1)}, u^{(2)}, u^{(3)}] = [e^{(2)}, e^{(1)}, e^{(3)}]$$

$$Au^{(1)} = u^{(1)} = e^{(2)}$$

$$Au^{(2)} = u^{(1)} + u^{(2)} = e^{(1)} + e^{(2)}$$

$$Au^{(3)} = u^{(2)} + u^{(3)} = e^{(1)} + e^{(3)}$$

$$AU = [e^{(2)}, e^{(1)} + e^{(2)}, e^{(1)} + e^{(3)}] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

In generale

$$U^{-1} A U = J \qquad \exp(At) = U \boxed{e^{Jt}} U^{-1}$$

$$A = UJU^{-1}$$

$$J = B(J_1, \dots, J_k) \qquad e^{Jt} = B(e^{J_1 t}, \dots, e^{\cdot})$$

$$\downarrow$$

$$\begin{pmatrix} \boxed{J_1}_{n_1} & & & & 0 \\ & \boxed{J_2}_{n_2} & & & \\ & & \ddots & & \\ & & & \boxed{J_k} & \\ 0 & & & & \boxed{J_n} \end{pmatrix}$$

$$J = \underline{\underline{\lambda I}} + \underline{N_k} \quad , \quad k \geq 2$$

$$e^{Jt} = e^{\lambda t} \cdot \underline{e^{N_k t}}$$

$$N := N_k = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{k} \Big\} k.$$

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & \ddots & \ddots & 1 \\ 0 & & \ddots & 0 \\ & & & 0 \end{pmatrix} , \dots , \quad N^k = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & \ddots & 0 \\ & 0 & & 0 \end{pmatrix}$$

$$\underline{\underline{\exp(N_k t)}} = I + Nt + N^2 \frac{t}{2} + \dots + N^k \frac{t^k}{k!}$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^k}{k!} \\ 0 & 1 & t & \ddots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 0 & 1 & t & \vdots \\ & & \ddots & \ddots & t \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^{\Delta}$$

ES 20. §8.5 Trovare una soluzione fondamentale di

$$\dot{x} = \underbrace{\begin{pmatrix} 3 & 4 & 1 \\ 1 & 3 & -2 \\ 0 & 0 & 3 \end{pmatrix}}\, x$$

$\dot{x}_3 = 3x_3$, $\underline{x_3 = e^{3t}}$ $\Rightarrow$ $(0, 0, e^{3t})$ è soluzione del sistema

$$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 4 & 1 \\ 1 & 3-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= \underbrace{(3-\lambda)} \cdot \left((3-\lambda)^2 - 4\right) \quad , \quad \lambda_1 = 3.$$

$$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 3 \pm \sqrt{9-5} = 3 \pm 2 = 5, 1.$$