mercoledì 28 ottobre 2020 15:57

Es studio $I = (T_-, T_+)$ $\begin{cases} \dot{x} = x^3 - 1 = f(x) \\ x(0) = x_0. \end{cases}$

$T_- = -\infty.$

1). punti di equilibrio $f(x_0, t) = 0, \forall t$

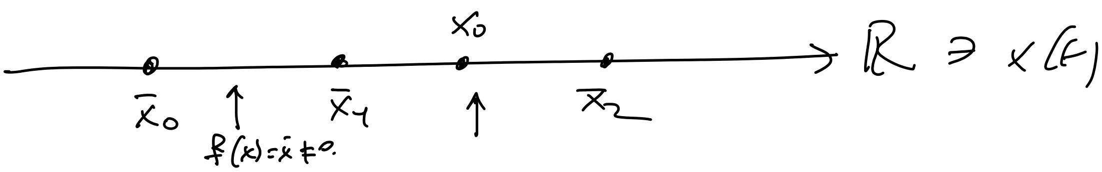
$f(x,t) = x^3 - 1 = 0 \iff x = x_0 = 1$

$f(x,t) = f_0(x) \cdot g(t)$

$f_0(x_i) = 0.$

$x(t) \in \mathbb{R}$

In generale.



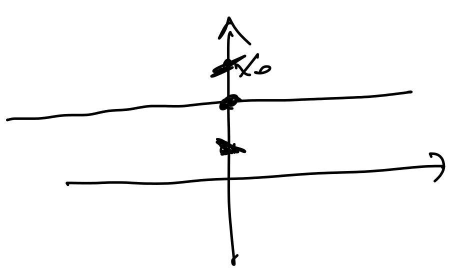
$f(x_i, t) = 0, \forall t.$ $x_i(t) \equiv \bar{x}_i$ sono soluzioni

negli intervalli in cui $f \neq 0$, x è monotona.

$\exists \lim_{x \to T_-} \phi^t(x_0) = x_- \lim_{t \to T_+} \phi^t(x_0)$

dove $T_\pm(x_0)$, $\phi^0(x_0) = x_0$,

2). $\forall x_0, T_-(x_0) = -\infty$ e $\lim_{t \to -\infty} \phi^t(x_0) = 1$.



$f = x^3 - 1$, $x_0 > 1$. $f > 0$.

$1 < \phi^t(x_0), \forall t$

$\Rightarrow x_- = \inf_{t \to T_-} x(t) \geq 1$

Se fosse $x_- > 1$ $\longrightarrow$ contradd. (per il teorema dell'asintoto).

$x_0 < 1$ $\quad f(x_0) = x_0^3 - 1 < 0$

$(-\infty, 1)$ è invariante per l'equazione

$x_0 < 1 \Rightarrow \phi^t(x_0) < 1$, $\forall\, t \in (T_-, T_+)$

$x(t) = \phi^t(x_0)$ è decrescente

$\lim_{t \to T_-} x(t) = \sup_{(T_-, T_+)} x(t) = x_-$

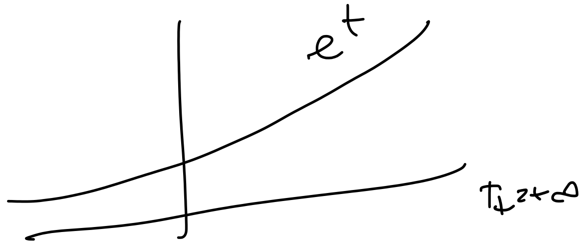
e si ragiona come prima $\Rightarrow x_- = 1$.

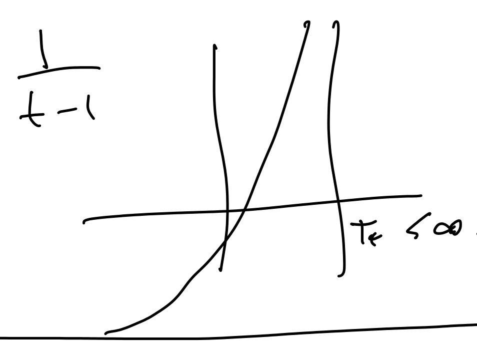
Discutiamo T_+, $T_+ = +\infty$ o no?

$x_0 > 1$, $\sup_{(-\infty, T_+)} \phi^t(x_0) = +\infty$ (non può avere limite finito $x_+ < \infty$ altrimenti per il teorema dell'asintoto $\dot x = f(x)$

$\lim_{t \to T_+} \dot x(t) = f(x_+) = 0$.

$x_+ > x_0$, $f(x_+) = 0$.)





$$\boxed{\begin{cases} \dot x = x^\alpha \\ x(0) = x_0 > 0. \end{cases}}, \quad \alpha > 0.$$

$\sqrt{|x|}$ è Hölder. con esp. $\frac{1}{2}$ in $x = 0$.

$$\int_{x_0}^{x} \frac{d\xi}{\xi^\alpha} = \int_0^t ds = t$$

ma è C^∞ nell'intorno di $x_0 \neq 0$.

MORALE: $\alpha \geq 1$, $T_+ = +\infty$

$0 < \alpha < 1$, $T_+ < \infty$

asint. verticale.

$$\left[\frac{\xi^{-\alpha+1}}{-\alpha+1}\right]_{x_0}^{x} \quad \text{se } \alpha \neq 1$$

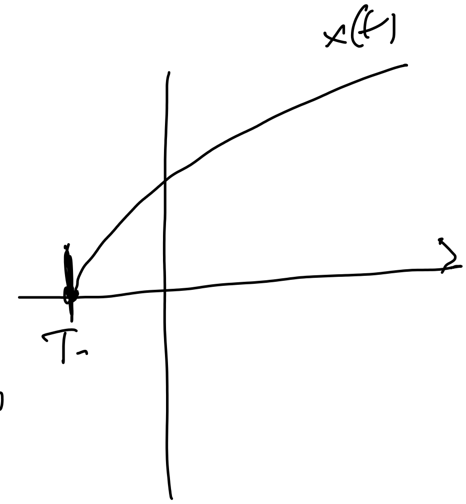
$$\left[\log \xi\right]_{x_0}^{x} \quad \text{se } \alpha = 1$$

Caso $0 < \alpha < 1$

$$\frac{1}{1-\alpha}\left(x^{1-\alpha} - x_0^{1-\alpha}\right) = t$$

$$x^{1-\alpha} = x_0^{1-\alpha} + (1-\alpha)t$$

$$x(t) = \left(x_0^{1-\alpha} + (1-\alpha)t\right)^{\frac{1}{1-\alpha}}$$



$T_+ = +\infty$ e $\lim_{t \to T_+} x(t) = +\infty$

$$T_-(1-\alpha) + x_0^{1-\alpha} = 0.$$

$$T_- = -\left(\frac{x_0^{1-\alpha}}{1-\alpha}\right)$$

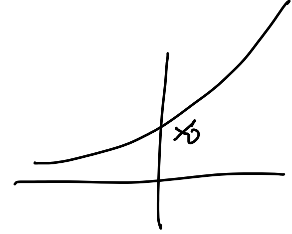
$$\lim_{t \to T_-} x(t) = 0$$

Caso $\alpha = 1$,

$\log x - \log x_0 = t$, $\log x = \log x_0 + t$

$$x = x_0 e^t$$

$$\begin{cases} \dot{x} = x \\ x(0) = x_0. \end{cases}$$



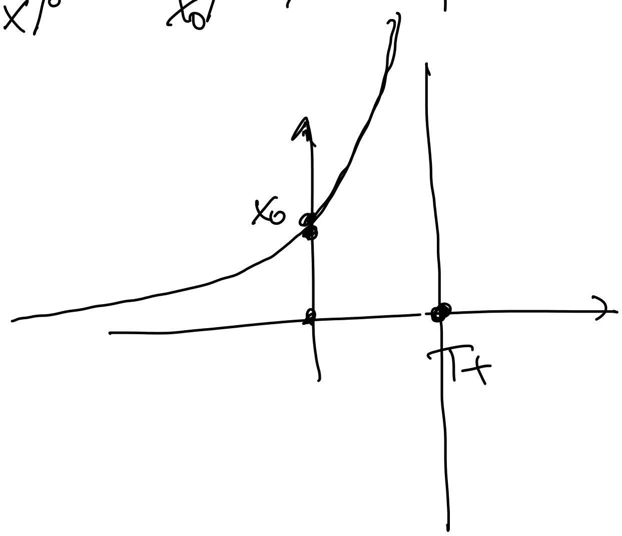
Caso $\alpha > 1$.

$$\frac{x^{-\alpha+1}}{-\alpha+1} - \frac{x_0^{-\alpha+1}}{-\alpha+1} = t \quad , \quad -\alpha+1 = -(\alpha-1) =: -\beta \quad \beta > 0$$

$$\beta = \alpha - 1$$

$$\frac{1}{\beta}\left(\frac{1}{x_0^\beta} - \frac{1}{x^\beta}\right) = t$$

$$\frac{1}{x^\beta} = \frac{1}{x_0^\beta} - \beta t \quad , \quad x = \left(\frac{1}{x_0^\beta} - \beta t\right)^{-\frac{1}{\beta}}$$



$$T_+ = \frac{1}{\beta x_0^\beta} < \infty$$

$$T_- = -\infty$$

$$\lim_{t \to -\infty} x(t) = 0.$$

$$\lim_{t \to T_+} x = +\infty.$$

Ricordiamo il teorema del confronto (Corollario 2 di Gronwall)

Siano $x(t)$ e $y(t)$ funzioni C^1 t.c.

$$\dot{x} \leq f(x,t)$$

$$\dot{y} \geq g(y,t)$$

$$f(x(t),t) \leq g(x(t),t).$$

Finché $(x(t),t) \in$ dom f, dom g, allora $(y(t),t) \in$ dom g.

$u(t_0) \leq v(t_0) \Rightarrow u(t) \leq v(t) \ \forall t \geq t_0$

$u(t_0) \geq v(t_0) \Rightarrow u(t) \geq v(t) \ \forall t \leq t_0$.

Corollario. Siano $x(t)$ e $y(t)$ funzioni C^1 t.c.

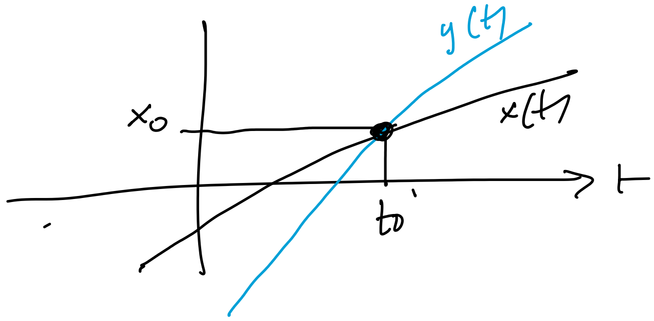
$$\dot x = f(x,t), \quad x(t_0)=x_0$$
$$\dot y = g(y,t), \quad y(t_0)=y_0$$

$\longrightarrow$ $f(x,t) \le g(x,t)$, $\forall\ x,t \in \text{dom } f = \text{dom } g$.

(i) $x_0 \le y_0 \Rightarrow x(t) \le y(t)$, $\forall\ t \ge t_0$

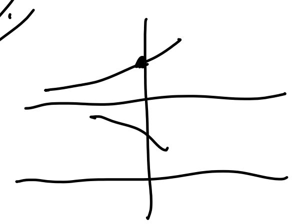
(ii) $x_0 \ge y_0 \Rightarrow x(t) \ge y(t)$ $\forall\ t \le t_0$

$\rightarrow$ (iii) $x_0 = y_0 \Rightarrow \begin{cases} x(t) \ge y(t) & \forall\ t \le t_0 \\ x(t) \le y(t) & \forall\ t \ge t_0 \end{cases}$



$$\begin{cases} \dot x = x^3 - 1 \\ x(0) = x_0 \ne 1 \end{cases}$$

Tt.



$x^3 - 1 \ge h(x) = x^\alpha$, $\alpha > 1$, p.e. x^2

$$\underline{x^3 - 1 \ge x^2}$$
$$\| $$
$$\underbrace{(x-1)}_{\ge 1}\ \underbrace{(x^2+x+1)}_{\ge 3}$$

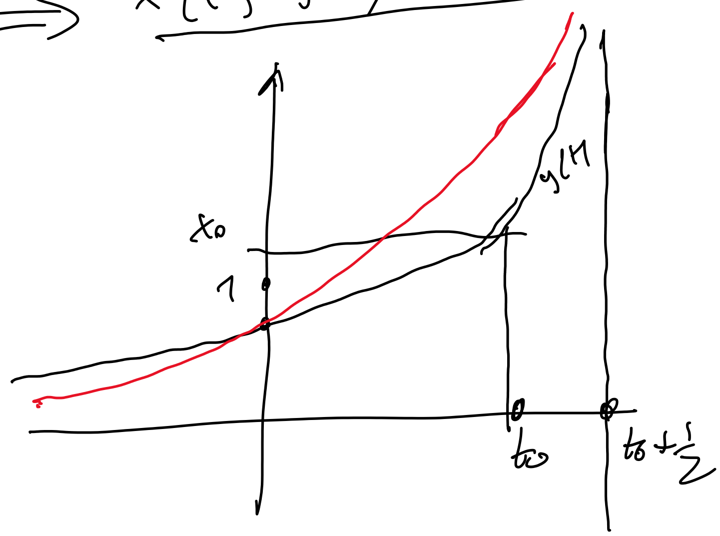
se $x \ge 2$ $x^3 - 1 > x^2$

$x_0 > 1$ $\exists\ t_0 \mid \phi^{t_0}(x_0) = x(t_0) = 2$

$$\begin{cases} \dot{x} = x^3 - 1 \\ x(t_0) = 2 \end{cases} \quad \leftarrow \quad x(t) = \varphi(x_0)$$

$$\begin{cases} \dot{y} = y^2 \\ y(t_0) = 2 \end{cases} \qquad y(t) = \left(\frac{1}{2} - (t-t_0)\right)^{-1} = \frac{1}{\frac{1}{2} - (t-t_0)} = \frac{1}{t_0 + \frac{1}{2} - t}$$

T.d.C. $\Longrightarrow$ $x(t) \geq y(t), \ \forall t \geq t_0$

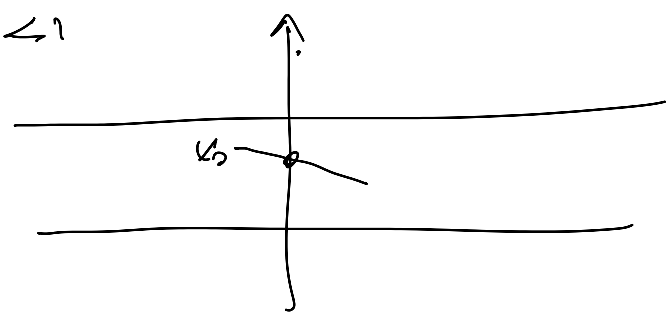


$\Longrightarrow$ $T_+ < +\infty$.

$T_+ \leq t_0 + \frac{1}{2}$

$$\sup_{(-\infty, T_+)} x(t) = +\infty \quad \text{e} \quad T_+ \leq t_0 + \frac{1}{2}$$

$x_0 < 1$

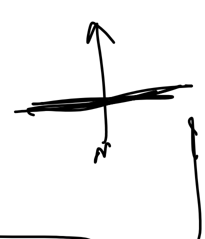


$\dot{x} = x^3 - 1$

$$\lim_{t \to T_+} x(t) = -\infty.$$

$x^3 - 1 < x^3$

$\dot{x} = x^3$



$$x(t) \leq y(t) \to -\infty$$
$t \geq t_0$

$3 \qquad dx = dt$

$$\begin{cases} \dot{x} = x \\ x(0) = x_0 \end{cases} \qquad \overline{x^3}$$

$$\left[-\frac{1}{2}\frac{1}{x^2}\right]_{x_0}^{x} = t$$

$$\frac{1}{2}\left(\frac{1}{x_0^2} - \frac{1}{x^2}\right) = t \qquad \frac{1}{x_0^2} - \frac{1}{x^2} = 2t$$

$$\frac{1}{x_0^2} - 2t = \frac{1}{x^2} = \operatorname{sgn} x_0 \cdot \left(\frac{1}{x_0^2} - 2t\right)^{-\frac{1}{2}}$$

$$(T_-, T_+) = (-\infty, T_+) \qquad T_+ = \frac{1}{2x_0^2} > 0.$$

$$\begin{cases} \dot{x} = x^3 - 1 \\ x(0) = x_0 \end{cases}, \quad x_0 < 1, \quad \lim_{t \to T_+} x(t) = -\infty.$$

fin t_0: $x(t_0) = -1$.

T.d.C. $\quad x(t) \leq -\left(1 - 2(t - t_0)\right)^{-\frac{1}{2}}$

$$= -\frac{1}{\sqrt{1 + 2(t_0 - t)}}$$

$$\longrightarrow T_+ < t_0 + \frac{1}{2}.$$

$$1 + 2(t_0 - t) = 0.$$
$$1 + 2t_0 = 2t$$
$$\frac{1}{2} + t_0 = \overline{t}_+$$

$$\lim_{t \to T_+} x(t) = -\infty.$$