- if $V_0 - \tau\overline{V} > 0$, i. e., $V_0 > \tau\overline{V}$, then $(V_0 - \tau\overline{V})e^{-t/\tau}$ is positive and decreasing. Thus $V(t) > \tau\overline{V}$ for all $t \geq 0$ and, in particular, $V(t) \to \tau\overline{V}$ from above, as $t \to +\infty$; see Fig. 2.7;
- if $V_0 - \tau\overline{V} = 0$, i. e., $V_0 = \tau\overline{V}$, then $V(t) = \tau\overline{V}$ for all $t \geq 0$.

Other equations arising in the electric circuit theory will be discussed later.

2.5 Exercises

1. Solve $x' = -2x$.
2. Solve $x' + (\ln 2)x = 2$.
3. Solve $x' + 2tx = 6t$.
4. Solve $x' - t = x + 1$.
5. Solve $x' + 2x = 4t$, $x(0) = 2$.
6. Solve $x' + (t \sin t)x = 0$, $x(\frac{\pi}{2}) = 1$.
7. Solve the initial value problem $x' + 2tx = 4t$, $x(0) = 1$.
8. * Find a number k such that $x' + kx = 0$, $x(1) = 2$, $x(2) = 1$ has a solution.
   Note: Such problems where initial conditions are specified at two different values of t, are called *boundary value problems.*
9. Show that if $k < 0$ there is no solution of $x' + kx = h$ such that $\lim_{t\to+\infty} x(t)$ is finite, but the constant one.
10. Show that for any $x_0 \in \mathbb{R}$ the solution of $x' - x = h$, $x(0) = x_0$ is such that $\lim_{t\to-\infty} x(t) = -h$.
11. Find the limits as $t \to \pm\infty$ of the solution of $x' = \frac{2t}{1+t^2}x$, $x(0) = 1$.
12. Find x_0 such that the solution of $x' = \frac{1}{1+t}x$, $x(0) = x_0$, satisfies $\lim_{t\to\pm\infty} x(t) = 0$.
13. Find $p \neq -1$ such that all the non-constant solutions of $x' + \frac{1}{p+1}t^p x = 0$ tend to zero as $t \to +\infty$.
14. * Explain that the boundary value problem $x' + kx = 0$, $x(1) = 2$, $x(3) = -1$ has no solution. This shows that, unlike the initial value problems, boundary value problems for first order homogeneous equations do not always have solutions.
15. Show that the boundary value problem $x' + p(t)x = q(t)$, $x(t_1) = x_1$, $x(t_2) = x_2$, $p(t)$ and $q(t)$ continuous, cannot have more than one solution.
16. Show that if $x(t)$ is a nontrivial solution to the initial value problem $x' - (\sin t)x = \sin t$, $x(\frac{\pi}{2}) = 1$, then $x(t)$ oscillates on $(0, \infty)$, i. e., it vanishes infinitely often (like the function $\sin t$).
17. Find the largest interval in which the solution to the following Cauchy problems exist:
    1. $x' + \frac{1}{t^2-4} = 0$, $x(-1) = 10$.
    2. $x' + \frac{1}{t^2-4} = 0$, $x(3) = 1$.
    3. $x' + \frac{1}{t^2-4} = 0$, $x(-10) = 1$.

18. Solve $t^3x' - 2t^2x = t^7$, $x(1) = 1$.
19. Solve $x' = t^2 - 1 - x$, $x(0) = -1$.
20. Solve $x' + (\cot t)x = \cos t$, $t \neq n\pi$.
21. Show that if x_1 and x_2 are solutions of the nonhomogeneous equation $x' + p(t)x = q(t) \neq 0$, then $x(t) = x_1 + x_2$ is not a solution.
22. Show that if x_1 and x_2 are solutions of the homogeneous equation $x' + p(t)x = 0$, then so is the linear combination $x(t) = c_1x_1 + c_2x_2$.
23. Solve $x' + x = \sin t + \cos t$.
24. (a) Show that if $f(t)$ is a differentiable function, then all solutions of $x' + x = f(t) + f'(t)$ are asymptotic to $f(t)$, that is, they approach $f(t)$ as $t \to \infty$.
    (b) Find a differential equation such that all of its solutions are asymptotic to $t^2 - 1$.
25. * Let $x_1(t)$ be any solution of (2.1) $x' + p(t)x = q(t)$. Recall that $x(t) = ce^{-\int_{t_0}^{t} p(t)\,dt}$ is the general solution of (2) $x' + p(t)x = 0$. Show that $x(t) = x_1(t) + ce^{-\int_{t_0}^{t} p(t)\,dt}$ is the general solution of the nonhomogeneous equation (2.1).
26. Use the method outlined in problem 18 to find the general solution of $x' - x = \cos x - \sin x$ by finding the general solution of the corresponding homogeneous equation and then a particular solution of this nonhomogeneous equation by inspection.
27. Explain why the solution in the preceding problem contradicts both properties (a) and (b) of Corollary 2.1.
28. Solve $\frac{1}{2}x' - x = \cos 2t - \sin 2t$.
29. Show that the ivp $tx' + x = 0$, $x(0) = a \neq 0$ has no solution. Explain why this does not contradict the existence and uniqueness Theorem 2.1.
30. Explain why the function $x(t) = e^t - 1$ cannot be a solution of any homogeneous equation $x' + p(t)x = 0$, where $p(t)$ is some continuous function.
31. Consider the equation of a RC electric circuit $V' + \frac{1}{RC}V = 0$. Find the time constant $\tau = RC$ such that $V(4) = \dfrac{1}{6}V(0)$.
32. Consider the equation of a RC electric circuit $V' + \frac{1}{RC}V = \overline{V}$. Find the time constant $\tau = RC$ such that $\lim_{t\to+\infty} V(t) = \frac{1}{4}\overline{V}$.