3.5 Exercises

1. Find a function $f(x)$ such that
$$x(t) = \begin{cases} e^t & \text{if } t \ge 0, \\ t+1 & \text{if } t \le 0, \end{cases}$$
is the solution of the Cauchy problem $x' = f(x)$, $x(0) = 1$.
2. Verify that the local existence and uniqueness theorem applies to the ivp $x' = \sqrt{t} + e^x$, $x(0) = 0$.
3. Verify that for any $(t_0, x_0) \in \mathbb{R}^2$ the local existence and uniqueness theorem applies to the ivp $x' = \ln(1 + x^4)$, $x(t_0) = x_0$.
4. Find the first three terms of the Picard approximation scheme for $x' = tx$, $x(0) = 1$.
5. Find the domain of definition of the solution to the ivp $x' = x^2$, $x(0) = a$
6. Verify that $x(t) = (t + \sqrt{a})^2$ for $t \ge -\sqrt{a}$ and $x(t) = 0$ for $t < -\sqrt{a}$ is the unique solution to the ivp $x' = 2\sqrt{x}$, $x(0) = a > 0$.
7. Show that the Cauchy problem $x' = x^{1/3}$, $x(0) = 0$, does not have a unique solution.
8. Show that the solutions of $x' = \cos^2 x$ are defined for all $t \in \mathbb{R}$.
9. Show that the solutions of $x' = \sqrt{1 + x^2}$ are defined on all $t \in \mathbb{R}$.
10. Show that the solution $x(t)$ of the Cauchy problem $x' = 1 + x^2$, $x(0) = 0$, cannot vanish for $t > 0$.
11. Show that the solution of the Cauchy problem $x' = \sin x$, $x(0) = 1$, is such that $0 < x(t) < \pi$ and is increasing.
12. Prove that if $f(x)$ is odd and $x(t)$ is a solution of $x' = f(x)$, then $-x(t)$ is also a solution.
13. Let $f(x)$ be an even $C^1(\mathbb{R})$ function and let $x_0(t)$ be a solution of $x' = f(x)$ such that $x(0) = 0$. Prove that $x_0(t)$ is an odd function.
14. If f is an odd function, show that the solutions of $x' = f(tx)$ are even.
15. Solve $x' = |x| - 1$, $x(0) = 0$.
16. Show that for any $x_0 \in \mathbb{R}$ the solutions to $x' = x^3 - 1$, $x(0) = x_0$, tend to 1 as $t \to -\infty$.
17. Find the $\lim_{t \to +\infty} x(t)$, where $x(t)$ is the solution of the Cauchy problem $x' = 1 - e^x$, $x(0) = 1$.
18. Find the $\lim_{t \to +\infty} x(t)$, where $x(t)$ is the solution of the Cauchy problem $x' = 1 - \arctan x$, $x(0) = 0$.
19. Let $x(t)$ be the solution of the Cauchy problem $x' = \ln(1 + x^2)$, $x(0) = 0$. Show that $\lim_{t \to +\infty} x(t) = +\infty$ and $\lim_{t \to -\infty} x(t) = -\infty$.
20. Study the qualitative behavior of the solution to $x' = 3x - x^2$, $x(0) = 1$.
21. Study the convexity of the solutions to $x' = x^3 - 1$.
22. Study the convexity of the solutions to $x' = x(2 - x)$.
23. Study the convexity of the solutions to $x' = x(1 + x)$.
24. Show that the solution to $x' = x^2 - t^3$, $x(1) = 1$ has a strict maximum at $t = 1$.
25. Find the locus of maxima of $x' = x^3 - t$.
26. Show that the solutions to $x' = e^x - t - t^5$ cannot have minima.

27. Show that the solution of $x' = x^4$, $x(0) = x_0 \neq 0$ is strictly convex if $x_0 > 0$ and strictly concave if $x_0 < 0$. Extend the result to $x' = x^p$, $x(0) = x_0 \neq 0$, $p \geq 1$.
28. Show that for any x_0 the solution of the ivp $x' = 1 + 2t + \sin^2 x$, $x(0) = x_0$, is such that $\lim_{t\to+\infty} x(t) = +\infty$.
29. Let $x(t)$ be the solution to $x' = f(t,x)$, $x(0) = x_0 > 0$, where f is smooth for $(t,x) \in \mathbb{R}^2$. If $x(t)$ is defined on $[0,+\infty)$ and $f(t,x) > x$, show that $\lim_{t\to+\infty} x(t) = +\infty$.
30. Let $x(t)$ be the solution of $x' = -1 - t + h(x)$, $x(0) = a$, with $0 < a < 2$, where h is a smooth function such that $h(x) \leq -x$. If $x(t)$ is defined for all $t \geq 0$, show that the equation $x(t) = 0$ has at least one solution contained in $(0,1)$.