problem

$$x = tx' + h(x'), \quad x(a) = b, \tag{4.17}$$

(1) there are points $(a, b) \in \mathbb{R}^2$ where there is a lack of uniqueness since (4.17) has two solutions—in particular, through any point of the singular solution passes the singular solution itself and the lines $x = ct + h(c)$ tangent to the singular solution at that point;

(2) there are points $(a, b) \in \mathbb{R}^2$ such that (4.17) has no solution.

For example, let us solve the ivp $x = tx' - \frac{1}{2}(x')^2$, $x(0) = b$ (see Example 4.8(i)). Inserting the initial condition into the general solution yields $b = -\frac{1}{2}c^2$ namely $c^2 = -2b$. Thus for $b < 0$ there are two solutions $x = \sqrt{(-2b)} \cdot t + b$ and $x = -\sqrt{(-2b)} \cdot t + b$; for $b = 0$ we find $x = 0$, to which we have to add the singular solution $x = \frac{1}{2}t^2$. Finally, for $b > 0$ there are neither solutions given by the general integral, nor the singular solution. □

4.6 Exercises

4.6.1 Exercises on separable equations

1. Find the general solution of $x' = x^k$, $k > 0$.
2. Find the general solution of $x' = x(x + 1)$.
3. Find the general solution of the logistic equation $x' = x(1 - x)$.
4. Find the general solution of $x' = \frac{2t^p+1}{3x^q+1}$, p, q positive numbers.
5. * Solve $xx' = \sqrt{x^2 - 1}$.
6. Solve $\sqrt{1 - t^2}x' = x$.
7. Solve $x' = (1 + 2\tilde{t} - t^2) \cdot \frac{x+x^2}{1+2x}$.
8. Solve $x' = \frac{x+1}{1+t^2}$.
9. Solve $x' = \frac{t-1}{t} \cdot \frac{x^3-x^2+x-1}{x^2+x}$.
10. Solve $x' = 4t\sqrt{x}$, $x > 0$, $x(0) = 1$.
11. Solve $x' = 4t^3\sqrt{x}$, $x \geq 0$, $x(0) = 1$.
12. Find the limits as $t \to \pm\infty$ of the solution of $x' = t^{-2}x^2$, $x(1) = -1$.
13. Solve $x' = x(2 - x)$, $x(0) = -1$ and $x(0) = 1$. Draw the graphs of the solutions.

4.6.2 Exercises on homogeneous equations

14. Solve $x' = \frac{x}{t} - (\frac{x}{t})^2$.
15. Solve $x' = \frac{x}{t} + 2(\frac{x}{t})^3$.
16. Solve $x' = \frac{x}{t} + \tan(\frac{x}{t})$.

17. Solve $x' = \frac{x^2+tx}{t^2}$.
18. Solve $x' = \frac{x^2+tx+t^2}{t^2}$.
19. Solve $x' = \frac{x^3-t^3}{tx^2}$.

4.6.3 Exercises on Bernoulli equations

20. Solve the Bernoulli equation $x' = 4x - 2t \cdot \sqrt{x}$, $x \geq 0$.
21. Solve the Bernoulli equation $x' = x + 2tx^{-1}$, $x(0) = 1$.
22. Solve the Bernoulli equation $x' = \frac{x}{t} + 3x^3$.
23. Solve the Bernoulli equation $x' = -\frac{x}{t} + x^{-2}$, $x(1) = 1$.
24. Solve the Riccati equation $x' = -x + x^2 - \frac{1}{t}$. [Hint: check that $z = -\frac{1}{t}$ is a particular solution.]

4.6.4 Exercises on Clairaut equations

25. Solve the Clairaut equation $x = (t-1)x' - (x')^2$ and find the singular solution.
26. Solve the Clairaut equation $x = tx' - \frac{1}{4}(x')^4$ and find the singular solution.
27. Solve the Clairaut equation $x = tx' - e^{x'}$ and find the singular solution.
28. Solve the Clairaut equation $x = tx' + \frac{1}{3(x')^3}$ and find the singular solution.
29. Solve the Cauchy problem $x = tx' + (x')^2$, $x(0) = b$.
30. Solve the Cauchy problem $x = tx' + \frac{1}{x'}$, $x(1) = b$.