Theorem 5.2 (Local existence and uniqueness for exact equations). *Let M, N be continuous on $S \subseteq \mathbb{R}^2$ and suppose that the equation $M(x,y)dx + N(x,y)dy = 0$ is exact. Let $P_0 = (x_0, y_0) \in S$ be such that $N(x_0, y_0) \neq 0$, or $M(x_0, y_0) \neq 0$. Then the exact equation $M(x,y)dx + N(x,y)dy = 0$ has one and only one solution passing through P_0.*

Proof. Since $Mdx + Ndy = 0$ is exact, there exists a C^1 function $F(x,y)$, such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$. If $N(x_0,y_0) \neq 0$, then $F_y(x_0,y_0) = N(x_0,y_0) \neq 0$ and we can apply the implicit function theorem to $F(x,y) = c_0 = F(x_0,y_0)$ at $P_0 = (x_0,y_0)$, yielding a unique differentiable function $y = g(x)$, defined in a neighborhood I of x_0 such that

$$F(x, g(x)) = c_0, \quad \forall x \in I, \quad g(x_0) = y_0.$$

Differentiating the preceding identity we find $F_x(x,g(x)) + F_y(x,g(x))\frac{dg(x)}{dx} = 0$, namely $M(x,g(x)) + N(x,g(x))\frac{dg(x)}{dx} = 0$, $x \in I$. This shows that $y = g(x)$ is a solution of (5.2). Since, in addition, $g(x_0) = y_0$, it follows that $y = g(x)$ is the unique solution of the ivp for (5.1) at P_0 we were looking for.

Similarly, if $M(x_0,y_0) \neq 0$ then $F_x(x_0,y_0) = M(x_0,y_0) \neq 0$ and the implicit function theorem yields a unique $x = h(y)$ such that $F(h(y),y) = c_0$, $h(y_0) = x_0$. Repeating the previous arguments it follows that $x = h(y)$ solves the ivp for (5.1) at P_0.

Notice that the result is local, in the sense that $g(x)$, resp. $h(y)$, is defined (in general) near x_0, resp. y_0. □

5.6 Exercises

1. Find the solution of $\cos x dx + e^y dy = 0$ passing through $(0,0)$ by solving it as an exact equation.
2. Solve $(4x^3 + 6x^5)dx - 2ydy = 0$, and find a such that there is a unique solution passing through $(0,a)$.
3. Solve $2axdx + 2bydy = 0$, $a \cdot b \neq 0$, and find (x_0,y_0) through which passes a unique solution.
4. Solve $2xy\,dx + (x^2 + y^2)dy = 0$.
5. Solve $(2x + y)dx + (x + 2y)dy = 0$.
6. Solve $x^2 + ye^x + (y + e^x)y' = 0$.
7. Solve $(x^2 + 2y)dx + (2x - y^3)dy = 0$.
8. Solve $(12x^5 - 2y)dx + (6y^5 - 2x)dy = 0$.
9. Solve $(y + \frac{1}{x})dx + (x - \frac{1}{y})dy = 0$.
10. Find a number a such that $(x^3 + 3axy^2)dx + (x^2y + y^4)dy = 0$ is exact and solve it.
11. Find numbers a and b such that $(xy + ay^3)dx + (bx^2 + xy^2)dy = 0$ is exact and solve it.
12. Find the solutions of $2xdx + 3(1 - y^2)dy = 0$ passing through $(0,2)$.
13. Solve $2xy^3 + 1 + (3x^2y^2)y' = 0$, $y(1) = 1$.

14. Solve $(y + 8x^3)dx + (x + 3y^2)dy = 0$, $y(1) = -1$.
15. Find the solution of $(x^2 - 1)dx + ydy = 0$ passing through $(-1, b)$ with $b > 0$ and show that it can be given in the form $y = y(x)$.
16. * Solve $(3y^4 - 1)dy - (2x + 1)dx = 0$, $y(0) = 1$ and show that, taking a sufficiently small neighborhood of $(0, 1)$, there exists a unique solution which can be written as $y = y(x)$ or as $x = x(y)$.
17. * Find the solutions of $2xdx + 3(1 - y^2)dy = 0$ passing through $(0, 1)$ and show that it has a node.
18. Solve $ydx - 3xdy = 0$.
19. Solve $(y^3 + 1)dx + 3y^2dy = 0$.
20. Solve $(x^2y + x^2)dx + x^2dy = 0$.
21 (A) Solve $(xy + x)dx + x^2dy = 0$ by finding an integrating factor $\mu(x)$.
(B) Solve $(xy + x)dx + x^2dy = 0$ by finding an integrating factor $\mu(y)$, and compare the answer to that in part (A).
22. Solve $(y + \frac{1}{2}xy^2 + x^2y)dx + (x + y)dy = 0$.
23. Solve $y(\cos x + \sin^2 x)dx + \sin x dy = 0$.
24. Show that there exists an integrating factor $\mu = \mu(y)$ for the equation $(1 + f(y))dx + (xg(y) + y^2)dy = 0$, where f and g are some differentiable functions $f \neq -1$.
25. Solve $(3y + x)dx + xdy = 0$.
26. Solve $[(1 + x)y + x]dx + xdy = 0$.
27. Solve $(x - 2y)dx + (xy + 1)dy = 0$.
28. Solve $(y + xy + y^2)dx + (x + 2y)dy = 0$.
29. Solve $2ydx + (x + \sqrt{y})dy = 0$, $(y \geq 0)$.
30. * Solve $(2x + hy)dx - (kx + 2y)dy = 0$, where $h, k \in \mathbb{R} - \{0\}$.