Coming back to the variable $t = e^s$ we obtain

$$x(t) = c\sin(\omega \ln t + \theta) + \frac{1}{1+\omega^2}t, \quad t > 0.$$

In Chapter 11 we will discuss another method to solve Euler's equation, by using solutions as power series.

6.5 Exercises

6.5.1 Exercises on linear dependence and independence

1. Show that $x_1 = 2\sin t$ and $x_2 = \sin 2t$ are linearly independent.
2. Show that $x_1 = e^{2t}$, $x_2 = e^t$ are linearly independent.
3. Show that $x_1 = (t^2-1)^5$ and $x_2 = (2-2t^2)^5$ are linearly dependent.
4. Find θ such that $\sin t$ and $\sin(t+\theta)$ are linearly dependent.
5. Show that $x_1 = \sin^2 t$ and $x_2 = \cos^2 t$ are linearly independent.
6. Show that $x_1 = 2t$, $x_2 = 3|t|$ are linearly dependent on $(-\infty, 0)$ but linearly independent on $\mathbb{R}$.
7. Let $y(t)$ be any function, $y(t) \neq 0$ for $t \in \mathbb{R}$. Show that if $x_1(t)$ and $x_2(t)$ are linearly independent, then so are $y(t)x_1(t)$ and $y(t)x_2(t)$.
8. Let f, g be linearly independent. Show that, for any $a, b \in \mathbb{R}$, $a \neq 0$, $x_1 = af + bg$ and $x_2 = g$ are linearly independent.
9. Show that if $p \neq q$ then $u_1 = t^p$ and $u_2 = t^q$ cannot be solutions of the same second order equation $L[x] = 0$.
10. Let u_1, u_2 be solutions of $L[x] = 0$ on I such that $u_1(t_0) = u_2(t_0) = 0$ for some $t_0 \in I$. Show that they are linearly dependent and hence they have all their zeros in common.
11. Using variations of parameters, find a particular solution of

$$x'' + x = \frac{1}{\cos t}, \quad t \neq \pm\frac{\pi}{2} + 2k\pi.$$

6.5.2 Exercises on equations with constant coefficients

12. Find the general solution of $x'' - 2x' - 4x = 0$.
13. Find the general solution of $x'' + 2x' + 10x = 0$.
14. Find the general solution of $x'' - x' + 2x = 0$.
15. Solve $x'' + 3x' = 0$, $x(0) = 1$, $x'(0) = -1$.
16. Solve $x'' - 2x' + x = 0$, $x(0) = 1$, $x'(0) = 2$.
17. Find the general solution of $9x'' + 6x' + x = 0$.
18. Solve the ivp $x'' + 4x' - 5x = 0$, $x(0) = 1$, $x'(0) = -1$.

19. Solve the ivp $x'' + 2x + 2 = 0$, $x(0) = 2$, $x'(0) = 0$.
20. Show that the nonconstant solutions of $x'' - kx' = 0$, $x(0) = 0$, $x'(0) = k$ are increasing or decreasing.
21. Let $x_a(t)$ be the solution of the ivp $x'' - x' - 2x = 0$, $x(0) = 0$, $x'(0) = a$. Find the $\lim_{t\to+\infty} x_a(t)$.
22. Show that for every $k \geq 0$ at least one solution of $x'' + 2x' - kx = 0$ tends to 0 as $t \to +\infty$.
23. Find $k \in \mathbb{R}$ such that all solutions of $x'' + 2kx' + 2k^2x = 0$ tend to 0 as $t \to +\infty$.
24. Solve $x'' + x = t^2 + 1$.
25. Solve $x'' - 4x' + 3x = \sin t$.
26. Solve $x'' - x = te^t$.
27. Solve $x'' + 4x = \sin \omega t$.
28. Solve $x'' + 4x = \sin t + \sin 2t$.

6.5.3 Exercises on Euler equations

29. Solve the Euler equation $2t^2x'' - tx' + x = 0$, $t > 0$.
30. Solve $t^2x'' - tx' + x = 0$.
31. Solve $t^2x'' - 4tx' + 6x = 0$.
32. Solve $t^2x'' - 5tx' + 5x = 0$.
33. Solve
$$t^2x'' - 2tx' + 2x = 0, \quad x(1) = 1, \quad x'(1) = 0.$$
34. Consider the Euler equation
$$at^2x'' + btx' + cx = 0, \quad t > 0.$$
Show that if $a > 0$ and $c < 0$, then the general solution of the Euler equation, is of the form $c_1t^p + c_2t^q$.
35. Solve $t^2x'' + tx' + x = 0$, $x(1) = 1$, $x'(1) = -1$.