contained in Ω, which is impossible by assumption (a). Then one can prove that (9.8) possesses a closed orbit (hence a periodic solution) such that the nearby trajectories look like spirals approaching the closed orbit from outside and from inside, which is therefore a limit cycle.

For example, Theorem 9.6 might be used to find the periodic solution of van der Pol equation.

For a broader treatment of the Poincaré–Bendixon theory, we refer, e. g., to the aforementioned book by H. Amann.

9.7 Exercises

1. Using the phase plane analysis, show that the solution of $x'' - 3x^2 = 0$ $x(0) = 1$, $x'(0) = 0$ is positive, has a minimum at $t = 0$ and is convex. Sketch a qualitative graph of the solution. [Hint: see Example 9.1.]

$$\frac{1}{2}y^2 - x^3 = -1 \Rightarrow yy' = 3x^2x' \Rightarrow x'x'' = 3x^2x' \Rightarrow x'' = 3x^2 \quad (\text{if } x' \neq 0).$$

2. Using the phase plane analysis, show that the solution of $x'' - 4x^3 = 0$ $x(0) = 0$, $x'(0) = -1$ is increasing, convex for $t < 0$ and concave for $t > 0$. Sketch a qualitative graph of the solution.
3. Show that the solution of $x'' + x + 4x^3 = 0$, $x(0) = 1$, $x'(0) = 0$ is periodic.
4. Show that the solution of $x'' - x + 4x^3 = 0$, $x(0) = 1$, $x'(0) = 0$ is periodic.
5. Prove that the solutions of the nonlinear harmonic oscillator $x'' + x - x^3 = 0$ with energy $c = 1$ are monotonic and unbounded.
6. Let $x_a(t)$ be the solution of the nonlinear harmonic oscillator $x'' + \omega^2 x - x^3 = 0$, $x_a(0) = 0$, $x_a'(0) = a$. Find $a > 0$ such that $x_a(t)$ is periodic.
7. Knowing that the boundary value problem $x'' + x^3 = 0$, $x(-1) = x(1) = 0$ has a solution $x(t)$ such that $M = \max_{[-1,1]} x(t) = 3$, find $x'(-1)$.
8. Show that the system

$$\begin{cases} x' = x + 2y, \\ y' = -2x - y, \end{cases}$$

is a hamiltonian system and prove that all non-trivial solutions are periodic.
9. Let $x_a(t), y_a(t)$ be the solution of the hamiltonian system

$$\begin{cases} x' = ay + y, \\ y' = -x - ax, \end{cases}$$

such that $x(0) = 0$, $y(0) = 1$. Find a such that $x_a(t), y_a(t)$ is periodic.
10. Show that for $a > 0$ the equation $x'' + ax^3 = 0$ has no homoclinic.
11. Show that $x'' = x - x^5$ has a positive and a negative homoclinic at $x^* = 0$ and find its maximum, resp. minimum, value.

12. Find k such that the equation $x'' = k^2x - x^3$ has a homoclinic $x(t)$ to 0 such that $\max_{\mathbb{R}} x(t) = 2$.
13. Show that $x'' = x - x^4$ has one and only one positive homoclinic at $x^* = 0$.
14. Show that for $k < 0$ the equation $x'' = kx - x^3$ has no homoclinic at 0.
15. Show that $x'' + V'(x) = 0$ cannot have homoclinics if V' does not change sign.
16. Show that if $x'' + V'(x) = 0$ has a homoclinic to 0, then V cannot have a minimum or a maximum at $x = 0$.
17. Show that for $p \geq 1$ the equation $x'' + x^p = 0$ has no heteroclinic.
18. Find a such that $x'' + 2x - 2x^3 = 0$, $x(0) = 0$, $x'(0) = a$ is a heteroclinic.
19. Find the integers $k > 1$ such that $x'' + x - x^k = 0$ has a heteroclinic.
20. * Let $T = T(a)$ the period of the periodic solution of the nonlinear harmonic oscillator $x'' + \omega^2 x - x^3 = 0$ such that $x(0) = 0$, $x'(0) = a > 0$. Show that $\lim_{a\to 0} T(a) = \frac{2\pi}{\omega}$.
21. * Show that there exists a unique $\lambda > 0$ such that the boundary value problem $x'' + \lambda x^3 = 0$, $x(0) = x(\pi) = 0$ has a positive solution.
22. * Describe the behavior of the solution of the Kepler problem with energy (i) $c = U_{\text{eff}}(r_0) = \min U_{\text{eff}}(r)$, (ii) $c = 0$ and (iii) $c > 0$.
23. Find a, c such that the equilibrium of the Lotka–Volterra system
$$\begin{cases} x' = ax - 3xy, \\ y' = -cy + xy, \end{cases}$$
is $(2, 3)$.
24. Let $x_0(\epsilon)$, $y_0(\epsilon)$ be the equilibrium of
$$\begin{cases} x' = ax - bxy - \epsilon x, \\ y' = -cy + dxy - \epsilon y. \end{cases}$$
Show that if $\epsilon > 0$ then $x_0(\epsilon) > x(0)$, $y_0(\epsilon) < y(0)$. Explain this result in terms of prey and predators.
25. Using Theorem 9.6, prove that the system
$$\begin{cases} x' = -y + x(1 - (x^2 + y^2)), \\ y' = x + y(1 - (x^2 + y^2)), \end{cases}$$
has a unique limit cycle and find it.
26. Given $f, g \in C^1(\mathbb{R}^2)$ consider the planar system
$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases}$$
and suppose that the vector field $F = (f, g)$ is such that $\operatorname{div} F > 0$. Prove that the system has no periodic solution.